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Existence of Solution of Urysohn Integral Equation Through Generalized Contractive Mapping

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abstract: In this note, we establish the existence of fixed point through fixed point theorems in the setting of partially ordered complex valued b- metric spaces. Then this fixed point is co-related as solution of equivalent operator equation of the Urysohn integral equation. In this process to make our results more authentic and meaningful we adopt an innovative way through visualling the given example supporting our findings. Naturally our results generalize some existing results.

Key Words: Fixed point, Partially ordered complex valued b-metric space, Cauchy sequence, Urysohn integral equation.

Contents

1. Introduction and preliminaries

In 1989, Bakhtin [5] introduced and studied the concept of b-metric spaces as a generalization of metric spaces. Also he proved the Banach contraction principle in b-metric spaces. After that many researchers obtained fixed point results in b-metric spaces (see [2], [3], [6], [8]).

In 2011, Azam et al. [4] introduced the concept of complex valued metric spaces as a generalization of the classical metric spaces and established some fixed point theorems for a pair of mappings for contraction condition satisfying a rational expression. After the establishment of complex valued metric spaces, Rao et al. [15] introduced the complex valued b-metric spaces and then several authors have contributed with different concepts in these spaces. One can see in ([1], [9]- [13], [16]- [21]).

On the other hand, many authors generalized the Banach contraction theorem in ordered metric spaces. The first result in ordered metric spaces was given by Ran and Reurings [14] who presented its applications to the linear and nonlinear metric spaces.

In this study, we have presented some fixed point theorems having rational type contraction conditions in the notion of partially ordered complex valued b-metric space. Furthermore, an application to establish the solution of Urysohn integral

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equation is also presented, utilizing our investigated results.
In what follows, we recall some definitions and notations that will be used in our note.
Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$and $Im(z_1) \leq Im(z_2)$.
It follows that $\precsim$ exists if one of the followings conditions is satisfied:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied. Note that
The following definition is due to Azam et al. [4].

Definition 1.1. : *Let X be a non empty set. A mapping $d : X \times X \to C$ is called a complex valued metric on X if d satisfies the following conditions :*

(CM1) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) \Leftrightarrow x = y$;

(CM2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called complex valued metric space.

Example 1.2. *Let $X = C$ be a set of complex number. Define the mapping $d : X \times X \to C$ by*

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|,$$

where $k \in R$. Then (X, d) is a complex valued metric space.

Acknowledging the concepts of Bakhtin [5] and Azam et al. [4], K.P.R. Rao et al. [15] introduced the notion of complex valued b-metric spaces as follows.

Definition 1.3. *[15] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to C$ satisfies the following conditions:*

(CVBM1) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(CVBM2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$;

(CVBM3) $d(x, y) \precsim s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then d is called a complex valued b-metric on X and (X, d) is called a complex valued b-metric space.

Example 1.4. *Let $X = [0, 1]$. Define the mapping $d : X \times X \to C$ by*

$$d(x, y) = |x - y|^2 + i|x - y|^2, \quad for \;\; all \;\; x, y \in X.$$

Then (X, d) is a complex valued b- metric space with $s = 2$.

For the routine definitions like convergent sequence, Cauchy sequence, complete complex valued b-metric space we refer [15].

Lemma 1.5. *[15] Let (X, d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.6. *[15] Let (X, d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$*

Definition 1.7. *Let (X, d) be a complex valued metric space, $T : X \to X$ and $x \in X$. Then the function T is continuous at x if for any sequence $\{x_n\}$ in X,*

$$x_n \to x \Rightarrow Tx_n \to Tx.$$

Definition 1.8. *Let $(X, \precsim)$ be a partially ordered set and $T : X \to X$. The mapping T is said to be nondecreasing if for all $x_1, x_2 \in X, x_1 \precsim x_2$ implies $Tx_1 \precsim Tx_2$ and nonincreasing if for all $x_1, x_2 \in X, x_1 \precsim x_2$ implies $Tx_1 \succsim Tx_2$.*

2. Main Result

In this section, some fixed point theorems for contraction conditions described by rational expressions are proved.

