On the M-hypercyclicity of Cosine Function on Banach Spaces

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ABSTRACT: In this paper we introduce and study the M-hypercyclicity of strongly continuous cosine function on separable complex Banach space, and we give the criteria for cosine function to be M-hypercyclic. We also prove that every separable infinite dimensional complex Banach space admits a uniformly continuous cosine function.

Key Words: Cosine function, Hypercyclicity, Topologically transitive, M-hypercyclicity, M-transitive.

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1. Introduction

A sequence of bounded operators \((T_n)_{n \geq 0}\) on Banach space \(X\) is called hypercyclic if there exists a vector \(x \in X\) such that the \(\{T_n x, n \geq 0\}\) is dense in \(X\). We note that \((T_n)_{n \geq 0}\) is said to be topologically transitive if for every non-empty open sets \(U\) and \(V\) of \(X\), there exists \(n \in \mathbb{N}\) such that \(T_n^{-1}(U) \cap V \neq \emptyset\). In [14] has been shown that in separable infinite dimensional Banach space \(X\), \((T_n)_{n \geq 0}\) is hypercyclic if and only if it is topologically transitive, in this case the family \((T_n)_{n \geq 0}\) has a dense set of hypercyclic vectors. When \(T_n := T^n\) for some \(T \in B(X)\) and for all \(n \in \mathbb{N}\), we say that \(T\) is hypercyclic. In this case, the set \(\{T^n x, n \geq 0\}\) is known as the orbit of the element \(x\) by the operator \(T\). Rolewicz [16] gave the first example of a hypercyclic operator on a Banach space; he showed that if the backward shift \(B\) on \(l^2(\mathbb{N})\) then \(\lambda B\) is hypercyclic if and only if \(|\lambda| > 1\), he also proved that in every separable infinite dimensional Banach space there exists a hypercyclic operator, for the existence of hypercyclic operator had been studied by [1], [4] and [5]. We recall that \(\tau = (T_t)_{t \geq 0} \subset B(X)\) is a \(C_0\)-semigroup if \(T_0 = I\), \(T_t T_s = T_{t+s}\) for all \(t, s \geq 0\) and \(\lim_{t \to 0} T_t x = x\) for all \(x \in X\). Given an arbitrary \(C_0\)-semigroup \(\tau = (T_t)_{t \geq 0}\) on Banach space \(X\), it can be shown that \(Ax = \lim_{t \to 0} \frac{T_t x - x}{t}\) exists on a dense subspace of \(X\). The set of these \(x\) is the domain of \(A\), that it is denoted by \(D(A) = \{x \in X/ \lim_{t \to 0} \frac{T_t x - x}{t}\}\) exists. Then \(A\) is called the infinitesimal generator.
of \( \tau \), moreover, \( T_tAx = AT_t x \) for all \( x \in D(A) \) and \( t \geq 0 \). Another important property is provided by the point spectral mapping theorem for \( C_0 \)-semigroups. If \( X \) is a Banach space, then for every \( x \in X \), \( \lambda \in \mathbb{C} \), \( Ax = \lambda x \) implies that \( T_t x = e^{\lambda t} x \) for every \( t \geq 0 \). In 1997 Desch and al [10] introduced and studied the hypercyclicity of \( C_0 \)-semigroup. A \( C_0 \)-semigroup \( \tau = (T_t)_{t \geq 0} \) is called hypercyclic if there exists a vector \( x \in X \) such that \( \text{Orb}(\tau, x) = \{ T_t x, t \geq 0 \} \) is dense in \( X \), they showed that if \( \tau = (T_t)_{t \geq 0} \) is a \( C_0 \)-semigroup then for all \( t > 0 \) the operator \( T_t \) has a dense range in \( X \), and \( \sigma_p(T_t) = \emptyset \), in [9] Conejero, A. Peris and Muller proved that there is a relationship between hypercyclicity of \( \tau = (T_t)_{t \geq 0} \) and hypercyclicity of every \( T_t \) for all \( t > 0 \) which is \( \tau = (T_t)_{t \geq 0} \) hypercyclic if and only if \( T_t \) is also for all \( t > 0 \). A discussion and references to earlier work on hypercyclic semigroup can be found in [11] and [13].

