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A Topological Conjugacy of Invariant Flows on Some Class of Lie Groups

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ABSTRACT: The aim of this paper is to classify invariant flows on Lie group G whose Lie algebra $\mathfrak{g}$ is associative or semisimple. Specifically, we present this classification from the hyperbolicity of the lift flows on $G \times \mathfrak{g}$. Then we apply this construction to some special cases as $\mathrm{Gl}(n, \mathbb{R})$ and affine Lie group.

Key Words: Topological conjugacy, Invariant flows, Lie groups.

Contents

1. Introduction

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and take $A \in \mathfrak{g}$. Consider the dynamical system

$$\dot{g} = A_g = R_{g*}(A), \quad g \in G. \tag{1.1}$$

We are interested in establishing a condition to classify systems of type (1.1) via topological conjugacy, i.e., to find a homeomorphism between the state spaces that maps trajectories of one system to trajectories of the other system, preserving the parametrization by time. We restrict to the cases where $\mathfrak{g}$ is an associative or semisimple Lie algebra. First, we study the associative case and then we apply it in the semisimple case. To develop our results we adopt the idea, due to Ayala, Colonious and Kliemann in [1], of we work with vector bundles.

In the classical case, $\dot{x} = A_i x$, $x \in \mathbb{R}^n$, $A_i \in \mathfrak{gl}(n, \mathbb{R})$ for $i = 1, 2$, a fundamental hypothesis to obtain the above homeomorphism is the hyperbolicity of the systems, that is, $\|e^{A_i t} x\| \leq C_i e^{-a_i t}\|x\|$ for some $a_i > 0$, $C_i > 1$ and for all $t > 0$ (see e.g. Robinson [5, ch.IV]). As in Lie groups there is not a normed space structure we lift the flows to $G \times \mathfrak{g}$, which inherits a normed vector structure from the Lie algebra $\mathfrak{g}$.

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In the following section, from the study of the lift flows on $G \times \mathfrak{g}$, we present a topological classification of flows on Lie group G. To be specific, we lift two systems of type (1.1) in two systems (on $G \times \mathfrak{g}$) of type

$$\begin{pmatrix} \dot{g} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} R_{g*}A \\ Av \end{pmatrix}, \tag{1.2}$$

whose solution is $(e^{At}g, e^{At}v)$. Supposing that these two systems on $G \times \mathfrak{g}$ are hyperbolic we have that they are topologically conjugate, and, as a consequence, we show that $e^{At}g$ and $e^{Bt}g$ are topologically conjugate on G (see Theorem 2.3). As application in the third section, we take G as the linear group $\mathrm{GL}(n, \mathbb{R})$, that is, we consider dynamical systems of kind $\dot{X} = AX$ where $X \in \mathrm{GL}(n, \mathbb{R})$ and from Theorem 2.3 we recover the classical result of dynamical system: if the real parts of generalized eigenvalues of A and B are negative, then e^{At} and e^{Bt} are topologically conjugate. To finish this section, we give a partial classification in case of $\mathrm{GL}(2, \mathbb{R})$. In the last section, we consider (1.1) on a semisimple Lie group and the purpose is to study the dynamical system (1.1) in the adjoint group $\mathrm{Ad}(G)$. In fact, we show that if the real parts of the generalized eigenvalues of $\mathrm{ad}(A)$ and $\mathrm{ad}(B)$ are negative, then e^{At} and e^{Bt} are topologically conjugate. From this, we can classify dynamical systems on affine groups $H \rtimes V$, where H is a semisimple Lie group and V a n-dimensional vector space, or rather, we establish a condition to topological conjugacy of flows on $H \rtimes V$.