Theorem 2.1. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Let the mapping $A : X \to X$ be a continuous and non decreasing mapping. Suppose there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s}$ such that, for all $x, y \in X$ with $x \precsim y$,*

$$\begin{aligned} d(Ax, Ay) \precsim & \alpha d(x, y) + \beta \frac{d(y, Ay)[1 + d(x, Ax)]}{1 + d(x, y)} + \gamma \frac{d(y, Ax)[1 + d(x, Ay)]}{1 + d(x, y)} \\ & + \delta[d(y, Ax) + d(x, Ay)] \end{aligned} \tag{2.1}$$

if there exist $x_0 \in X$ with $x_0 \precsim Ax_0$, then A has a fixed point.

Proof: If $x_0 = Ax_0$, then we have the result.
Suppose that $x_0 \prec Ax_0$. Then we construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Ax_n, \quad for \;\; every \;\; n \geq 0. \tag{2.2}$$

Since A is a non-decreasing mapping, we obtain by induction that

$$x_0 \prec Ax_0 = x_1 \precsim Ax_1 = x_2 \precsim Ax_1 = x_2 \precsim \cdots Ax_{n-1} = x_n \precsim Ax_n = x_{n+1}. \tag{2.3}$$

If there exist some $N \geq 1$ such that $x_{N+1} = x_N$, then from (2.2), $x_{N+1} = Ax_N = x_N$; that is, x_N is a fixed point of A and the proof is finished.
Now we suppose that $x_{N+1} \neq x_N$ for all $n \geq 1$. Since $x_n \prec x_{n+1}$, for all $n \geq 1$, applying (2.1) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) =& d(Ax_n, Ax_{n+1}) \\ \precsim& \alpha d(x_n, x_{n+1}) + \beta \frac{d(x_{n+1}, Ax_{n+1})[1 + d(x_n, Ax_n)]}{1 + d(x_n, x_{n+1})} \\ &+ \gamma \frac{d(x_{n+1}, Ax_n)[1 + d(x_n, Ax_{n+1})]}{1 + d(x_n, x_{n+1})} \\ &+ \delta[d(x_{n+1}, Ax_n) + d(x_n, Ax_{n+1})] \\ \precsim& \alpha d(x_n, x_{n+1}) + \beta \frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})} \\ &+ \gamma \frac{d(x_{n+1}, x_{n+1})[1 + d(x_n, x_{n+2})]}{1 + d(x_n, x_{n+1})} + \delta[d(x_{n+1}, x_{n+1}) \\ &+ d(x_n, x_{n+2})] \\ \precsim& \alpha d(x_n, x_{n+1}) + \beta d(x_{n+1}, x_{n+2}) + \delta d(x_n, x_{n+2}) \\ \precsim& \alpha d(x_n, x_{n+1}) + \beta d(x_{n+1}, x_{n+2}) + s\delta[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \end{aligned}$$

thus one can get

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \precsim& \Big(\frac{\alpha + s\delta}{1 - \beta - s\delta}\Big) d(x_n, x_{n+1}) \\ \precsim& h d(x_n, x_{n+1}), \quad where \quad h = \Big(\frac{\alpha + s\delta}{1 - \beta - s\delta}\Big) < \frac{1}{s}. \end{aligned} \tag{2.4}$$

This follows immediately

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \precsim& h d(x_n, x_{n+1}) \precsim h^2 d(x_{n-1}, x_n) \\ \precsim& h^3 d(x_{n-2}, x_{n-1}) \precsim \cdots \precsim h^{n+1} d(x_0, x_1) \end{aligned}$$

for $m > n$

$$\begin{aligned} d(x_n, x_m) \precsim& s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ \precsim& s d(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ \precsim& s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) \\ &+ s^3 d(x_{n+2}, x_m) + \cdots + s^{m-n} d(x_{m-1}, x_m) \\ \precsim& (sh^n + s^2 h^{n+1} + \cdots + s^{m-n} h^{m-1}) d(x_0, x_1) \\ \precsim& sh^n [1 + (sh) + (sh)^2 \cdots + (sh)^{m-n-1}] d(x_0, x_1) \\ \precsim& \frac{sh^n}{1 - sh} d(x_0, x_1). \end{aligned}$$

