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On the M-hypercyclicity of Cosine Function on Banach Spaces

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ABSTRACT: In this paper we introduce and study the M-hypercyclicity of strongly continuous cosine function on separable complex Banach space, and we give the criteria for cosine function to be M-hypercyclic. We also prove that every separable infinite dimensional complex Banach space admits a uniformly continuous cosine function.

Key Words: Cosine function, Hypercyclicity, Topologically transitive, M-hypercyclicity, M-transitive.

Contents

1. Introduction

A sequence of bounded operators $(T_n)_{n\geq 0}$ on Banach space X is called hypercyclic if there exists a vector $x \in X$ such that the $\{T_n x, n \geq 0\}$ is dense in X. We note that $(T_n)_{n\geq 0}$ is said to be topologically transitive if for every non-empty open sets U and V of X, there exists $n \in \mathbb{N}$ such that $T_n^{-1}(U) \cap V \neq \emptyset$. In [14] has been shown that in separable infinite dimensional Banach space X , $(T_n)_{n\geq 0}$ is hypercyclic if and only if it is topologically transitive, in this case the family $(T_n)_{n\geq 0}$ has a dense set of hypercyclic vectors. When $T_n := T^n$ for some $T \in B(X)$ and for all $n \in \mathbb{N}$, we say that T is hypercyclic. In this case, the set $\{T^n x, n \geq 0\}$ is known as the orbit of the element x by the operator T. Rolewicz [16] gave the first example of a hypercyclic operator on a Banach space; he showed that if the backward shift B on $l^2(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$, he also proved that in every separable infinite dimensional Banach space there exists a hypercyclic operator, for the existence of hypercyclic operator had been studied by [1], [4] and [5]. We recall that $\tau = (T_t)_{t\geq 0} \subset B(X)$ is a C_0-semigroup if $T_0 = I$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$ and $\lim\limits_{t\mapsto 0} T_t x = x$ for all $x \in X$. Given an arbitrary C_0-semigroup $\tau = (T_t)_{t\geq 0}$ on Banach space X, it can be shown that $Ax = \lim\limits_{t\mapsto 0} \dfrac{T_t x - x}{t}$ exists on a dense subspace of X. The set of these x is the domain of A, that it is denoted by $D(A) = \{x \in X / \lim\limits_{t\mapsto 0} \dfrac{T_t x - x}{t}$ exists$\}$. Then A, is called the infinitesimal generator

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of τ, moreover, $T_t Ax = AT_t x$ for all $x \in D(A)$ and $t \geq 0$. Another important property is provided by the point spectral mapping theorem for C_0-semigroups. If X is a Banach space, then for every $x \in X$ and $\lambda \in \mathbb{C}$, $Ax = \lambda x$ implies that $T_t x = e^{\lambda t}x$ for every $t \geq 0$. In 1997 Desch and al [10] introduced and studied the hypercyclicity of C_0-semigroup. A C_0-semigroup $\tau = (T_t)_{t\geq 0}$ is called hypercyclic if there exists a vector $x \in X$ such that $Orb(\tau, x) = \{T_t x, t \geq 0\}$ is dense in X, they showed that if $\tau = (T_t)_{t\geq 0}$ is a C_0-semigroup then for all $t > 0$ the operator T_t has a dense range in X, and $\sigma_p(T_t^*) = \emptyset$, in [9] Conejero, A. Peris and Muller proved that there is a relationship between hypercyclicity of $\tau = (T_t)_{t\geq 0}$ and hypercyclicity of every T_t for all $t > 0$ which is $\tau = (T_t)_{t\geq 0}$ hypercyclic if and only if T_t is also for all $t > 0$. A discussion and references to earlier work on hypercyclic semigroup can be found in [11] and [13].

