On Fully-Convex Harmonic Functions and their Extension

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ABSTRACT: Uniformly convex univalent functions that introduced by Goodman, maps every circular arc contained in the open unit disk with center in it into a convex curve. On the other hand, a fully-convex harmonic function, maps each subdisk \(|z| = r < 1\) onto a convex curve. Here we synthesis these two ideas and introduce a family of univalent harmonic functions which are fully-convex and uniformly convex also. In the following we will mention some examples of this subclass and obtain a necessary and sufficient conditions and finally a coefficient condition is given as an application of some convolution results.

Key Words: Uniformly convex function, Fully-Convex function, Harmonic function, Convolution.

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1. Introduction and Preliminaries

Let \(\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}\) be the open unit disk in complex plane. Let \(A\) be the familiar class of all analytic functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

in the open unit disk \(\mathbb{D}\). Let \(S\) denotes the family of all functions \(f(z)\) of the form (1.1) that are univalent in \(\mathbb{D}\) and normalized with \(f(0) = 0\) and \(f'(0) = 1\).

A conformal function \(f(z)\) is said to be starlike if every point of its range can be connected to the origin by a radial line that lies entirely in that region. The class of all starlike functions in \(S\) is shown by \(S^*\) [9] and \(f(z) \in S^*\) if and only if \(\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > 0\). Starlikeness is a hereditary property for conformal mappings, so if \(f(z) \in S\), and if \(f\) maps \(\mathbb{D}\) onto a domain that is starlike with respect to the origin, then the image of every subdisk \(|z| < r < 1\) is also starlike with respect to the origin.

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An analytic function \( f(z) \) is said to be convex if its range \( f(\mathbb{D}) \) is a convex set. It has shown that every convex function \( f \) in \( \mathbb{S} \) satisfy following analytic property

\[
\text{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > 0
\]

The class of all convex functions in \( \mathbb{S} \) is denoted by \( \mathcal{K} \) \[9\].

The subclass of uniformly starlike functions, \( \mathcal{UST} \) introduced by Goodman \[6\] and studied in analytic and geometric view.

**Definition 1.1.** \[6\] A function \( f(z) \in \mathcal{S}^* \) is said to be uniformly starlike in \( \mathbb{D} \) if it has the property that for every circular arc \( \gamma \) contained in \( \mathbb{D} \), with center \( \zeta \in \mathbb{D} \), the arc \( f(\gamma) \) be starlike with respect to \( f(\zeta) \). We denote the family of all uniformly starlike functions by \( \mathcal{UST} \) and we have,

\[
\mathcal{UST} = \left\{ f(z) \in \mathbb{S} : \text{Re} \left\{ \frac{(z - \zeta)f'(z)}{f(z) - f(\zeta)} > 0 \right\}, (z, \zeta) \in \mathbb{D}^2 \right\}
\]

(1.2)

It’s clear that \( \mathcal{UST} \subset \mathcal{S}^* \) and every function in \( \mathcal{UST} \) maps each subdisk \( \{ |z - \zeta| < \rho \} \subset \mathbb{D} \) onto a domain starlike with respect to \( f(\zeta) \). Goodman \[5\] also defined the subclass of convex functions with this property that for every \( f(\zeta) \) contained in \( \mathbb{D} \), with center \( \zeta \in \mathbb{D} \), the arc \( f(\gamma) \) be a convex arc. We have,

\[
\mathcal{UCV} = \left\{ f(z) \in \mathbb{S} : \text{Re} \left\{ 1 + (z - \zeta)\frac{f''(z)}{f'(z)} \right\} \geq 0 \right\}, (z, \zeta) \in \mathbb{D}^2
\]

(1.3)

A summary of early works on uniformly starlike and uniformly convex functions can be found in \[10\].

The complex-valued function \( f(x, y) = u(x, y) + iv(x, y) \) is complex-valued harmonic function in \( \mathbb{D} \) if \( f \) is continuous and \( u \) and \( v \) are real harmonic in \( \mathbb{D} \). We denote \( H \) the family of continuous complex-valued functions which are harmonic in the open unit disk \( \mathbb{D} \). In simply-connected domain \( \mathbb{D} \), \( f \in H \) has a canonical representation \( f = h + g \), where \( h \) and \( g \) analytic in \( \mathbb{D} \) \[3,4\]. Then, \( g \) and \( h \) have expansions in Taylor series as \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), so we may represent \( f \) by a power series of the form

\[
f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n
\]

(1.4)