Since $0 \leq h < \frac{1}{s}$, we conclude that $\frac{sh^n}{1-sh} \to 0$ as $n \to \infty$. Which implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exist a point $z \in X$ such that

$$x_n \to z \;\; as \;\; n \to \infty. \tag{2.5}$$

The continuity of A implies that $Az = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = z$.
That is z is a fixed point of A. □

Now question arised if some how continuity is dropped for underlying mapping then how it impacts on existence of fixed point, this question is answered in the following result.

Theorem 2.2. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \precsim x$, for all $n \in N$. Let the mapping $A : X \to X$ be a non decreasing mapping. Suppose that (2.1) holds for all $x, y \in X$, with $x \precsim y$. If there exist $x_0 \in X$ with $x_0 \precsim Ax_0$, then A has a fixed point.*

Proof: We take the same pattern of sequence $\{x_n\}$ as in the proof of Theorem 2.1 and with Similar approach we prove that $\{x_n\}$ is a non-decreasing sequence such that $x_n \to z \in X$. Then $x_n \precsim z$, for all $n \in N$.
Applying Inequality (2.1), we have

$$\begin{aligned}
d(x_{n+1}, Az) =& d(Ax_n, Az)\\
&\precsim \alpha d(x_n, z) + \beta \frac{d(z, Az)[1 + d(x_n, Ax_n)]}{1 + d(x_n, z)}\\
&\quad + \gamma \frac{d(z, Ax_n)[1 + d(x_n, Az)]}{1 + d(x_n, z)}\\
&\quad + \delta[d(z, Ax_n) + d(x_n, Az)]\\
&\precsim \alpha d(x_n, z) + \beta \frac{d(z, Az)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, z)}\\
&\quad + \gamma \frac{d(z, x_{n+1})[1 + d(x_n, Az)]}{1 + d(x_n, z)}\\
&\quad + \delta[d(z, x_{n+1}) + d(x_n, Az)].
\end{aligned}$$

Taking the limit as $n \to \infty$ and using (2.5), we have

$$\begin{aligned}
d(z, Az) \precsim& \alpha d(z, z) + \beta \frac{d(z, Az)[1 + d(z, z)]}{1 + d(z, z)} + \gamma \frac{d(z, z)[1 + d(z, Az)]}{1 + d(z, z)}\\
&+ \delta[d(z, z) + d(z, Az)]\\
\precsim& \beta d(z, Az) + \delta d(z, Az)\\
\precsim& (\beta + \delta) d(z, Az).
\end{aligned}$$

Since $\beta + \delta < 1$, it is a contradiction unless $d(z, Az) = 0$. This amounts to say that $Az = z$, therefore z is a fixed point of A. □

In the following result we proved uniqueness of fixed point with the application of order relation, justifying the setting of working space.

Theorem 2.3. *In addition to the hypothesis of Theorem 2.1 or Theorem 2.2, suppose that for every $x, y \in X$, there exist $u \in X$ such that $u \precsim x$ and $u \precsim y$, then A has a unique fixed point.*

Proof: It follows from the Theorem 2.1 or Theorem 2.2 that the set of fixed point of A is non-empty. We shall show that if x^* and y^* two fixed point of A, that is, if $x^* = Ax^*$ and $y^* = Ay^*$ then $x^* = y^*$.
By the assumption, there exist $u_0 \in X$ such that $u_0 \precsim x^*$ and $u_0 \precsim y^*$. Then similarly as in the proof of Theorem 2.1, we define the sequence $\{u_n\}$ such that

$$u_{n+1} = Au_n = A^{n+1}u_0, \ \ n = 0, 1, 2, ... \tag{2.6}$$

monotonicity of A implies that

$$A^n u_0 = u_n \precsim x^* = A^n x^* \ \ and \ \ A^n u_0 = Au_n \precsim y^* = A^n y^*.$$