2. Hyperbolic condition for topological conjugacy on Lie groups

Let G be a Lie group such that its Lie algebra $\mathfrak{g}$ is an associative algebra. This structure is natural in matrix groups. We adopt in the trivial vector bundle $G \times \mathfrak{g}$ the direct product

$$(g, v) \cdot (h, u) = (g \cdot h, v \cdot u).$$

Our intention is to present a hyperbolic sufficient condition on $\mathfrak{g}$ to obtain topological conjugacy between flows on G. Take the following dynamical system

$$\dot{g} = R_{g*}A, \quad A \in \mathfrak{g}$$

on the Lie group G. Consider, on the trivial bundle $G \times \mathfrak{g}$, the system

$$\begin{pmatrix} \dot{g} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} R_{g*}A \\ Av \end{pmatrix}. \tag{2.1}$$

A direct account shows that $(e^{At}g, e^{At}v)$ is a solution of (2.1) with $g \in G$ and $v \in \mathfrak{g}$ (see for instance [6]).

Remark 2.1. *Being $\mathfrak{g}$ an associative algebra it is true that $G \times g$ is a Lie group and projections of flows on $G \times \mathfrak{g}$ are natural on G and $\mathfrak{g}$. However, for our work, the importance of associativity property is that solution of $\dot{v} = Av$ is $e^{At}v = \sum_{i=1}^{\infty} \frac{(tA)^i}{i!}(v)$, which allow us to work with hyperbolic property in a easy way.*

Before we introduce the hyperbolic property of dynamical system of form (2.1), we say what means topological conjugacy of dynamical systems in our context. Take $A, B \in \mathfrak{g}$. Two dynamical systems $\dot{g} = R_{g*}A$ and $\dot{g} = R_{g*}B$ on G are topologically conjugate if there exist a homeomorphism $\psi : G \to G$ such that $\psi(e^{At}g) = e^{Bt}\psi(g)$ for $g \in G$. On $G \times \mathfrak{g}$, two dynamical system of form (2.1) are topological conjugate if there is a homeomorphism $\Psi : G \times \mathfrak{g} \to G \times \mathfrak{g}$ such that $\Psi((e^{At}, e^{At})(g, v)) = (e^{Bt}, e^{Bt})\Psi(g, v)$ for $(g, v) \in G \times \mathfrak{g}$.

Since hyperbolic property needs a norm, we begin by adopting a norm $\| \cdot \|_g$ in $\mathfrak{g}$. Then, in the trivial vector bundle $G \times \mathfrak{g}$, we consider the following norm

$$\|(g, v)\| = \|v\|_{\mathfrak{g}}.$$

For this reason,

$$\|(e^{At}g, e^{At}v)\| = \|e^{At}v\|_{\mathfrak{g}}.$$

Hence, the behavior of the flow $e^{At}v$ drives the behavior of the flow $(e^{At}g, e^{At}v)$ on $G \times \mathfrak{g}$. As a direct consequence, if $e^{At}v$ has the hyperbolic property, that is, there exists $a > 0$ and $C > 1$ such that for every $t \geq 0$ we have $\|e^{At}v\|_{\mathfrak{g}} \leq Ce^{-at}\|v\|_{\mathfrak{g}}$, then $(e^{At}g, e^{At}v)$ has the same property. Since $\mathfrak{g}$ is a vector space, it follows that there exists a norm $\| \cdot \|_A$ on $\mathfrak{g}$ such that $\|e^{At}v\|_A \leq e^{-at}\|v\|_A$ (see for instance Theorem 5.1 in [5, ch.IV]). We also denote by $\|(\cdot, \cdot)\|_A$ the norm on $G\times\mathfrak{g}$ associated with $\| \cdot \|_A$.

In the following we show a technical lemma that is necessary to show our main theorem.

Lemma 2.2. *Let $A \in \mathfrak{g}$ and assume that hyperbolic condition is satisfied, that is, there exists $a > 0$ such that for every $t \geq 0$ we have*

$$\|e^{At}v\|_A \leq Ce^{-at}\|v\|_A. \tag{2.2}$$

Then there exists a unique time τ such that $\|(e^{A\tau}g, e^{A\tau}v)\|_A = 1$.