The Jacobian of a function \( f = u + iv \) is \( J_f(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = |h'(z)|^2 - |g'(z)|^2 \), and \( f(z) = h(z) + g(z) \) is sense-preserving if \( J_f(z) > 0 \). In 1984, Clunie and Sheil-Small
investigated the class \(S_H\), consisting of sense-preserving univalent harmonic functions \(f(z) = h(z) + g(z)\) in simply-connected domain \(D\) which normalized by \(f(0) = 0\) and \(f_z(0) = 1\) with the form,

\[
f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n
t(1.5)
\]

The subclass \(S_0^H\) of \(S_H\) includes all functions \(f \in S_H\) with \(f_z(0) = 0\), so \(S \subset S_0^H \subset S_H\). Clunie and Sheil-Small also considered convex functions in \(S_H\), denoted by \(K_H\). The hereditary property of convexity for conformal maps does not generalize to univalent harmonic mappings. If \(f\) is a univalent harmonic map of \(D\) onto a convex domain, then the image of the disk \(|z| < r\) is convex for each radius \(r \leq \sqrt{2} - 1\), but not necessarily for any radius in the interval \(\sqrt{2} - 1 < r < 1\). In fact, the function

\[
f(z) = \text{Re} z + i\text{Im} \frac{z}{(1 - z)^2}
\]

(1.6)

is a harmonic mapping of the disk \(D\) onto the half-plane \(\text{Re } w > -\frac{1}{2}\), but the image of the disk \(|z| \leq r\) fails to be convex for every \(r\) in the interval \(\sqrt{2} - 1 < r < 1\) [4]. Thus we need a property to explain convexity of a map in a hereditary form in whole disk. We have following definition.

**Definition 1.3.** [2] A harmonic mapping \(f\) with \(f(0) = 0\) of the unit disk is said to be fully-convex if it maps every circle \(|z| = r < 1\) in a one-to-one manner onto a convex curve.

For \(f \in S_H\), the family of fully-convex harmonic functions denotes by \(\mathcal{F}_H\). In 1980 Mocanu gave a relation between fully-starlikeness and a differential operator of a non-analytic function [7]. Let

\[
Df = zf_z - \overline{zf_z}
\]

(1.7)

be the differential operator and

\[
D^2f = D(Df) = zzf_{zz} + \overline{zf_z} + zf_z + \overline{zf_z}
\]

(1.8)

**Lemma 1.4.** [7] Let \(f \in C^4(D)\) is a complex-valued function such that \(f(0) = 0\), \(f(z) \neq 0\) for all \(z \in D - \{0\}\), and \(J_f(z) > 0\) in \(D\) and \(\text{Re} \frac{Df(z)}{f(z)} > 0\) then \(f\) is univalent and fully-starlike in \(D\).

**Lemma 1.5.** Let \(f \in C^4(D)\) is a complex-valued function such that \(f(0) = 0\), \(f(z) \neq 0\) for all \(z \in D - \{0\}\), and \(J_f(z) > 0\) in \(D\) and \(\text{Re} \frac{D^2f(z)}{Df(z)} > 0\) then \(f\) is univalent and fully-convex in \(D\).
Since for a sense-preserving complex-valued function \( f(z) \), \( Df \neq 0 \), if \( f(z) \in \mathcal{S}_H \) and satisfies condition such as \( \text{Re} \frac{Df(z)}{f(z)} > 0 \) or \( \text{Re} \frac{D^2 f(z)}{Df(z)} > 0 \) for all \( z \in \mathbb{D} - \{0\} \), then \( f \) maps every circle \( 0 < |z| = r < 1 \) onto a simple closed curve \([7]\). However, a fully-starlike mapping need not be univalent \([2]\), we restrict our discussion to \( \mathcal{S}_H \).

2. Definition and Examples

For a harmonic function \( f(z) = h(z) + g(z) \in \mathcal{S}_H \), and \( \zeta \in \mathbb{D} \) we define the operator

\[
Df(z, \zeta) = (z - \zeta)f_z(z) - (z - \zeta)g_z(z)
\]

is harmonic also. For \( \zeta = 0 \) the operator \( Df(z, 0) = zf_z(z) - zg_z(z) = Df(z) \) is previous operator \((1.7)\). Differentiating of the operator \( Df(z, \zeta) \) gives us

\[
D^2 f(z, \zeta) = D(Df(z, \zeta)) = D((z - \zeta)h_z(z) - (z - \zeta)g_z(z)) = (z - \zeta)h''(z) + (z - \zeta)^2 h'(z) + (z - \zeta)h'(z) + (z - \zeta)g'(z)
\]