2. Preliminaries

A cosine operator function on a Banach space X is a strongly continuous $\mathcal{C} = (C_t)_{t\in\mathbb{R}} \subset B(X)$ satisfying $C_0 = I$, and the d'Alembert function equation $2C_tC_s = C_{t+s} + C_{t-s}$, for all $t, s \in \mathbb{R}$, which implies $C_t = C_{-t}$ for all $t \in \mathbb{R}$.
$\mathcal{C} = (C_t)_{t\in\mathbb{R}}$ is called strongly continuous if $t \mapsto C_t(x)$ for all $x \in X$ is a continuous function from $\mathbb{R}$ to X.The infinitesimal generator of a cosine function is defined by $Ax = 2\lim\limits_{t\mapsto 0}\dfrac{(C_t - I)(x)}{t^2}$ for all $x \in D(A)$ where:
$D(A) = \{x \in X/\lim\limits_{t\mapsto 0}\dfrac{C_t(x) - x}{t^2}$ exists$\}$. In [2] W. Arendt and al proved that if A is a generator of cosine operator function, then A is a closed, densely defined operator($\overline{D(A)} = X$), and there exists constants $M > 0, w \geq 0$ such that $\|C_t\| \leq Me^{w|t|}$ for all $t \in \mathbb{R}$; moreover if $z \in \mathbb{C}$ such that $Re(z) > w$ we have $z^2 \in \rho(A)$. If $\lim\limits_{t\mapsto 0}\|C_t - I\| = 0$ we say that $\mathcal{C} = (C_t)_{t\in\mathbb{R}}$ is uniformly continuous cosine function, in this case the generator is a bounded operator A, and $\mathcal{C}$ admits the following representation: $C_t = I + \sum\limits_{n\geq 1}\dfrac{t^{2n}}{(2n!)}A^n$ for all $t \in \mathbb{R}$, we note that the cosine operator functions are associated with the solution of the second order Cauchy problem: $\frac{d^2}{dt^2}u(t) = Au(t)\ t \in \mathbb{R}; u^k(0) = f_k \in D(A), k = 0, 1$, we mention that the Cauchy problem is well posed if and only if A generates a cosine operator function $(C_t)_{t\in\mathbb{R}}$, with the solution given by $u(t) = C_t f_0 + \int_0^t C_u f_1 du,\ t \in \mathbb{R}$, for more details about this theory see [2]. For example if $\tau = (T_t)_{t\geq 0}$ is a C_0-group on Banach space X with generator B, it is easily to see that $C_t = \frac{1}{2}(T_t + T_{-t})$ for all $t \in \mathbb{R}$, defines a cosine operator generated by $A = B^2$.
In [7] A. Bonilla and P.J. Miana introduced and studied the hypercyclicity of a cosine operator function on a Banach space. A cosine operator function $\mathcal{C} = (C_t)_{t\in\mathbb{R}}$ is hypercyclic if there exsits a vector $x \in X$ such that $\{C_t(x), t \in \mathbb{R}\}$ is dense in X; the same authors gave the sufficient conditions for the hyperyclicity and topological mixing of a strongly continuous function, and showed that every infinite dimensional complex Banach space admits a topologically mixing uniformly cosine operator function. In [12] T. Kalmes gave the characterization for cosine opera-

tor function generated by second order partial differential operator on $l^p(\omega, \mu)$ and $C_{0,\rho}(\omega)$ with $\omega \subset \mathbb{R}^d$ is open, to be transitive and [8] the authors characterized the cosine operator function generated by unilateral and bilateral weighted shift on $l^p(\mathbb{N})$, and $l^p(\mathbb{Z})$ with $1 \leq p \leq \infty$.
Let M be a closed subspace of Banach space X, and $\tau = (T_t)_{t\geq 0}$ be a C_0-semigroup, we say that τ is M-hypercyclic if there exists a vector $x \in X$ such that $\{T_t x, t \geq 0\} \cap M$ is dense in M. The M-hypercyclicity of C_0-semigroups are crucial for the investigation of hypercyclicity of C_0-semigroups; we refer to [15], [17], [18] for some references. The motivation for the study of M-hypercyclicity of cosine operator function is inspired by the work of A. Bonilla and P.J. Miana [7].
In this work, we introduce and study the M-hypercyclicity of cosine operator function on separable complex Banach space, and we give the sufficient conditions for the M-hypercyclicity and M-transitivity of strongly continuous cosine function, we also prove that in every separable infinite dimensional complex Banach space X there exists a M-hypercyclic uniformly continuous cosine function, with M is a non-trivial closed subspace of X.