2. Preliminaries

A cosine operator function on a Banach space \( X \) is a strongly continuous \( \mathcal{C} = (C_t)_{t \in \mathbb{R}} \subset B(X) \) satisfying \( C_0 = I \), and the d’Alembert function equation \( 2C_tC_s = C_{t+s} + C_{t-s} \), for all \( s, t \in \mathbb{R} \), which implies \( C_t \) is \( C_0 \)-semigroup for all \( t \in \mathbb{R} \).

The infinitesimal generator of a cosine function is defined by \( \mathcal{C} = \{ \mathcal{C}_t \}_{t \in \mathbb{R}} \) is called strongly continuous if \( t \mapsto C_t(x) \) for all \( x \in X \) is a continuous function from \( \mathbb{R} \) to \( X \). The infinitesimal generator of a cosine function is defined by \( A x = 2 \lim_{t \to 0} \frac{(C_t - I)(x)}{t^2} \) for all \( x \in D(A) \) where:

\[
D(A) = \{ x \in X / \lim_{t \to 0} \frac{C_t(x) - x}{t^2} \text{ exists} \}. \]

In [2] W. Arendt and al proved that if \( A \) is a generator of cosine operator function, then \( A \) is a closed, densely defined operator, and there exists constants \( M > 0, w > 0 \) such that \( ||C_t|| \leq Me^{wt} \) for all \( t \in \mathbb{R} \); moreover if \( z \in \mathbb{C} \) such that \( Re(z) > w \) we have \( z^2 \in \rho(A) \). If \( \lim_{t \to 0} ||C_t - I|| = 0 \) we say that \( \mathcal{C} = (C_t)_{t \in \mathbb{R}} \) is uniformly continuous cosine function, in this case the generator is a bounded operator \( A \), and \( \mathcal{C} \) admits the following representation: \( C_t = I + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} A^n \) for all \( t \in \mathbb{R} \), we note that the cosine operator functions are associated with the solution of the second order Cauchy problem:

\[ \frac{d}{dt} u(t) = Au(t) \quad t \in \mathbb{R}; \quad u^k(0) = f_k \in D(A), k = 0, 1, \]

we mention that the Cauchy problem is well posed if and only if \( A \) generates a cosine operator function \( (C_t)_{t \in \mathbb{R}} \), with the solution given by \( u(t) = C_t f_0 + \int_0^t C_s f_1 ds \), \( t \in \mathbb{R} \), for more details about this theory see [2]. For example if \( \tau = (T_t)_{t \geq 0} \) is a \( C_0 \)-group on Banach space \( X \) with generator \( B \), it is easily to see that \( C_t = \frac{1}{2}(T_t + T_{-t}) \) for all \( t \in \mathbb{R} \), defines a cosine operator generated by \( A = B^2 \).

In [7] A. Bonilla and P.J. Miana introduced and studied the hypercyclicity of a cosine operator function on a Banach space. A cosine operator function \( \mathcal{C} = (C_t)_{t \in \mathbb{R}} \) is hypercyclic if there exists a vector \( x \in X \) such that \( \{ C_t(x), t \in \mathbb{R} \} \) is dense in \( X \); the same authors gave the sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous function, and showed that every infinite dimensional complex Banach space admits a topologically mixing uniformly cosine operator function. In [12] T. Kalmes gave the characterization for cosine opera-
tor function generated by second order partial differential operator on $l^p(\omega, \mu)$ and $C_0, p(\omega)$ with $\omega \subset \mathbb{R}^d$ is open, to be transitive and [8] the authors characterized the cosine operator function generated by unilateral and bilateral weighted shift on $l^p(\mathbb{N})$, and $l^p(\mathbb{Z})$ with $1 \leq p \leq \infty$.

Let $M$ be a closed subspace of Banach space $X$, and $\tau = (T_t)_{t \geq 0}$ be a $C_0$-semigroup, we say that $\tau$ is $M$-hypercyclic if there exists a vector $x \in X$ such that $\{T_t x, t \geq 0\} \cap M$ is dense in $M$. The $M$-hypercyclicity of $C_0$-semigroups are crucial for the investigation of hypercyclicity of $C_0$-semigroups; we refer to [15], [17], [18] for some references. The motivation for the study of $M$-hypercyclicity of cosine operator function is inspired by the work of A. Bonilla and P.J. Miana [7].