Proof. We first assume that the hyperbolic condition is satisfied for $e^{At}v$. In consequence, the same occur with $(e^{At}g, e^{At}v)$, that is,

$$\|(e^{At}g, e^{At}v)\|_A \leq e^{-at}\|(g, v)\|_A, \quad t \geq 0. \tag{2.3}$$

On the other hand, if $t < 0$ then

$$\|e^{At}v\|_A \geq e^{a|t|}\|v\|_A,$$

and, consequently,

$$\|(e^{At}g, e^{At}v)\|_A \geq e^{a|t|}\|(g, v)\|_A.$$

Taking $t \to \infty$ or $t \to -\infty$ we obtain $\|(e^{At}g, e^{At}v)\|_A \to 0$ or $\|(e^{At}g, e^{At}v)\|_A \to \infty$, respectively. Since $\|(\cdot, \cdot)\|_A$ is a continuous function, there is a time τ such that

$$\|(e^{A\tau}g, e^{A\tau}v)\|_A = 1.$$

We claim that this time is unique. In fact, suppose that there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $\|(e^{A\tau_1}g, e^{A\tau_1}v)\|_A = \|(e^{A\tau_2}g, e^{A\tau_2}v)\|_A = 1$. Assume that $\tau_2 \geq \tau_1$. If we write $\tau_2 = (\tau_2 - \tau_1) + \tau_1$, then

$$1 = \|e^{A\tau_2}v\|_A = \|e^{A(\tau_2-\tau_1)}e^{A\tau_1}v\|_A.$$

From (2.3) it follows that

$$\|e^{A(\tau_2-\tau_1)}e^{A\tau_1}v\|_A \leq e^{-a(\tau_2-\tau_1)}\|e^{A\tau_1}v\|_A = e^{-a(\tau_2-\tau_2)}.$$

Therefore $1 = e^{-a(\tau_2-\tau_1)}$ and, consequently, $-a(\tau_2 - \tau_1) = 0$. Since $a > 0$, we conclude that $\tau_2 = \tau_1$. □ □

In this context, we can prove the following theorem which is an adaptation of a classical result (see for instance Theorem 2.2.8 in [2] or Theorem 7.1 in [5, ch.IV]).

Theorem 2.3. *Take $A, B \in \mathfrak{g}$ such that $e^{At}v$ and $e^{Bt}v$ satisfy the hyperbolic property. Then $(e^{At}g, e^{At}v)$ and $(e^{Bt}g, e^{Bt}v)$ are topologically conjugate in $G \times \mathfrak{g}$. In consequence, $e^{At}g$ and $e^{Bt}g$ are topologically conjugate in G.*

Proof. We first assume that there exist $a, b > 0$ such that for every $t \geq 0$ we have

$$\begin{array}{rcl} \|(e^{At}g, e^{At}v)\|_A & \leq & e^{-at}\|(g,v)\|_A \\ \|(e^{Bt}g, e^{Bt}v)\|_B & \leq & e^{-bt}\|(g,v)\|_B. \end{array}$$

For every $t < 0$, it follows that

$$\begin{array}{rcl} \|(e^{At}g, e^{At}v)\|_A & \geq & e^{a|t|}\|(g,v)\|_A \\ \|(e^{Bt}g, e^{Bt}v)\|_B & \geq & e^{b|t|}\|(g,v)\|_B. \end{array}$$

Denote by $\mathbb{S}_A$ and $\mathbb{S}_B$ the unit spheres in $G \times \mathfrak{g}$

$$\mathbb{S}_A = \{(g,v) : \|(g,v)\|_A = 1\} \text{ and } \mathbb{S}_B = \{(g,v) : \|(g,v)\|_B = 1\}. \tag{2.4}$$

It is clear that the flows $(e^{At}g, e^{At}v)$ and $(e^{Bt}g, e^{Bt}v)$ cross $\mathbb{S}_A$ and $\mathbb{S}_B$, respectively, in an unique point, by Lemma (2.2).