If there exist a positive integer m such that

$$x^* = u_m, \ \ then \ \ x^* = Ax^* = Au_n = c_{n+1},$$

for all $n \geq m$. Then $u_n \to x^*$ as $n \to \infty$. Suppose that $x^* \neq u_n$, for all $n \geq 0$. So $u_n \prec x^*$ for all $n \geq 0$. Applying inequality (2.1), we have

$$\begin{aligned}
d(u_{n+1}, x^*) =& d(Au_n, Ax^*) \\
\precsim& \alpha d(u_n, x^*) + \beta \frac{d(x^*, Ax^*)[1 + d(u_n, Ax_n)]}{1 + d(u_n, x^*)} \\
&+ \gamma \frac{d(x^*, Au_n)[1 + d(u_n, Ax^*)]}{1 + d(u_n, x^*)} \\
&+ \delta[d(x^*, Au_n) + d(u_n, Ax^*)] \\
\precsim& \alpha d(u_n, x^*) + \beta \frac{d(x^*, x^*)[1 + d(u_n, Ax_n)]}{1 + d(u_n, x^*)} \\
&+ \gamma \frac{d(x^*, u_{n+1})[1 + d(u_n, x^*)]}{1 + d(u_n, x^*)} \\
&+ \delta[d(x^*, u_{n+1}) + d(u_n, x^*)] \\
\precsim& \alpha d(u_n, x^*) + \gamma d(x^*, u_{n+1}) + \delta[d(x^*, u_{n+1}) + d(u_n, x^*)] \\
\precsim& \Big(\frac{\alpha + \delta}{1 - \gamma - \delta}\Big) d(u_n, x^*),
\end{aligned}$$

put $\left(\frac{\alpha+\gamma}{1-\gamma-\delta}\right) = k < \frac{1}{s}$. Thus we get

$$d(u_{n+1}, x^*) \precsim kd(u_n, x^*) \precsim k^2 d(u_{n-1}, x^*) \precsim \cdots k^{n+1} d(u_0, x^*) \ \to 0 \ \ as \ \ n \to \infty.$$

Hence $d(u_n, x^*) \to 0$ as $n \to \infty$,
or $u_n \to x^*$ as $n \to \infty$.
Using a similar argument, we can prove that
$u_n \to y^*$ as $n \to \infty$.
Finally, the uniqueness of the limit implies that $x^* = y^*$.
Hence A has a unique fixed point. □

If we put $\delta = 0$ *and* $s = 1$ in inequality (2.1) of Theorem 2.1 then we get following corollary which coincides the result due to Choudhury et al. [7].

Corollary 2.4. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued metric d on X such that (X, d) is a complete complex valued metric space. Let the mapping $A : X \to X$ be a continuous and non decreasing mapping. Suppose there exist non-negative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that, for all $x, y \in X$ with $x \precsim y$,*

$$d(Ax, Ay) \precsim \alpha d(x, y) + \beta \frac{d(y, Ay)[1 + d(x, Ax)]}{1 + d(x, y)} + \gamma \frac{d(y, Ax)[1 + d(x, Ay)]}{1 + d(x, y)} \tag{2.7}$$

if there exist $x_0 \in X$ with $x_0 \precsim Ax_0$, then A has a fixed point.

If we set $\beta = \gamma = 0$ in inequality (2.1) of Theorem 2.1 then we get following corollary.

Corollary 2.5. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Let the mapping $A : X \to X$ be a continuous and non decreasing mapping. Suppose there exist non-negative real numbers α, δ with $\alpha + 2s\delta < \frac{1}{s}$ such that, for all $x, y \in X$ with $x \precsim y$,*

$$d(Ax, Ay) \precsim \alpha d(x, y) + \delta[d(y, Ax) + d(x, Ay)] \tag{2.8}$$

if there exist $x_0 \in X$ with $x_0 \precsim Ax_0$, then A has a fixed point.

