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Convergence of a Series Leading to an Analogue of Ramanujan's Assertion on Squarefree Inetegers*

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ABSTRACT: Let d be a squarefree integer. We prove that
(i) $\sum_n \frac{\mu(n)}{n} d(n')$ converges to zero, where n' is the product of prime divisors of n with $(\frac{d}{n}) = +1$. We use the Prime Number Theorem.
(ii) $\prod_{(\frac{d}{p})=+1}(1 - \frac{1}{p^s})$ is not analytic at s=1, nor is $\prod_{(\frac{d}{p})=-1}(1 - \frac{1}{p^s})$.
(iii) The convergence (i) leads to a proof that asymptotically half the squarefree ideals have an even number of prime ideal factors (analogue of Ramanujan's assertion).

Key Words: Dirichlet series; Prime Number Theorem.

Contents

1. Introduction

Ramanujan's Assertion was that asymptotically half the squarefree integers had an even number of prime factors ([2],pp 64-65).We prove the convergence of a certain series using the Prime Number Theorem.(Lemma B).This fills a gap in the proof of the analogue of Ramanujan's Assertion for Squarefree ideals in a Quadratic Field"([6]).

2. Mathematics

Notation: If $n = (p_1 p_2 \cdots p_k)(q_1 q_2 \cdots q_l)$

$$(\frac{d}{p_i}) = +1\ \forall i,\ \ (\frac{d}{q_j}) = -1\ or\ 0\ \forall j$$

write $n' = p_1 p_2 \cdots p_k$, $n'' = q_1 q_2 \cdots q_l$, $\mu(n)$ and $d(n)$ denote Moebius and divisor functions respectively.

Lemma B:

$$\sum_n \frac{\mu(n) d(n')}{n}\ converges\ (to\ 0)$$

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Proof. Recall that for a multiplicative function $f(n)$ one has ([1], Theorem 11.7)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\ prime} (1 + \frac{f(p)}{p^s} + \cdots + \frac{f(p^k)}{p^{ks}} + \cdots)$$

Choose $f(n) = \mu(n)d(n')$, f is indeed multiplicative

$$f(p) = \mu(p)d(p') = \begin{cases} -2 & \text{if } (\frac{d}{p}) = +1 \\ -1 & \text{if } (\frac{d}{p}) = -1 \ or \ 0 \end{cases}$$

$$\therefore \sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n^s} = \prod_{p,+1} (1 - \frac{2}{p^s}) \prod_{q,-1,0} (1 - \frac{1}{q^s}), \ (Re\ s > 1)$$

with the obvious notation
Now ([5], p321)

$$\prod_{p,+1} (1 - \frac{2}{p^s}) = \prod_{p,+1} (1 - \frac{1}{p^s})^2 \prod_{p,+1} (1 - \frac{1}{p^{2s}(1 - \frac{1}{p^s})^2})$$

so that by combining terms

$$\sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n^s} = (\prod_{p,+1} (1 - \frac{1}{p^s}))(\prod_{p}(1 - \frac{1}{p^s}))(\sum_{n=1}^{\infty} \frac{a_n}{n^s}) \ (\star)$$

where the last series converges absolutely for Re $s > \frac{1}{2}$.

Indeed for Re $s > 1$

$$\prod_{p}(1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

We check that $(\star)$ extends to $s = 1$ as a convergent Cauchy product of series. Firstly, since $\sum \frac{1}{p} = \infty$

$$\prod_{p}(1 - \frac{1}{p}) = 0$$

and the Prime Number Theorem ([1],p 97) is equivalent to $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$. Thus at $s = 1$ the middle term above in $(\star\ )$ is a series converging to zero
Claim:The first term in $(\star)$ is also zero i.e.

$$\prod_{p,+1} (1 - \frac{1}{p}) = \sum_{n'} \frac{\mu(n')}{n'} = 0$$

where the sum is over all numbers n' which are products of primes of type $+1$. We prove this Claim by means of Lemmas 2.1 and 2.2 below. □

Lemma 2.1.

$$\sum_{n' \leq x} \mu(n') = \circ(x)$$

Proof.

$$\begin{aligned}
|Lim_{x\to\infty}\frac{1}{x}\sum_{n'\leq x}\mu(n')| &\leq Lim_{x\to\infty}\frac{1}{x}(\sum_{n'\leq x}|\mu(n')|) \\
&\leq Lim_{x\to\infty}\{\frac{(\sum\limits_{n'\leq x}|\mu(n')|)}{\sum\limits_{n\leq x}|\mu(n)|}\frac{\sum\limits_{n\leq x}|\mu(n)|}{x}\} \\
&= Lim_{x\to\infty}\{\frac{\sum\limits_{n'\leq x}|\mu(n')|}{\sum\limits_{n\leq x}|\mu(n)|}\}.\frac{1}{\zeta(2)}
\end{aligned}$$