Let $t_A : G \times \mathfrak{g} \to \mathbb{R}$ be the map that associates for every $(g,v) \in G \times \mathfrak{g}$ the unique time $t_A(g,v)$ such that $\|(e^{At_A(g,v)}g, e^{At_A(g,v)}v)\|_A = 1$. This map is well defined by Lemma (2.2). We claim that t_A is continuous. In fact, take $(g,v) \in G \times \mathfrak{g}$ and a sequence $(g_n, v_n) \in G \times \mathfrak{g}$ such that $(g_n, v_n) \to (g,v)$. We observe that $v_n \to v$, that is, $\|v_n - v\|_A \to 0$. For simplicity, suppose that $t_A(g_n, v_n) > 0$ for every $n \in \mathbb{N}$. Thus we have

$$\|e^{At_A(g,v_n)}(v_n - v)\|_A \leq e^{-at_A(g_n,v_n)}\|v_n - v\|_A \to 0.$$

This gives

$$\lim_{n\to\infty} e^{At_A(g_n,v_n)}(v_n) = \lim_{n\to\infty} e^{At_A(g_n,v_n)}v.$$

Since $\|\cdot\|_A$ is a continuous map,

$$\lim_{n\to\infty} \|e^{At_A(g_n,v_n)}(v_n)\|_A = \|e^{A\lim_{n\to\infty} t_A(g_n,v_n)}v\|_A$$

As $\|e^{At_A(g_n,v_n)}(v_n)\|_A = 1$ we have $\|e^{A\lim_{n\to\infty} t_A(g_n,v_n)}v\|_A = 1$. According to uniqueness, we get $\lim_{n\to\infty} t_A(g_n, v_n) = t_A(g, v)$. Therefore t_A is a continuous map.

Before we construct a homeomorphism that conjugate the flows, we show the following property: $t_A(e^{At}g, e^{At}v) = t_A(g, v) - t$. In fact,

$$1 = \|e^{At_A(g,v)}v\|_A = \|e^{At_A(g,v)}e^{A(-t)}e^{At}v\|_A = \|e^{A(t_A(g,v)-t)}e^{At}v\|_A.$$

By uniqueness, we conclude that $t_A(e^{At}g, e^{At}v) = t_A(g, v) - t$.

We define the maps $\psi_0 : \mathbb{S}_A \to \mathbb{S}_B$ by $\psi_o(g, v) = (g, \frac{v}{\|v\|_B})$ and $\phi_0 : \mathbb{S}_B \to \mathbb{S}_A$ by $\phi_o(g, v) = (g^{-1}, \frac{v}{\|v\|_A})$. Note that $\phi_0 = \psi_0^{-1}$ and that ψ_0 and ϕ_0 are continuous. Hence ψ_0 is a homeomorphism.

Now we extend the homeomorphism ψ_0 to $G \times \mathfrak{g}$. We begin by defining the map $\psi : G \times \mathfrak{g} \to G \times \mathfrak{g}$ by

$$\psi(g, v) = \begin{cases} (e^{-B\,t_A(g,v)}e^{A\,t_A(g,v)}g, e^{-B\,t_A(g,v)}\psi_0(g, e^{At_A(g,v)}v)) & \text{if } v \neq 0 \\ (g, 0) & \text{if } v = 0. \end{cases}$$

To simplify, we write $\psi(g, v) = (h_1(g), h_2(v))$ where

$$h_1(g) = e^{-B\,t_A(g,v)}e^{A\,t_A(g,v)}g$$

and

$$h_2(v) = \begin{cases} e^{-B\,t_A(g,v)}\psi_0(g, e^{At_A(g,v)}v) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$$

We now proceed to show the conjugacy of the flows. In fact,

$$\psi(e^{At}g, e^{At}v) =$$
$$(e^{-B\,t_A(e^{At}g,e^{At}v)}e^{At_A(e^{At}g,e^{At}v)}e^{At}g,\ e^{-B\,t_A(e^{At}g,e^{At}v)}\psi_0(g, e^{At_A(e^{At}g,e^{At}v)}e^{At}v).$$