In this work, we introduce and study the $M$-hypercyclicity of cosine operator function generated by unilateral and bilateral weighted shift on separable complex Banach space, and we give the sufficient conditions for $M$-hypercyclicity of cosine operator function. We also prove that in every separable infinite dimensional complex Banach space, there exists a non-trivial closed subspace of $X$.

3. Main Results

Definition 3.1. Let $(C_t)_{t \in \mathbb{R}}$ be a cosine function on separable Banach space $X$, and $M$ be a nonzero subspace of $X$, we say that $(C_t)_{t \in \mathbb{R}}$ is $M$-hypercyclic if there exists a vector $x \in X$, such that $\{C_t x, t \in \mathbb{R}\} \cap M$ is dense in $M$, in this case the vector $x \in X$ is called vector $M$-hypercyclic for $(C_t)_{t \in \mathbb{R}}$.

Remark 3.2.

1. If $M = X$ it is clear that the above definition coincides with the hypercyclicity of cosine function $(C_t)_{t \in \mathbb{R}}$.

2. Let $(C_t)_{t \in \mathbb{R}}$ be a cosine function on Banach space $X$. Observe by taking $t = 0$ in the d’Alembert equation $2C_tC_s = C_{t+s} + C_{t-s}$ we have $C_t = C_{-t}$ for all $t \geq 0$, then $(C_t)_{t \in \mathbb{R}}$ is $M$-hypercyclic if and only if $(C_t)_{t \geq 0}$ is $M$-hypercyclic.

Example 3.3. If $(A_t)_{t \in \mathbb{R}}$ is a hypercyclic cosine function on Banach space $X$, then $C_t := A_t \oplus I$ for all $t \in \mathbb{R}$ is $M$-hypercyclic cosine function with $M = X \oplus \{0\}$.

Indeed firstly we prove that $(C_t)_{t \geq 0}$ is cosine function on $X \oplus X$. Let $x \oplus y \in X \oplus X$ we have:

\[
2C_tC_s(x \oplus y) = 2C_t(A_s \oplus I)(x \oplus y) = 2C_t(A_sx \oplus y) = 2(A_t \oplus I)(A_sx \oplus y) = 2A_tA_sx \oplus 2y = A_{t+s}(x) \oplus A_{t-s}(x) \oplus 2y = (A_{t+s}(x) \oplus y) + (A_{t-s}(x) \oplus y) = C_{t+s}(x \oplus y) + C_{t-s}(x \oplus y) = (C_{t+s} + C_{t-s})(x \oplus y)
\]

for all $t \geq s \geq 0$ then $2C_tC_s = C_{t+s} + C_{t-s}$, and we have: $C_0 = A_0 \oplus I = I \oplus I = I_{X \oplus X}$ finally $(C_t)_{t \geq 0}$ is a cosine function on $X \oplus X$, since $(A_t)_{t \geq 0}$ is hypercyclic, let
Lemma 3.8. Let $M$ be a nonzero subspace of $X$.

In general if $(A_t)_{t \in \mathbb{R}}$ and $(B_t)_{t \in \mathbb{R}}$ be two hypercyclic cosines function on Banach space $X$, then $C_t := A_t \oplus B_t; \forall t \in \mathbb{R}$ is a hypercyclic cosine function on $X \oplus X$.

Example 3.4. If $(T_t)_{t \in \mathbb{R}}$ is $C_0$-group on Banach space $X$, then $C_t = \frac{1}{2}(T_t + T_{-t}) \oplus I$ for all $t \in \mathbb{R}$ is $M$-hypercyclic cosine function with $M = X \oplus \{0\}$.

Definition 3.5. Let $(C_t)_{t \in \mathbb{R}}$ be a strongly continuous cosine function on separable Banach space $X$, and $M$ be a nonzero subspace of $X$, $(C_t)_{t \in \mathbb{R}}$ is called $M$-transitive if for every non-empty sets $U, V$ of $M$ there exists $t \in \mathbb{R}$, such that $C_t^{-1}(U) \cap V$ contains a non-empty open set of $M$.

Remark 3.6. By $C_t = C_{-t}$ for all $t \in \mathbb{R}$, then $(C_t)_{t \in \mathbb{R}}$ is $M$-transitive if and only if $(C_t)_{t \geq 0}$ is also.