Now we claim the ratio $\frac{(\sum_{n'\leq x}|\mu(n')|)}{(\sum_{n\leq x}|\mu(n)|)}$ tends to 0 as $x \to \infty$ because it can be rewritten as the ratio of the number of squarefree integers formed from the first t primes of symbol +1: $\frac{|\{p_{i_1}p_{i_2}\ldots p_{i_m}|i_j\leq t,(\frac{d}{p_{i_j}})=+1\}|}{|\{p_{i_1}p_{i_2}\ldots p_{i_m}|i_j\leq t\}|}$ t depending on x. But counting the "favourable cases" and using the above Theorem on density, this ratio is asymptotically $(\frac{3}{4})^t$ which tends to 0 as $t \to \infty$ or $x \to \infty$. This proves Lemma 2.1. □

Lemma 2.2.

$$Lim_{x\to\infty}\sum_{n'\leq x}\frac{\mu(n')}{n'} = 0$$

Proof. Note that the argument of Hlawka et al ([3],p200) applies since $|\mu(n')| \leq 1$ (boundedness) and $\sum_{n'\leq x}\mu(n') = \circ(x)$. Thus with $f(n)$ defined by

$$f(n) = \begin{cases} \mu(n') & \text{if } n = n' \\ 0 & \text{if not} \end{cases}$$

we have

$$\begin{aligned}
\sum_{n\leq x}\frac{f(n)}{n} &= \sum_{n'\leq x}\frac{\mu(n')}{n'} \\
&= \frac{1}{x}\sum_{n\leq x}(\sum_{k|n}f(k)) + \circ(1) \\
&= \frac{1}{x}\sum_{n\leq x, n squarefree}(\sum_{k|n'}f(k)) + \circ(1)
\end{aligned}$$

$$= \frac{1}{x}\sum_{n\leq x}(\sum_{k|n'}\mu(k)) + \circ(1)$$
$$= \frac{1}{x}\sum_{n\leq x, n'=1} 1$$
$$= \frac{1}{x}|\{n|n \leq x, n = n''\}| + \circ(1)$$

Let $x \to \infty$ so that (since the fraction of n'' tends to 0)

$$Lim_{x\to\infty}\sum_{n'\leq x}\frac{\mu(n')}{n'} = 0$$

Now

$$\sum_{n=1}^{\infty}\frac{\mu(n)d(n')}{n} = [\sum_{n'}\frac{\mu(n')}{n'}][\sum_{n=1}^{\infty}\frac{\mu(n)}{n}][\sum_{n=1}^{\infty}\frac{a_n}{n}]$$

Now one notes that the Cauchy product of the first two series on the RHS is zero by adapting the argument for $(\sum\frac{\mu(n)}{n})^2$ ([7], Lemma 3). Let $\sum_{n=1}^{\infty}\frac{b_n}{n}$ be this Cauchy product.Thus $\sum_{n=1}^{\infty}\frac{\mu(n)d(n')}{n} = (\sum\frac{b_n}{n})(\sum\frac{a_n}{n}) = (0)(A) = 0$ by Mertens'. □

Corollary 2.3. *Writing (for Re $s > 1$)*

$$\frac{1}{\zeta(s)} = \prod_p(1-\frac{1}{p^s}) = \prod_{p,+1}(1-\frac{1}{p^s})\prod_{p,-1,0}(1-\frac{1}{p^s})$$

we cannot have both the products on the right to be analytic at $s = 1$.

Proof. $\frac{1}{\zeta(s)}$ has a simple zero at $s = 1$ ([1],Theorem 12.4) and by Lemma 2.1 and Remark 2.1 the two products have limits 0 as $s \to 1^+$. If both products extend analytically to $s = 1$ then the order of zero for the function is at least 2, contradicting known behaviour of $\zeta(s)$. □

Remark 2.4. *The Dedekind Zeta function of $K = Q(\sqrt{d})$ has a factorization* ($Re\ s > 1$)

$$\zeta_K(s) = \zeta(s)L_d(s)$$

where ζ is the Riemann Zeta function and L_d is the L- series ([3]). On the other hand

$$\zeta_K(s) = \prod_{(\frac{d}{p})=+1}(1-\frac{1}{p^s})^{-2}\prod_{(\frac{d}{p})=-1}(1-\frac{1}{p^{2s}})^{-1}\prod_{p|d}(1-\frac{1}{p^s})^{-1}$$
$$= \prod_{(\frac{d}{p})=+1}(1-\frac{1}{p^s})^{-2}f(s)$$

with f analytic for Re $s > \frac{1}{2}$. By the above ζ_K has a simple pole at $s = 1$ coming from the pole of ζ. Thus if there is a zero at $s = 1$ for the "analytic" function

$$\prod_{(\frac{d}{p})=+1} (1 - \frac{1}{p^s})$$

then ζ_K has a double pole at $s = 1$, contradicting the above. Hence $\prod_{(\frac{d}{p})=+1}(1-\frac{1}{p^s})$ is not analytic at $s = 1$.

Now if $\prod_{(\frac{d}{p})=-1,0}(1 - \frac{1}{p^s})$ is analytic at $s = 1$ then in Cor 2.3 by cross multiplication, we have the function $\prod_{(\frac{d}{p})=+1}(1 - \frac{1}{p^s})$ to be meromorphic at $s = 1$. But this is not the case, by above, there is no pole at s=1 and it is not analytic at s=1.

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