3. Main Results

Definition 3.1. *Let $(C_t)_{t\in\mathbb{R}}$ be a cosine function on separable Banach space X, and M be a nonzero subspace of X, we say that $(C_t)_{t\in\mathbb{R}}$ is M-hypercyclic if there exists a vector $x \in X$, such that $\{C_t x, t \in \mathbb{R}\} \cap M$ is dense in M, in this case the vector $x \in X$ is called vector M-hypercyclic for $(C_t)_{t\in\mathbb{R}}$.*

Remark 3.2.

1. *If $M = X$ it is clear that the above definition coincides with the hypercyclicity of cosine function $(C_t)_{t\in\mathbb{R}}$.*
2. *Let $(C_t)_{t\in\mathbb{R}}$ be a cosine function on Banach space X. Observe by taking $t = 0$ in the d'Alembert equation $2C_tC_s = C_{t+s} + C_{t-s}$ we have $C_t = C_{-t}$ for all $t \geq 0$, then $(C_t)_{t\in\mathbb{R}}$ is M-hypercyclic if and only if $(C_t)_{t\geq 0}$ is M-hypercyclic.*

Example 3.3. *If $(A_t)_{t\in\mathbb{R}}$ is a hypercyclic cosine function on Banach space X, then $C_t := A_t \oplus I$ for all $t \in \mathbb{R}$ is M-hypercyclic cosine function with $M = X \oplus \{0\}$.*
Indeed firstly we prove that $(C_t)_{t\geq 0}$ is cosine function on $X \oplus X$. Let $x \oplus y \in X \oplus X$ we have:

$$\begin{aligned}
2C_tC_s(x \oplus y) &= 2C_t(A_s \oplus I)(x \oplus y) = 2C_t(A_s x \oplus y) \\
&= 2(A_t \oplus I)(A_s x \oplus y) = 2(A_tA_s x \oplus y) \\
&= 2A_tA_s x \oplus 2y = A_{t+s}(x) + A_{t-s}(x) \oplus 2y \\
&= (A_{t+s}(x) \oplus y) + (A_{t-s}(x) \oplus y) = C_{t+s}(x \oplus y) + C_{t-s}(x \oplus y) \\
&= (C_{t+s} + C_{t-s})(x \oplus y)
\end{aligned}$$

for all $t \geq s \geq 0$ then $2C_tC_s = C_{t+s} + C_{t-s}$, and we have: $C_0 = A_0 \oplus I = I \oplus I = I_{X\oplus X}$
finally $(C_t)_{t\geq 0}$ is a cosine function on $X \oplus X$, since $(A_t)_{t\geq 0}$ is hypercyclic , let

$x \in X$ be a hypercyclic vector of $(A_t)_{t\geq 0}$ then $\{A_t x, t \geq 0\}$ is dense in X, then $\{C_t(x \oplus 0); t \geq 0\} = \{A_t x; t \geq 0\} \oplus \{0\}$ is dense in $M = X \oplus \{0\}$, wich implies that $x \oplus 0$ is M-hypercyclic vector for $(C_t)_{t\geq 0}$, but not hypercyclic on $X \oplus X$.
In general if $(A_t)_{t\in\mathbb{R}}$ and $(B_t)_{t\in\mathbb{R}}$ be two hypercyclic cosines function on Banach space X, then $C_t := A_t \oplus B_t; \forall t \in \mathbb{R}$ is a hypercyclic cosine function on $X \oplus X$.