Theorem 3.7. Let $(C_t)_{t \geq 0}$ be a strongly continuous cosine function on separable Banach space, and $M$ be a nonzero subspace of $X$, then the following conditions are equivalent:

1. $(C_t)_{t \geq 0}$ is $M$-transitive.

2. For every non-empty open sets $U$ and $V$ of $M$, there exists $t > 0$ such that $C_t^{-1}(U) \cap V$ is non-empty open set of $M$.

3. For every non-empty open sets $U$ and $V$ of $M$, there exists $t > 0$ such that $C_t^{-1}(U) \cap V$ is non-empty and $C_t(M) \subset M$.

Proof: (2) $\iff$ (1) is clear.

$(3) \Rightarrow (2)$. Let $U$ and $V$ be non-empty open subsets of $M$, by (3) there is $t_0 \geq 0$ such that $C_{t_0}^{-1}(U) \cap V \neq \emptyset$ and $C_{t_0}(M) \subset M$.

Since $C_{t_0}: M \rightarrow M$ is continuous, then $C_{t_0}^{-1}(U)$ is open in $M$, therefore $C_{t_0}^{-1}(U) \cap V$ is non-empty open of $M$.

$(1) \Rightarrow (3)$. Let $U$ and $V$ be two non-empty open subsets of $M$. By (1) there exists $t_0 \geq 0$ such that $C_{t_0}^{-1}(U) \cap V$ contains a non-empty open $W$ of $M$, it gives $W \subset C_{t_0}^{-1}(U) \cap V$ and $C_{t_0}^{-1}(U) \cap V \neq \emptyset$.

Next, we prove that $C_{t_0}(M) \subset M$.

Let $x \in M$, we have $W \subset C_{t_0}^{-1}(U) \cap V$, this implies that $C_{t_0}(W) \subset U \subset M$. Let $x_0 \in W$, since $W$ is open of $M$ then for all $r$ small enough we have $x_0 + rx \in W$, therefore $C_{t_0}(x_0 + rx) = C_{t_0}x_0 + rC_{t_0}x \in C_{t_0}(W) \subset M$. From $C_{t_0}x_0 \in M$ it follows that $C_{t_0}x \in M$.

We then conclude that $C_{t_0}(M) \subset M$.

Lemma 3.8. Let $(C_t)_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space $X$, and $M$ be a nonzero subspace of $X$. If $(C_t)_{t \geq 0}$ is $M$-transitive, then $(C_t)_{t \geq 0}$ has a dense set in $M$ of $M$-hypercyclic vectors.
Proof: Denote by $H_c(\mathcal{C}, M)$ the set of all $M$-hypercyclic vectors of $\mathcal{C} = (C_t)_{t \in \mathbb{R}}$, and since $X$ is separable let $(B_k)_{k \geq 0}$ be a countable open basis for the relative topology of $M$. We have $x \in H_c(\mathcal{C}, M)$ if and only if $\text{Orb}(\mathcal{C}, x) \cap M$ is dense in $M$ if and only if for each $k \geq 0$; there are $t \geq 0$ such that $C_t x \in B_k$, if and only if $x \in \bigcap_{k \geq 0} \bigcup_{t \geq 0} C_t^{-1}(B_k)$. (*)

then $H_c(\mathcal{C}, M) = \bigcap_{k \geq 0} \bigcup_{t \geq 0} C_t^{-1}(B_k)$. But $\mathcal{C}$ is $M$-transitive, then by theorem 3-7, for each $k, m \geq 0$ there exists $t = t_{k, m} \geq 0$ such that $C_t^{-1}(B_k) \cap B_m$ is non-empty open set, hence the set $A_k = \bigcup_{m \geq 0} C_t^{-1}(B_k) \cap B_m$ is non-empty and open set. Furthermore, for all $k \geq 0; A_k$ is dense in $M$, and by Baire category theorem we have that $\bigcap_{k \geq 0} A_k = \bigcap_{k \geq 0} \bigcup_{m \geq 0} C_t^{-1}(B_k) \cap B_m$ is dense in $M$, by (*) we have that $H_c(\mathcal{C}, M) = \bigcap_{k \geq 0} \bigcup_{t \geq 0} C_t^{-1}(B_k)$ is also dense in $M$. □

Theorem 3.9. Let $(C_t)_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space $X$, and $M$ be a nonzero subspace of $X$, if $(C_t)_{t \geq 0}$ is $M$-transitive, then $(C_t)_{t \geq 0}$ is $M$-hypercyclic.