For \( \zeta = 0 \) the operator \( D^2 f(z, 0) = z^2 h''(z) + z^2 g''(z) + zh'(z) + zg'(z) = D^2 f(z) \) has described by Al-Amiri and Mocanu \([1]\). Similar to definition \((1.1)\) we say that for an arbitrary function:

**Definition 2.1.** A function \( f \in \mathcal{S}_H \) is said to be uniformly fully-convex harmonic function in \( \mathbb{D} \) if it has the property that for every circular arc \( \gamma \) contained in \( \mathbb{D} \), with center \( \zeta \in \mathbb{D} \), the arc \( f(\gamma) \) is convex in \( f(\mathbb{D}) \).

We denote the set of all uniformly fully-convex harmonic functions in \( \mathbb{D} \) by \( \mathcal{UFK}_H \). The following theorem gives analytic equivalency for above definition:

**Theorem 2.2.** Let \( f \in \mathcal{S}_H \). \( f \in \mathcal{UFK}_H \) if and only if

\[
\text{Re} \frac{D^2 f(z, \zeta)}{Df(z, \zeta)} > 0 , \ (z, \zeta) \in \mathbb{D}^2
\]

**Proof:** Let \( \gamma : \zeta + re^{i\theta} \) with \( \theta_1 \leq \theta \leq \theta_2 \) be a circular arc centered at \( \zeta \) and contained in \( \mathbb{D} \), then the image of \( \gamma \) under \( f \) is convex if the argument of the tangent to the image be a non-decreasing function of \( \theta \), that is,

\[
\frac{\partial}{\partial \theta} \left( \text{arg} \frac{\partial}{\partial \theta} \{f(z) - f(\zeta)\} \right) \geq 0
\]

Hence

\[
\text{Im} \frac{\partial}{\partial \theta} \left( \log \frac{\partial}{\partial \theta} \{f(z) - f(\zeta)\} \right) \geq 0
\]
But for a circular arc $\gamma$, set $z = \zeta + re^{i\theta}$, then $\frac{\partial}{\partial \theta} z = i(z - \zeta)$ and a brief computation will give us

$$\frac{\partial}{\partial \theta} \{ f(z) - f(\zeta) \} = i \left\{ (z - \zeta) f_z(z) - \overline{(z - \zeta) f_{\overline{z}}(z)} \right\} = i D f(z, \zeta)$$

then

$$\frac{\partial}{\partial \theta} \log i D f(z, \zeta) = \frac{\partial}{\partial \theta} \log i \left\{ (z - \zeta) h'(z) - \overline{(z - \zeta) g'(z)} \right\}$$

$$= \frac{i h'(z) + (z - \zeta) h''(z)}{i D f(z, \zeta)} i(z - \zeta)$$

$$= \frac{i D^2 f(z, \zeta)}{D f(z, \zeta)}$$

Therefore, we must have

$$\text{Im} \frac{\partial}{\partial \theta} \log i D f(z, \zeta) = \text{Re} \frac{D^2 f(z, \zeta)}{D f(z, \zeta)} \geq 0$$

as we want. \qed

It should be noted that $\frac{D^2 f(z, \zeta)}{D f(z, \zeta)}(0, 0) = 1$, and

$$\mathcal{UF}_{H} = \left\{ f(z) \in \mathcal{S}_H : \text{Re} \frac{D^2 f(z, \zeta)}{D f(z, \zeta)} > 0, (z, \zeta) \in \mathbb{D}^2 \right\}$$

It’s simple that one checks the rotations, $e^{-i\alpha} f(e^{i\alpha} z)$ for some real $\alpha$, are preserve the class $\mathcal{UF}_{H}$ and the transformation $1/t f(t z)$ preserves this class also, where $0 < t \leq 1$. On the other hand, the class $\mathcal{UF}_{H}$ includes all fully-convex functions and uniformly convex functions. With $g = 0$ in (2.3), the analytic function $f(z) \in \mathcal{UF}_{H}$ by (2.1) and (2.2) satisfies condition

$$\text{Re} \frac{D^2 f(z, \zeta)}{D f(z, \zeta)} = \text{Re} \frac{(z - \zeta) h''(z) + (z - \zeta) h'(z)}{(z - \zeta) h'(z)} = \text{Re} \left( 1 + (z - \zeta) \frac{h''(z)}{h'(z)} \right) \geq 0$$

where $(z, \zeta) \in \mathbb{D}^2$. Then

**Corollary 2.3.** If $f \in \mathcal{UCV}$ be an analytic function, then $f \in \mathcal{UF}_{H}$. So, $\mathcal{UCV} \subset \mathcal{UF}_{H} \subset \mathcal{K}_{H}$. Goodman [5] shows the analytic function $f(z) = \frac{z}{1 - Az} \in \mathcal{UCV}$ if and only if $|A| \leq \frac{1}{3}$, thus the convex function $f(z) = \frac{z}{1 - z} \notin \mathcal{UF}_{H}$. 