Using $t_A(e^{At}g, e^{At}v) = t_A(g, v) - t$ we obtain

$$\begin{aligned}
\psi(e^{At}g, e^{At}v) &= (e^{-B\,[t_A(g,v)-t]}e^{A[t_A(g,v)-t]}e^{At}g,\ e^{-B\,[t_A(g,v)-t]}\psi_0(g, e^{A[t_A(g,v)-t]}e^{At}v)) \\
&= (e^{Bt}e^{-B\,t_A(g,v)}e^{At_A(g,v)}g,\ e^{Bt}e^{-B\,t_A(g,v)}\psi_0(g, e^{At_A(g,v)}v)) \\
&= (e^{Bt}h_1(g),\ e^{Bt}h_2(v)) \\
&= (e^{Bt}, e^{Bt})(h_1(g), h_2(v)) \\
&= (e^{Bt}, e^{Bt})\psi(g, v).
\end{aligned}$$

The next step is to show that ψ is continuous. We need only consider the case (g, v) with $v = 0$. In fact, the map $\psi = (h_1, h_2)$ is continuous if $v \neq 0$ because $(e^{At}g, e^{At}v)$, $(e^{Bt}g, e^{Bt}v)$, t_A and t_B are continuous. We begin by observing that

h_1 is continuous at $v = 0$, so we only need to show that h_2 is continuous at $v = 0$. Fix $g \in G$ and take a sequence v_n such that $v_n \to 0$ in $\mathfrak{g}$. Our work is to show that

$$h_2(v_n) \to h_2(0) = 0.$$

If $t_A(g, v_n)$ is the time such that $\|e^{At_A(g,v_n)}v_n\|_A = 1$, then $t_n = t_A(g, v_n) \to -\infty$. Let us denote $u_n = \psi_0(g, ^{At_n} v_n)$. Thus

$$\|u_n\|_A = \|\psi_0(g, e^{At_n}v_n)\|_A = \frac{\|e^{At_n}v_n\|_A}{\|e^{At_n}v_n\|_A} = 1$$

and, for this,

$$\begin{aligned}\|h_2(v_n)\|_A &= \|e^{-Bt_n}\psi_0(g, e^{At_n}v_n)\|_A \\ &\leq \|e^{-Bt_n}\|_A\|\psi_0(g, e^{At_n}v_n)\|_A = \|e^{-Bt_n}\|_A \leq e^{bt_n} \to 0.\end{aligned}$$

We conclude that $h_2(v_n) \to 0$, hence that h_2 is continuous at $0 \in \mathfrak{g}$, and finally that ψ is continuous.

To end, we define the map

$$\psi^{-1}(g, v) = \begin{cases} (e^{-A\,t_B(g,v)}e^{B\,t_B(g,v)}g, e^{-A\,t_B(g,v)}\phi_0(g, e^{Bt_B(g,v)}v)) & \text{if } v \neq 0 \\ (g^{-1}, 0) & \text{if } v = 0. \end{cases}$$

It is easily verified that ψ^{-1} is the inverse map of ψ. It follows from the arguments above that ψ is a homeomorphism that conjugate the flows $(e^{At}g, e^{At}v)$ and $(e^{Bt}g, e^{Bt}v)$ in $G \times \mathfrak{g}$.

It is not difficult to see that the projection of ψ on G is a homeomorphism that conjugates e^{At} and e^{Bt} on G. □ □

3. Topological conjugacy in $\mathrm{Gl}(n, \mathbb{R})$

Let A be a matrix in $\mathfrak{gl}(n, \mathbb{R})$ and consider the system

$$\dot{X} = AX, \ X \in \mathrm{Gl}(n, \mathbb{R}).$$

This system is well posted (see for instance [6, section 2.3]). Following the idea of previous section, we consider the trivial bundle $\mathrm{Gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R})$ and study the topological conjugacy of the flow of

$$\begin{pmatrix} \dot{X} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} AX \\ AV \end{pmatrix},$$

that is, the flow given by $(e^{At}X, e^{At}V)$, with $e^{At} = \sum_{i=1}^{\infty} \frac{(tA)^i}{i!}$. Thus, taking a norm $|\cdot|$ on $\mathbb{R}^n$ we consider the supremun norm in $\mathfrak{gl}(n, \mathbb{R})$

$$\|A\| = \sup\left\{\frac{|Ax|}{|x|}; 0 \neq x \in \mathbb{R}^n\right\}. \tag{3.1}$$

The choice of the norm $\|\cdot\|$ allows us to show a well-known result that relates hyperbolic property and eigenvalues of A (see e.g. Theorem 5.1 in [5, ch. IV]).