Criteria of $M$-hypercyclicity of cosine function

In general it is difficult to find the $M$-hypercyclic vectors of cosine function, on Banach space, so we look for the criteria of $M$-hypercyclicity for cosine function.

Theorem 3.10. Let $(C_t)_{t \in \mathbb{R}}$ be a strongly continuous cosine function on separable Banach space $X$, and $M$ be a subspace of $X$, suppose that there is $D_0$ and $D_1$ two dense set in $M$, and an increasing positively sequence $(t_n)_{n \geq 0}$ such that:

- (i) $C_{t_n} x \to 0$ for all $x \in D_0$.
- (ii) For all $y \in D_1$, there is a sequence $(x_n)_n \subset M$, such that $x_n \to 0$ and $C_{t_n} x_n \to y$.
- (iii) $C_{t_n}(M) \subset M$.

Then $(C_t)_{t \geq 0}$ is $M$-transitive, it is $M$-hypercyclic.

Proof: Let $U$ and $V$ be two non-empty open set in $M$, we prove that there is $t \in \mathbb{R}$ such that $C_t^{-1}(U) \cap V \neq \emptyset$ and $C_t(M) \subset M$.

Since $D_0$ and $D_1$ are dense in $M$ then $V \cap D_0 \neq \emptyset$ and $D_1 \cap U \neq \emptyset$. Let $a \in U \cap D_1$ and $b \in D_0 \cap V$, hence there is $\epsilon > 0$ such that $B(a, \epsilon) \subset U$ and $B(b, \epsilon) \subset V$. From $b \in D_0$ and $a \in D_1$, we have $C_{t_n} b \to b$ and there exists $(x_n)_n \subset M$ such that $x_n \to 0$ and $C_{t_n} x_n \to a$.

Consequently there exists $N \in \mathbb{N}$ such that: $||x_n|| < \epsilon$, $||C_{t_n}(x_n) - a|| < \frac{\epsilon}{2}$ and
\[ \|C_{t_n}(b)\| < \frac{\epsilon}{\epsilon} \text{ for all } n \geq N. \]
Therefore \[ \|b + x_n - b\| = \|x_n\| < \epsilon \Rightarrow b + x_n \in B(b, \epsilon) \subset V \Rightarrow b + x_n \in V. \]
On the other hand
\[ \|C_{t_n}(b + x_n) - a\| = \|C_{t_n}(b) + C_{t_n}(x_n) - a\| \leq \|C_{t_n}(b)\| + \|C_{t_n}(x_n) - a\| < \frac{\epsilon}{\epsilon} + \frac{\epsilon}{\epsilon} = \epsilon \]
this implies that \[ C_{t_n}(b + x_n) \in B(a, \epsilon) \subset U \]
hence \[ b + x_n \in C_{t_n}^{-1}(U) \]. We obtain that \[ b + x_n \in C_{t_n}^{-1}(U) \cap V \] and \[ C_{t_n}^{-1}(U) \cap V \neq \emptyset \], since \[ C_{t_n}(M) \subset M \] then \( (C_t)_{t \geq 0} \) is \( M \)-transitive.
Let \( (C_t)_{t \in \mathbb{R}} \) be a strongly continuous cosine function on Banach space \( X \), and \( (t_n) \) be a sequence of positive real we put: \[ X_0 := \{ x \in X / \lim_{n \to +\infty} C_{t_n}x = 0 \} \]
and \( X_\infty := \{ y \in X / \exists u_n \to 0 ; \lim_{n \to +\infty} C_{t_n}(u_n) = y \}. \]
In [7] Antonio Bonilla and Pero J. Miana, proved that if there is \( t_n \to +\infty \) such that \( X_0 \) and \( X_\infty \) are dense, then \( (C_t)_{t \geq 0} \) is a hypercyclic cosine function. \( \square \)

**Theorem 3.11.** Let \( (C_t)_{t \geq 0} \) be a strongly continuous cosine function on separable Banach space \( X \), and \( M \) be a nonzero subspace of \( X \), if there exists a sequence of positive real \( (t_n) ; t_n \to +\infty \) such that \( X_0 \cap M \) and \( X_\infty \cap M \) are dense in \( M \) and \( C_{t_n}(M) \subset M \), then \( (C_t)_{t \geq 0} \) is \( M \)-transitive in particular it is \( M \)-hypercyclic.