Example 2.1. For $|\beta| < 1$ the affine mappings $f(z) = z + \overline{\beta}z \in \mathcal{UF}_K$, since

$$\text{Re}\left(\frac{(z - \zeta) + (z - \zeta)\beta}{(z - \zeta) - (z - \zeta)\beta}\right)\geq 0$$

is equivalent to

$$\text{Re}\left((z - \zeta) + (z - \zeta)\beta\right)\left((z - \zeta) - (z - \zeta)\beta\right)\geq 0$$

that is $(1 - |\beta|^2)|z - \zeta|^2 \geq 0$.

Corollary 2.4. For $\zeta = 0$ in (2.3), the harmonic function $f \in \mathcal{UF}_K$ will be univalent and fully-convex in $\mathbb{D}$ by Lemma 1.5. Thus it’s clear any non fully-convex harmonic function is not in $\mathcal{UF}_K$. The harmonic function $f(z) = \text{Re}\left(\frac{z}{1 - z} + i\text{Im}\frac{z}{(1 - z)^2}\right)$ isn’t fully-convex ([4], p.46), then $f \notin \mathcal{UF}_K$.

In the following we will give a necessary and sufficient condition for that $f \in \mathcal{UF}_K$. This condition is a generalization form of a theorem about fully-convex functions mentioned by Chuaqui et al. in [2], p.139.

Theorem 2.5. Let $f(z) \in \mathcal{S}_H$, $f \in \mathcal{UF}_K$ if and only if

$$|(z - \zeta)h'(z)|^2\text{Re}Q_h \geq$$

$$|\text{Im}f(z)|^2\text{Re}Q_g + \text{Re}\left\{(z - \zeta)^3h''(z)\overline{h'(z)} - h'(z)g''(z)\right\}$$

where $Q_h = 1 + (z - \zeta)\frac{h''(z)}{h'(z)}$ and $Q_g = 1 + (z - \zeta)\frac{g''(z)}{g'(z)}$ for $(z, \zeta)$ in polydisk $\mathbb{D}^2$.

Proof: According to the definition, $f \in \mathcal{UF}_K$ if and only if $\text{Re}\frac{D^2f(z, \zeta)}{Df(z, \zeta)} > 0$ for $(z, \zeta) \in \mathbb{D}^2$, and if and only if $\text{Re}\left\{D^2f(z, \zeta)\overline{Df(z, \zeta)}\right\} > 0$ for $(z, \zeta) \in \mathbb{D}^2$, then a simple calculation gives us (2.5). □

Lemma 2.6. $f = h + \overline{\beta}h \in \mathcal{UF}_K$ if and only if $h \in \mathcal{UCE}$, where $|\beta| < 1$. □

Proof: Let $f = h + \overline{\beta}h \in \mathcal{S}_H$ and $g = \beta h$ with $|\beta| < 1$, then $f \in \mathcal{UF}_K$ if and only if (2.5) holds. Since in this case, $h$ and $g$ satisfy equality $Q_h = Q_g$ so (2.5) holds if and only if $|(z - \zeta)h'(z)|^2\text{Re}Q_h(1 - |\beta|^2) \geq 0$, or $\text{Re}Q_h \geq 0$ that shows $h \in \mathcal{UCE}$. □

Example 2.2. The analytic function $h = z + Az^2$ is in $\mathcal{UCE}$ if and only if $|A| \leq \frac{1}{6}$ [5]. By Lemma 2.6 we get $f(z) = z + Az^2 + \overline{\beta}z + \beta Az^2 \in \mathcal{UF}_K$ with $|\beta| < 1$ and $|A| \leq \frac{1}{6}$.

For example, let $A = \frac{1}{6}$, $\beta = -\frac{i}{2}$ then $f = z + \frac{1}{6}z^2 - \frac{1}{2}z - \frac{i}{12}z^2 \in \mathcal{UF}_K$.