Proposition 3.1. *Take $A \in \mathfrak{gl}(n, \mathbb{R})$ and consider the equation $\dot{X} = AX$. The following statements are equivalent:*

1. *There are a norm $\| \cdot \|_A$ and a constant $a > 0$ such that for any initial condition $X \in \mathrm{Gl}(n, \mathbb{R})$, the solution satisfies*
$$\|e^{At}X\|_A \leq e^{-at}\|X\|_A, \quad \text{for all} \quad t > 0.$$
2. *for every generalized eigenvalue μ of A we have $Re(\mu) < 0$.*

Proof. We first observe that every generalized eigenvalue μ of A has $Re(\mu) < 0$ if and only if there is a norm $| \cdot |_A$ on $\mathbb{R}^n$ and a constant $a > 0$ such that for any initial condition $x \in \mathbb{R}^n$, the solution of $\dot{x} = Ax$ satisfies

$$|e^{A}x|_A \leq e^{-at}|x|_A, \quad \text{for all} \quad t > 0.$$

Thus it is sufficient to show that

$$|e^{At}x|_A \leq e^{-at}|x|_A \Leftrightarrow \|e^{At}X\|_A \leq e^{-at}\|X\|_A.$$

In fact, suppose that $|e^{At}x|_A \leq e^{-at}|x|_A$ where $\dot{x} = Ax$. Since $e^{At}X(x)$ is a solution for $\dot{x} = Ax$, it follows that $|e^{At}X(x)|_A \leq e^{-at}|X(x)|_A$. Thus we obtain

$$\|e^{At}X\|_A \leq e^{-at}\|X\|_A.$$

Conversely, suppose that $\|e^{At}X\|_A \leq e^{-at}\|X\|_A$. Taking $x = e_1$ in Definition (3.1), it follows that

$$|e^{At}x|_A \leq e^{-at}|x|_A$$

with x satisfying $\dot{x} = Ax$, and it assures the last assertion is hold. □ □

Now we take a norm on $\mathrm{Gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R})$ as $\|(X, V)\|_A = \|V\|_A$. Thus, we are able to show a sufficient condition for topological conjugacy on $\mathrm{GL}(n, \mathbb{R})$.

Theorem 3.2. *Consider the systems $\dot{X} = AX$ and $\dot{Y} = BY$, where $A, B \in \mathfrak{gl}(n, \mathbb{R})$. If every generalized eigenvalue of A and B has negative real part, then e^{At} and e^{Bt} are topologically conjugate on $\mathrm{Gl}(n, \mathbb{R})$.*

Proof. By Proposition 3.1, we have that there exist norms $\|\cdot\|_A$, $\|\cdot\|_B$ and constants $a, b > 0$ such that

$$\begin{array}{rcl} \|e^{At}X\|_A & \leq & e^{-at}\|X\|_A \\ \|e^{Bt}Y\|_B & \leq & e^{-bt}\|Y\|_B \end{array}$$

in $\mathfrak{gl}(n, \mathbb{R})$. Using Theorem (2.3) we have that e^{At} and e^{Bt} are topologically conjugate on $\mathrm{Gl}(n, \mathbb{R})$. □

□

As every generalized eigenvalue of $-A$ and $-B$ has negative real part it follows that

Corollary 3.3. *Consider $\dot{X} = AX$ and $\dot{Y} = BY$, where $A, B \in gl(n, \mathbb{R})$, and suppose that every generalized eigenvalue of A and B has positive real part. Then e^{At} and e^{Bt} are topologically conjugate on* $\mathrm{Gl}(n, \mathbb{R})$.