Example 3.4. *If $(T_t)_{t\in\mathbb{R}}$ is C_0-group on Banach space X, then $C_t = \frac{1}{2}(T_t + T_{-t}) \oplus I$ for all $t \in \mathbb{R}$ is M-hypercyclic cosine function with $M = X \oplus \{0\}$.*

Definition 3.5. *Let $(C_t)_{t\in\mathbb{R}}$ be a strongly continuous cosine function on separable Banach space X, and M be a nonzero subspace of X, $(C_t)_{t\in\mathbb{R}}$ is called M-transitive if for every non-empty sets U, V of M there exists $t \in \mathbb{R}$, such that $C_t^{-1}(U) \cap V$ contains a non-empty open set of M.*

Remark 3.6. *By $C_t = C_{-t}$ for all $t \in \mathbb{R}$, then $(C_t)_{t\in\mathbb{R}}$ is M-transitive if and only if $(C_t)_{t\geq 0}$ is also.*

Theorem 3.7. *Let $(C_t)_{t\geq 0}$ be a strongly continuous cosine function on separable Banach space, and M be a nonzero subspace of X, then the following conditions are equivalent:*

1. *$(C_t)_{t\geq 0}$ is M-transitive.*
2. *For every non-empty open sets U and V of M, there exists $t > 0$ such that $C_t^{-1}(U) \cap V$ is non-empty open set of M.*
3. *For every non-empty open sets U and V of M, there exists $t > 0$ such that $C_t^{-1}(U) \cap V$ is non-empty and $C_t(M) \subset M$.*

Proof: $(2) \Leftrightarrow (1)$ is clear.
$(3) \Rightarrow (2)$. Let U and V be non-empty open subsets of M, by (3) there is $t_0 \geq 0$ such that $C_{t_0}^{-1}(U) \cap V \neq \phi$ and $C_{t_0}(M) \subset M$
Since $C_{t_0|M} : M \longrightarrow M$ is continuous, then $C_{t_0}^{-1}(U)$ is open in M, therefore $C_{t_0}^{-1}(U) \cap V$ is non-empty open of M.
$(1) \Rightarrow (3)$. Let U and V be two non-empty open subsets of M. By (1) there exists $t_0 \geq 0$ such that $C_{t_0}^{-1}(U) \cap V$ contains a non-empty open W of M, it gives $W \subset C_{t_0}^{-1}(U) \cap V$ and $C_{t_0}^{-1}(U) \cap V \neq \phi$.
Next, we prove that $C_{t_0}(M) \subset M$.
Let $x \in M$, we have $W \subset C_{t_0}^{-1}(U) \cap V$, this implies that $C_{t_0}(W) \subset U \subset M$. Let $x_0 \in W$, since W is open of M then for all r small enough we have $x_0 + rx \in W$, therefore $C_{t_0}(x_0 + rx) = C_{t_0}x_0 + rC_{t_0}x \in C_{t_0}(W) \subset M$. From $C_{t_0}x_0 \in M$ it follows that $C_{t_0}x \in M$.
We then conclude that $C_{t_0}(M) \subset M$. □

Lemma 3.8. *Let $(C_t)_{t\geq 0}$ be a strongly continuous cosine function on a separable Banach space X, and M be a nonzero subspace of X. If $(C_t)_{t\geq 0}$ is M-transitive, then $(C_t)_{t\geq 0}$ has a dense set in M of M-hypercyclic vectors.*

Proof: Denote by $Hc(\mathcal{C}, M)$ the set of all M-hypercyclic vectors of $\mathcal{C} = (C_t)_{t\in\mathbb{R}}$, and since X is separable let $(B_k)_{k\geq 0}$ be a countable open basis for the relative topology of M. We have $x \in Hc(\mathcal{C}, M)$ if and only if $Orb(\mathcal{C}, x) \cap M$ is dense in M if and only if for each $k \geq 0$; there are $t \geq 0$ such that $C_t x \in B_k$, if and only if $x \in \bigcap_{k\geq 0} \bigcup_{t\geq 0} C_t^{-1}(B_k)$.(*)

then $Hc(\mathcal{C}, M) = \bigcap_{k\geq 0} \bigcup_{t\geq 0} C_t^{-1}(B_k)$. But $\mathcal{C}$ is M-transitive, then by theorem 3-7, for each $k, m \geq 0$ there exists $t = t_{k,m} \geq 0$ such that $C_t^{-1}(B_k) \bigcap B_m$ is non-empty open set, hence the set $A_k = \bigcup_{m\geq 0} C_t^{-1}(B_k) \cap B_m$ is non-empty and open set. Furthermore, for all $k \geq 0$; A_k is dense in M, and by Baire category theorem we have that $\bigcap_{k\geq 0} A_k = \bigcap_{k\geq 0} \bigcup_{m\geq 0} C_t^{-1}(B_k) \cap B_m$ is dense in M, by (*) we have that $Hc(\mathcal{C}, M) = \bigcap_{k\geq 0} \bigcup_{t\geq 0} C_t^{-1}(B_k)$ is also dense in M. □