Example 2.6. *Let $X = [3, 4]$ with usual partial order $\leq$. Let the complex valued b-metric d be given by*

$$d(x, y) = |x - y|^2 e^{i\frac{\pi}{4}} \ \ with \ \ s = \sqrt{2}, \ \ for \ \ all \ \ x, y \in X.$$

Let $A : X \to X$ be defined as $Ax = \sqrt{x} + 2$.
First we check that there exist $x_0 \in X$ such that $x_0 \precsim Ax_0$ with usual partial order $\leq$.
Clearly, $\quad x \leq \sqrt{x} + 2, \quad \forall x \in [3, 4]$.
then this condition is satisfied.
In order to verify the condition (2.1), first we notice that

$$0 \precsim \frac{d(y, Ay)[1 + d(x, Ax)]}{1 + d(x, y)}, 0 \precsim \frac{d(y, Ax)[1 + d(x, Ay)]}{1 + d(x, y)}, 0 \precsim [d(y, Ax) + d(x, Ay)].$$

for all $x, y \in X$.
Thus it is sufficient to show that $d(Ax, Ay) \precsim ad(x, y)$ *with* $\alpha, \beta, \gamma, \delta \geq 0$ *and* $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s}$
Now,

$$\begin{aligned} d(Ax, Ay) &\precsim \alpha d(x, y) \\ \Rightarrow d(\sqrt{x}+2, \sqrt{y}+2) &\precsim \alpha d(x, y) \\ \Rightarrow |\sqrt{x}+2-\sqrt{y}+2)|^2 e^{\frac{i\pi}{4}} &\precsim \alpha |x-y|^2 e^{\frac{i\pi}{4}}. \end{aligned}$$

This implies that

$$|\sqrt{x}-\sqrt{y})|^2 e^{\frac{i\pi}{4}} \precsim \alpha |x-y|^2 e^{\frac{i\pi}{4}} \tag{2.9}$$

(2.9) *is true with a view that* $ae^{i\theta} \precsim be^{i\theta}$ *iff* $a \leq b$, *where* $a, b \in R$ *and in* (2.9), *we have*

$$|\sqrt{x}-\sqrt{y})|^2 \precsim \alpha |x-y)|^2, \tag{2.10}$$

for all $x, y \in [3, 4]$ *with* $\alpha = 0.5$ *and suitable values of* β, γ, δ *such that* $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s}$ *where* $s = \sqrt{2}$.
Following Figures 1 and 2 validate inequality (2.10) *graphically. In subsequent Figures 1 and 2 , surfaces with purple color represent the L.H.S. of* (2.10) *and surfaces with red color represent the R. H. S. of* (2.10). *Clearly red surfaces are dominating the purple surfaces. consequently, Condition* (2.1) *is satisfied.*