Example 3.4. *Take the matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathfrak{gl}(2, \mathbb{R})$. Then its eigenvalues are given by $\lambda = \frac{(a+d)\pm\sqrt{(a-d)^2+bc}}{2}$. When

$$(a-d)^2 + bc \leq 0 \tag{3.2}$$

the real part of the eigenvalues depend only on the trace $a + d$. Thus, supposing that two matrix A and B in $\mathfrak{gl}(2, \mathbb{R})$ satisfy the inequality (3.2) we have:

1. *if $\mathrm{tr(A)} < 0$ and $\mathrm{tr(B)} < 0$, then by previous theorem e^{At} and e^{Bt} are topologically conjugate; and*
2. *if $\mathrm{tr(A)} > 0$ and $\mathrm{tr(B)} > 0$, then by previous corollary e^{At} and e^{Bt} are topologically conjugate.*

4. Invariant flows on Semisimple and Affine Lie Groups

In this section, we study the topological conjugation on semisimple Lie group. Our idea is to transfer the dynamical system $\dot{g} = A_g$ on G to one on the adjoint Lie group $\mathrm{Ad}(G)$ and to find a condition to topological conjugation in this group, since $\mathrm{Ad}(G)$ is a matrix group.

We begin by assuming that G is a semisimple Lie group. We know that solution of $\dot{g} = A_g$ is given by

$$g(t) = e^{At} g_0.$$

Applying the adjoint operator in $g(t)$ we have

$$\mathrm{Ad}(g(t)) = \mathrm{Ad}(e^{At} g_0) = e^{\mathrm{ad}(At)} \mathrm{Ad}(g_0).$$

Taking the derivative of $\mathrm{Ad}(g(t))$ with respect to t we obtain

$$\mathrm{Ad}'(g(t)) \quad = \quad \mathrm{ad}(A) \cdot \mathrm{Ad}(g(t)).$$

Thus considering two systems $\dot{g} = A_g$ and $\dot{h} = B_h$ we have the dynamical systems on $\mathrm{Gl}(\mathfrak{g})$

$$\begin{aligned} \mathrm{Ad}'(g(t)) &= \mathrm{ad}(A)\mathrm{Ad}(g(t)) \\ \mathrm{Ad}'(h(t)) &= \mathrm{ad}(B)\mathrm{Ad}(h(t)). \end{aligned}$$

Suppose that every generalized eigenvalue of $\mathrm{ad}(A)$ and $\mathrm{ad}(B)$ has real part negative. From Theorem 3.2 there exists a homeomorphism $\psi : \mathrm{Gl}(\mathfrak{g}) \to \mathrm{Gl}(\mathfrak{g})$ such that

$$\psi(e^{\mathrm{ad}(At)}\mathrm{Ad}(g_0)) = e^{\mathrm{ad}(Bt)}\psi(\mathrm{Ad}(g_0)).$$

Since $\mathfrak{g}$ is semisimple, it follows that Ad is an isomorphism on its image (see for instance Corollaries 5.2 and 6.2 in [3, ch.II]). Thus we apply Ad^{-1} in the above equality to obtain

$$\begin{aligned}\mathrm{Ad}^{-1}(\psi(\mathrm{Ad}(e^{At})\mathrm{Ad}(g_0))) &= \mathrm{Ad}^{-1}(\mathrm{Ad}(e^{Bt})\psi(\mathrm{Ad}(g_0))) \\ \mathrm{Ad}^{-1}(\psi(\mathrm{Ad}(e^{At}g_0))) &= e^{Bt}\mathrm{Ad}^{-1}\psi(\mathrm{Ad}(g_0))).\end{aligned}$$

If we denote $\varphi = \mathrm{Ad}^{-1} \circ \psi \circ \mathrm{Ad}$ we have

$$\varphi(e^{At}g_0) = e^{Bt}\varphi(g_0).$$

Summarizing,

Theorem 4.1. *Let $A, B \in \mathfrak{g}$ and consider $\dot{g} = A_g$ and $\dot{h} = B_h$. Suppose that $\mathfrak{g}$ is a semisimple Lie algebra. If every generalized eigenvalue of* $\mathrm{ad}(A)$ *and* $\mathrm{ad}(B)$ *has negative real part then e^{At} and e^{Bt} are topologically conjugate on G.*

Our final step is to work with the class of Affine groups. This idea follows from work due to Kawan, Rocio and Santana [4]. We recall the affine Lie groups and study the topological conjugacy on it. First, we need to establish a result on semisimple Lie groups.