Theorem 3.9. *Let $(C_t)_{t\geq 0}$ be a strongly continuous cosine function on a separable Banach space X, and M be a nonzero subspace of X, if $(C_t)_{t\geq 0}$ is M-transitive, then $(C_t)_{t\geq 0}$ is M-hypercyclic.*

Criteria of M-hypercyclicity of cosine function

In general it is difficult to find the M-hypercyclic vectors of cosine function, on Banach space, so we look for the criteria of M-hypercyclicity for cosine function.

Theorem 3.10. *Let $(C_t)_{t\in\mathbb{R}}$ be a strongly continuous cosine function on separable Banach space X, and M be a subspace of X, suppose that there is D_0 and D_1 two dense set in M, and an increasing positively sequence $(t_n)_{n\geq 0}$ such that:*

- *(i) $C_{t_n} x \to 0$ for all $x \in D_0$.*
- *(ii) For all $y \in D_1$, there is a sequence $(x_n)_n \subset M$, such that $x_n \to 0$ and $C_{t_n} x_n \to y$.*
- *(iii) $C_{t_n}(M) \subset M$.*
  Then $(C_t)_{t\geq 0}$ is M-transitive, it is M-hypercyclic.

Proof: Let U and V be two non-empty open set in M, we prove that there is $t \in \mathbb{R}$ such that $C_t^{-1}(U) \cap V \neq \emptyset$ and $C_t(M) \subset M$.
Since D_0 and D_1 are dense in M then $V \cap D_0 \neq \emptyset$ and $D_1 \cap U \neq \emptyset$. Let $a \in U \cap D_1$ and $b \in D_0 \cap V$, hence there is $\epsilon > 0$ such that $B(a, \epsilon) \subset U$ and $B(b, \epsilon) \subset V$. From $b \in D_0$ and $a \in D_1$, we have $C_{t_n} \to b$ and there exists $(x_n)_n \subset M$ such that $x_n \to 0$ and $C_{t_n} x_n \to a$.
Consequently there exists $N \in \mathbb{N}$ such that: $\|x_n\| < \epsilon$, $\|C_{t_n}(x_n) - a\| < \frac{\epsilon}{2}$ and

$\|C_{t_n}(b)\| < \frac{\epsilon}{2}$ for all $n \geq N$.
Therefore $\|b + x_n - b\| = \|x_n\| < \epsilon \Rightarrow b + x_n \in B(b, \epsilon) \subset V \Rightarrow b + x_n \in V$.
On the other hand
$\|C_{t_n}(b+x_n) - a\| = \|C_{t_n}(b) + C_{t_n}(x_n) - a\| \leq \|C_{t_n}(b)\| + \|C_{t_n}(x_n) - a\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
this implies that $C_{t_n}(b + x_n) \in B(a, \epsilon) \subset U$ then $C_{t_n}(b + x_n) \in U$
hence $b + x_n \in C_{t_n}^{-1}(U)$. We obtain that $b + x_n \in C_{t_n}^{-1}(U) \cap V$ and $C_{t_n}^{-1}(U) \cap V \neq \emptyset$, since $C_{t_n}(M) \subset M$ then $(C_t)_{t\geq 0}$ is M-transitive.
Let $(C_t)_{t\in\mathbb{R}}$ be a strongly continuous cosine function on Banach space X, and (t_n) be a sequence of positive real we put: $X_0 := \{x \in X / \lim_{n\mapsto+\infty} C_{t_n} x = 0\}$
and $X_\infty := \{y \in X / \exists u_n \mapsto 0; \lim_{n\mapsto+\infty} C_{t_n}(u_n) = y\}$.
In [7] Antonio Bonilla and Pero J. Miana, proved that if there is $t_n \mapsto +\infty$ such that X_0 and X_∞ are dense, then $(C_t)_{t\geq 0}$ is a hypercyclic cosine function. □