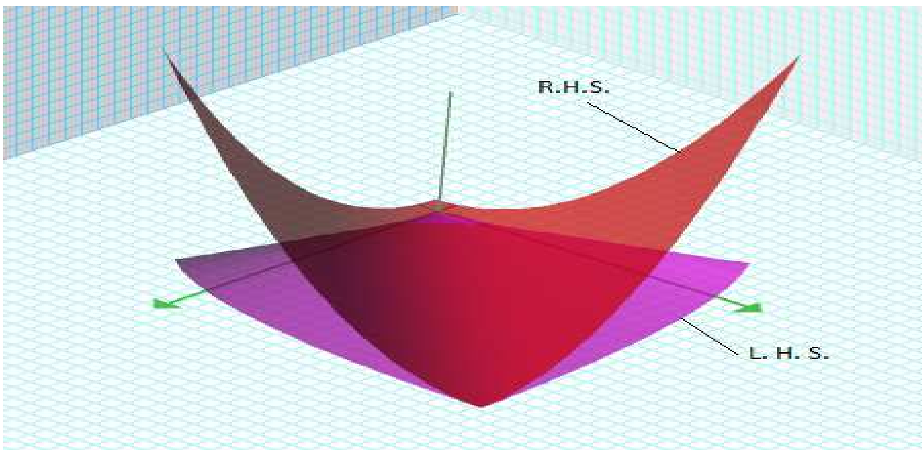

Figure 1: Plot of condition 2.11

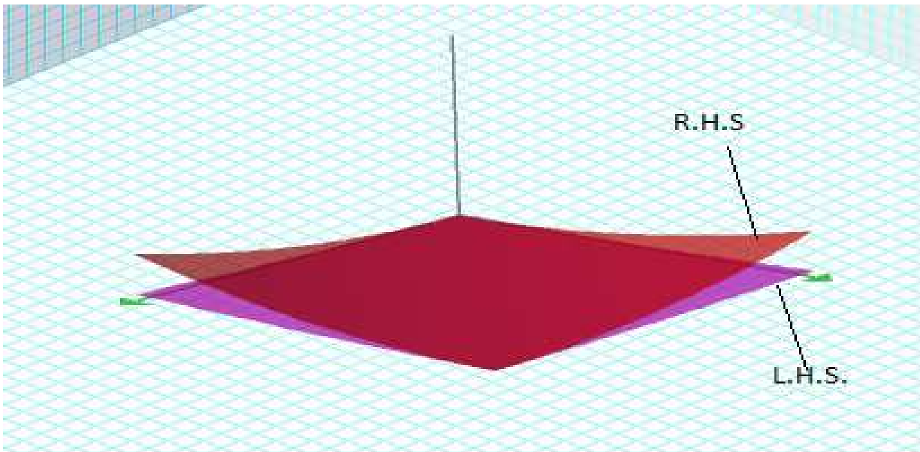

Figure 2: Zoom view of condition 2.11 exactly in [3, 4]

Also the function is continuous. Hence all the conditions of Theorem 2.1 are satisfied and $x = 4$ *is a unique fixed point of* A *which is demonstrated by the Figure 2. In the Figure 2 lines with red color represent function* $f(x) = \sqrt{x} + 2$ *and purple line represents* $y = x$ *for fixed point purpose. Clearly, we can see that line* $y = x$ *intersects functions* $f(x)$ *only at* $x = 4$*, this amounts to say that* $x = 4$ *is the unique fixed point of* $f(x) = \sqrt{x} + 2$*.*

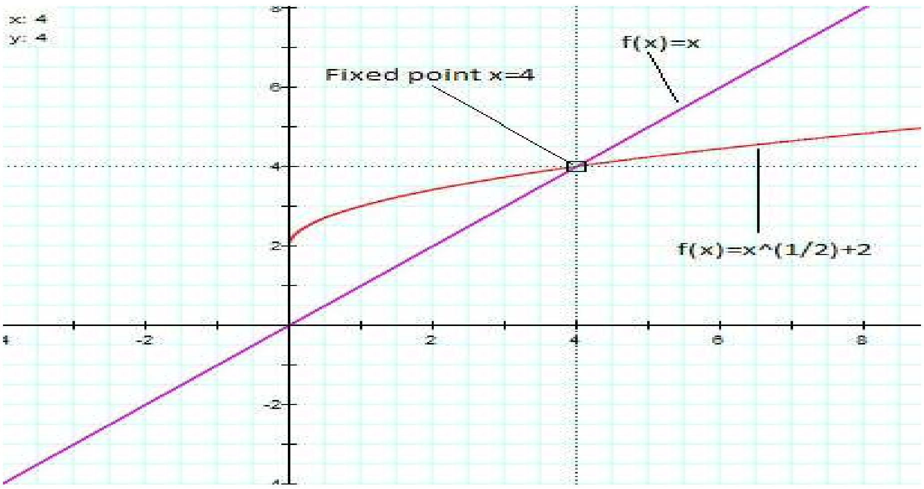

Figure 3: Fixed Point

3. Application of fixed point results

Let $X = C([a,b], \mathbb{R}^n),\ a > 0$ and $d : X \times X \to X$ be defined by