**Proof:** It is sufficient to take: \( D_0 = X_0 \cap M \) and \( D_1 = X_\infty \cap M \). \( \square \)

**Corollary 3.12.** Let \( (C_t)_{t \geq 0} \) be a strongly continuous cosine function on a separable Banach space \( X \), if there exists a sequence of real \( (t_n) ; t_n \to +\infty \) such that \( X_1 = \{ x \in X / \lim_{n \to +\infty} C_{t_n}(x) = \lim_{n \to +\infty} C_{2t_n}(x) = 0 \} \cap M \) is dense in \( M \) and \( C_{t_n}(M) \subset M \), then \( (C_t)_{t \geq 0} \) is a \( M \)-transitive cosine function.

**Proof:** we prove that \( X_1 \cap M \subset X_0 \cap M \) and \( X_1 \cap M \subset X_\infty \cap M \)
Let \( y \in X_1 \cap M \) then \( y \in X_1 \) and \( y \in M \), we define \( x_n = 2C_{t_n}(y) \) we have \( x_n \to 0 \) and \( C_{t_n}(y) = C_{t_n}(2C_{t_n}(y)) = 2C_{t_n}C_{t_n}(y) = C_{2t_n}(y) + y \to y \) then \( X_1 \cap M \subset X_0 \cap M \) and \( X_1 \cap M \subset X_\infty \cap M \), since \( X_1 \cap M \) dense in \( M \), then \( X_0 \cap M \) and \( X_\infty \cap M \) are also dense in \( M \) and since \( C_{t_n}(M) \subset M \) then \( (C_t)_{t \geq 0} \) is \( M \)-transitive. In [2, proposition 3.14.6] Arendt and al showed that if \( (C_t)_{t \in \mathbb{R}} \) is a strongly continuous cosine function on Banach space \( X \), such that \( \lim_{t \to +\infty} C_t(x) = 0 \), then \( x = 0 \). In [6], A. Bprowski and W. Chojnacki generalized this result by showing: if \( \lim_{t \to +\infty} C_t(x) \) exists for all \( x \in X \) then \( C_t = I \) for all \( t \in \mathbb{R} \), based on this we obtain the following result. \( \square \)

**Corollary 3.13.** Let \( (C_t)_{t \in \mathbb{R}} \) be a strongly continuous cosine function on Banach space \( X \), if \( \lim_{t \to +\infty} C_t(x) \) exists for all \( x \in X \), then \( (C_t)_{t \in \mathbb{R}} \) is not \( M \)-hypercyclic for any \( M \) subspace of \( X \), in particular it is not hypercyclic.

The existence of \( M \)-hypercyclic cosine function
It is natural to ask about the existence of $M$-hypercyclic cosine function on a separable Banach space.

**Lemma 3.14.** [3] If $E$ is a dense subset of Banach space $X$, then there exists a non-trivial closed subspace $M$ of $X$, such that $E \cap M$ is dense in $M$.

**Proposition 3.15.** If $(C_t)_{t \geq 0}$ is a hypercyclic strongly continuous cosine function on Banach space $X$, then $(C_t)_{t \geq 0}$ is $M$-hypercyclic cosine function with $M$ is a non-trivial closed subspace of $X$.

**Remark 3.16.** Every hypercyclic strongly continuous cosine function is $M$-hypercyclic for $M$ is non-trivial closed subspace of $X$, but there exists a $M$-hypercyclic cosine function, that is not hypercyclic on $X$ see example 1.

**Theorem 3.17.** [7] Every separable infinite dimensional complex Banach space $X$ admits a topologically mixing (hypercyclic) uniformly continuous cosine function.

**Corollary 3.18.** Every separable infinite dimensional complex Banach space $X$, admits a $M$-hypercyclic uniformly continuous cosine function with $M$ is a non-trivial closed subspace of $X$.

**References**


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