Theorem 3.11. *Let $(C_t)_{t\geq 0}$ be a strongly continuous cosine function on separable Banach space X, and Mbe a nonzero subspace of X, if there exists a sequence of positive real (t_n); $t_n \mapsto +\infty$ such that $X_0 \cap M$ and $X_\infty \cap M$ are dense in M and $C_{t_n}(M) \subset M$, then $(C_t)_{t\geq 0}$ is M-transitive in particular it is M-hypercyclic.*

Proof: It is sufficient to take: $D_0 = X_0 \cap M$ and $D_1 = X_\infty \cap M$. □

Corollary 3.12. *Let $(C_t)_{t\geq 0}$ be a strongly continuous cosine function on a separable Banach space X, if there exists a sequence of real (t_n); $t_n \mapsto +\infty$ such that $X_1 = \{x \in X / \lim_{n\mapsto+\infty} C_{t_n}(x) = \lim_{n\mapsto+\infty} C_{2t_n}(x) = 0\} \cap M$ is dense in M and $C_{t_n}(M) \subset M$, then $(C_t)_{t\geq 0}$ is a M-transitive cosine function.*

Proof: we prove that $X_1 \cap M \subset X_0 \cap M$ and $X_1 \cap M \subset X_\infty \cap M$
Let $y \in X_1 \cap M$ then $y \in X_1$ and $y \in M$, we define $x_n = 2C_{t_n}(y)$ we have $x_n \mapsto 0$ and
$C_{t_n}(x_n) = C_{t_n}(2C_{t_n}(y)) = 2C_{t_n}C_{t_n}(y) = C_{2t_n}(y) + y \mapsto y$ then $X_1 \cap M \subset X_0 \cap M$ and $X_1 \cap M \subset X_\infty \cap M$, since $X_1 \cap M$ dense in M, then $X_0 \cap M$ and $X_\infty \cap M$ are also dense in M and since $C_{t_n}(M) \subset M$ then $(C_t)_{t\geq 0}$ is M-transitive. In [2, proposition 3.14.6] Arendt and al showed that if $(C_t)_{t\in\mathbb{R}}$ is a strongly continuous cosine function on Banach space X, such that $\lim_{t\mapsto\infty} C_t(x) = 0$, then $x = 0$. In [6], A. Bpbrowski and W. Chojnacki generalized this result by showing: if $\lim_{t\mapsto\infty} C_t(x)$ exists for all $x \in X$ then $C_t = I$ for all $t \in \mathbb{R}$, based on this we obtain the following result. □

Corollary 3.13. *Let $(C_t)_{t\in\mathbb{R}}$ be a strongly continuous cosine function on Banach space X, if $\lim_{t\mapsto\infty} C_t(x)$ exists for all $x \in X$, then $(C_t)_{t\in\mathbb{R}}$ is not M-hypercyclic for any M subspace of X, in particular it is not hypercyclic.*

The existence of M-hypercyclic cosine function

It is natural to ask about the existence of M-hypercyclic cosine function on a separable Banach space.

Lemma 3.14. *[3] If E is a dense subset of Banach space X, then there exists a non-trivial closed subspace M of X, such that $E \cap M$ is dense in M.*

Proposition 3.15. *If $(C_t)_{t\geq 0}$ is a hypercyclic strongly continuous cosine function on Banach space X, then $(C_t)_{t\geq 0}$ is M-hypercyclic cosine function with M is a non-trivial closed subspace of X.*

Remark 3.16. *Every hypercyclic strongly continuous cosine function is M-hypercyclic for M is non-trivial closed subspace of X, but there exists a M-hypercyclic cosine function, that is not hypercyclic on X see example 1.*

Theorem 3.17. *[7] Every separable infinite dimensional complex Banach space X admits a topologically mixing (hypercyclic) uniformly continuous cosine function.*

Corollary 3.18. *Every separable infinite dimensional complex Banach space X, admits a M-hypercyclic uniformly continuous cosine function with M is a non-trivial closed subspace of X.*

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