$$d(x,y) = \Big[\max_{t\in[a,b]} ||x(t) - y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a} \Big]^q, \tag{3.1}$$

where $\mathtt{s} = 2^{q-1}$. Consider the Urysohn integral equation

$$x(t) = \int_a^b K(t,s,x(s))\, ds + g(t), \tag{3.2}$$

where $t \in [a,b] \subset \mathbb{R},\ x \in X$. Suppose that $K : [a,b] \times [a,b] \times \mathbb{R}^n$ such that $F_x \in X$, for each $x \in X$, where

$$F_x(t) = \int_a^b K(t,s,x(s))\, ds, \text{ for all } t \in [a,b].$$

Theorem 3.1. *If there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ such that for all $x, y \in X$ with $x \precsim y$*

(i) $\alpha + \beta + \gamma + 2\mathtt{s}\delta < \frac{1}{\mathtt{s}}$; (ii)

$$\Big[||F_x(t) - F_y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q \precsim \alpha\, S(x,y)(t) + \beta\, T(x,y)(t) + \gamma\, U(x,y)(t) + \delta\, V(x,y)(t),$$

where,

$$S(x,y)(t) = \Big[||x(t) - y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q;$$

$$T(x,y)(t) = \frac{\Big[||F_y(t) + g(t) - y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q \Big[1 + ||F_x(t) + g(t) - x(t)||_\infty^q\Big]}{1 + \max_{t\in[a,b]} S(x,y)(t)};$$

$$U(x,y)(t) = \frac{\Big[||F_x(t) + g(t) - y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q \Big[1 + ||F_y(t) + g(t) - x(t)||_\infty^q\Big]}{1 + \max_{t\in[a,b]} S(x,y)(t)};$$

$$V(x,y)(t) = \Big[||F_x(t) + x(t) - y(t)||_\infty^q + ||F_y(t) + y(t) - x(t)||_\infty^q\Big]\Big[\sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q;$$

(iii) there exists $x_0 \in X$ such that

$$x_0(t) \precsim \int_a^b K(t,s,x_0(s))\, ds + g(t), \textit{for all} t \in [a,b].$$

Then the integral equation defined in (3.1) has a solution in X.

Proof: Define $A : X \to X$ by $A_x = F_x + g$. It is easy to deduce that (X,d) is a complex valued b-metric space. Then

$$d(Ax, Ay) = \Big[\max_{t\in[a,b]} ||F_x(t) + g(t) - F_y(t) - g(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q$$
$$= \Big[\max_{t\in[a,b]} ||F_x(t) - F_y(t)||_\infty \sqrt{1+a^2}\ e^{i\tan^{-1}a}\Big]^q;$$
$$d(x,y) = \max_{t\in[a,b]} S(x,y)(t);$$
$$\frac{d(y,Ay)[1+d(x,Ax)]}{1+d(x,y)} = \max_{t\in[a,b]} T(x,y)(t);$$
$$\frac{d(y,Ax)[1+d(x,Ay)]}{1+d(x,y)} = \max_{t\in[a,b]} U(x,y)(t);$$
$$d(y,Ax) + d(x,Ay) = \max_{t\in[a,b]} V(x,y)(t).$$

It is easy to conclude that

$$d(Ax,Ay) \precsim \alpha d(x,y) + \beta\frac{d(y,Ay)[1+d(x,Ax)]}{1+d(x,y)} + \gamma\frac{d(y,Ax)[1+d(x,Ay)]}{1+d(x,y)} + \delta[d(y,Ax)+d(x,Ay)].$$

Clearly, the contractive condition of Theorem 2.1 is satisfied. From condition (iii), we have $x_0 \precsim Ax_0$.
Hence all the conditions of Theorem 2.1 are fulfilled. Therefore, by Theorem 2.1, Urysohn integral equation (3.1) has a solution in X. □

Competing Interests
The authors declare that they have no competing interests.

Author's contribution
All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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