Proposition 4.2. *Take $A \in \mathfrak{g}$ where $\mathfrak{g}$ is a semisimple Lie algebra. Consider the system $\dot{g} = A_g$. Suppose that G has a left invariant metric and denote by ρ the distance associated to this metric. If* $\mathrm{ad}(A)$ *has negative real part for every generalized eigenvalue, then there exists a positive constant a such that $\rho(e^{At}g_o, e) \leq e^{-at}\rho(g_o, e)$.*

Proof. Let $<,>$ be a left invariant metric on Lie group G. As Ad is an isomorphism we consider the following metric on $\mathrm{Ad}(G)$:

$$< X, Y >=< d(\mathrm{Ad}^{-1})X, d(\mathrm{Ad}^{-1})Y > .$$

Let us denote by ρ_1 the distance associated to metric on $\mathrm{Ad}(G)$. It follows that $\rho_1(\mathrm{Ad}(g(t)), Id) = \rho(g(t), e)$ for any smooth curve $g(t) \in G$. Suppose now that $\mathrm{Ad}(c(t))$ satisfies the differential equation $\mathrm{Ad}'(c(t)) = \mathrm{ad}(A) \cdot \mathrm{Ad}(c(t))$. Then Proposition 3.1 assures that there are a positive constant a such that

$$\rho_1(\mathrm{Ad}(g(t)), Id) \quad = \quad \rho_1(e^{\mathrm{ad}(A)t}\mathrm{Ad}(g_0), Id) \leq e^{-at}\rho_1(\mathrm{Ad}(g_0), Id).$$

Hence $\rho(g(t), e) \leq e^{-at}\rho(g_0, e)$. □ □

Now we recall the affine Lie groups. Let V be an n-dimensional real vector space and H a Lie group that acts on V. Take the group $G = H \rtimes V$ given by the semidirect product of H and V. If $\Phi(t,g) = \exp_G(tX)g$ is a flow on G, where $X = (A,b) \in \mathfrak{g} = \mathfrak{h} \rtimes V$ and $g \in G$, we denote by Φ^H the flow on H given by $\Phi^H(t,h) = \exp_H(At)h$ with $h \in H$.

Before we state and show the last result, we recall, for group actions, that a fundamental domain is a subset of the space on which the group acts such that this subset contains exactly one point of each orbit. For example, spheres $\mathbb{S}_A$ and $\mathbb{S}_B$ given by (2.4) are fundamental domains.

Proposition 4.3. *Consider the affine group $G = H \rtimes V$ with H being a semisimple Lie group with a left invariant metric. Let $\pi : G \to H$ be the canonical projection. Take flows $\Phi_i(t,g) = \exp(tX_i)g$ with $X_i = (A_i, b_i)$, $i = 1, 2$, on G, and suppose that every generalized eigenvalue of* $\mathrm{ad}(A_1)$ *and* $\mathrm{ad}(A_2)$ *has negative real part. Then Φ_1 and Φ_2 are topologically conjugate.*

Proof. We first suppose that the real part of all generalized eigenvalues of $\mathrm{ad}(A_1)$ and $\mathrm{ad}(A_2)$ are negative. From Proposition 4.2 we have two unitary spheres $\mathbb{S}^1_{A_1}$ and $\mathbb{S}^1_{A_2}$ in G with center g_0 and g_1, respectively, such that $\exp(A_1 t)g_0$ and $\exp(A_2 t)g_1$ cross these in a unique time, respectively. It means that these unitary spheres are fundamental domains to flows $\exp(A_1 t)g_0$ and $\exp(A_2 t)g_1$, respectively. In this way, Proposition 12 of [4] guarantees that Φ_1 and Φ_2 are topologically conjugate. □ □

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