In Figure 1, the disk $|z - 0.7| < 0.3$ is mapped under this uniformly fully-convex harmonic function to a convex elliptical shape with center $f(\zeta) = (0.78, 0.39)$.
3. Convolution and a sufficient condition

The convolution or Hadamard product of two harmonic functions $f(z)$ and $F(z)$ with canonical representations

$$f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$$

and

$$F(z) = H(z) + G(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n$$

is defined as

$$(f * F)(z) = (h * H)(z) + (g * G)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n z^n$$

The right half-plane mapping $\ell(z) = \frac{z}{1 - z}$ acts as the convolution identity and the Koebe map $k(z) = \frac{z}{(1 - z)^2}$ acts as derivative operation over functions convolution.

We have some properties for convolution over analytic functions $f$ and $g$:

$$f * g = g * f \quad , \quad \alpha(f * g) = \alpha f * g$$

$$f * \ell = f \quad , \quad zf'(z) = f * k(z)$$

where $\alpha \in \mathbb{C}$. For a given subset $\mathcal{V} \subset \mathcal{A}$, its dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{f * g(z)}{z} \neq 0, \quad \forall f \in \mathcal{V}, \quad \forall z \in \mathbb{D} \right\}$$
Nezhmetdinov (1997) proved that class $\mathcal{UCV}$ is dual set for certain family of functions from $\mathcal{A}$. He proved ([8], Theorem 2, p.43) that the class $\mathcal{UCV}$ is the dual set of a subset of $\mathcal{A}$ consisting of functions $\varphi : \mathbb{D} \to \mathbb{C}$ given by

$$\varphi(z) = \frac{z}{(1 - z)^3} \left[ 1 - z - \frac{4z}{(\alpha + i)^2} \right]$$

(3.5)

where $\alpha \in \mathbb{R}$. He determined the uniform estimate $|a_n(\varphi)| \leq n(2n - 1)$ for the $n$-th Taylor coefficient of $\varphi(z)$:

**Lemma 3.1.** [8] Let $G$ is all function $\varphi \in \mathcal{A}$ of the form (3.5), then $\mathcal{UCV} = G^*$ and $|a_n(\varphi)| \leq n(2n - 1)$ for all $n \geq 2$.

For obtaining a sufficient condition in class $\mathcal{UFK}_H$, we define the dual set of a harmonic function. Let $\mathcal{A}_H$ be the class of complex-valued harmonic functions $f(z) = h(z) + g(z)$ in simply connected domain $\mathbb{D}$ of the form (1.5) which are not necessarily sense-preserving univalent on $\mathbb{D}$. We define the dual set of a subset of $\mathcal{A}_H$:

**Definition 3.2.** For a given subset $\mathcal{V}_H \subset \mathcal{A}_H$, the dual set $\mathcal{V}_H^*$ is

$$\mathcal{V}_H^* = \left\{ F = H + \overline{G} \in \mathcal{A}_H : \frac{h \ast H}{z} + \frac{g \ast G}{z} \neq 0, \forall f = h + \overline{g} \in \mathcal{V}_H, \forall z \in \mathbb{D} \right\} \quad (3.6)$$

**Theorem 3.3.** Let $\alpha \in \mathbb{R}$, $|w| = 1$ and

$$G_H = \left\{ \varphi - \sigma \overline{\varphi} : \varphi(z) = \frac{z}{(1 - z)^3} \left( 1 - \frac{w - i\alpha}{2 - w - i\alpha} z \right), \quad \sigma = \frac{(1 - w)(2 - w - i\alpha)}{(1 - w)(2 - w - i\alpha)}, z \in \mathbb{D} \right\}$$

then $\mathcal{UFK}_H = G_H^*$. Furthermore If $\sum_{n=2}^{\infty} n(2n-1) |a_n| + n(2n-1) |b_n| < 1 - |b_1|$ then $f \in \mathcal{UFK}_H$.

It’s clear that the analytic function $\varphi$ is the same (3.5), but $\sigma$ with $|\sigma| = 1$ isn’t an arbitrary number and depend on both $w$ and $\alpha$ in $\varphi$.

**Proof:** Let $f = h + \overline{g} \in \mathcal{UFK}_H$, that is

$$\Re \left( \frac{(z - \zeta)^2 h''(z) + (z - \zeta)^2 g''(z) + (z - \zeta) h'(z) + (z - \zeta) g'(z)}{(z - \zeta) h'(z) - (z - \zeta) g'(z)} \right) > 0 , \quad (3.7)$$

$(z, \zeta) \in \mathbb{D}^2$. For $\zeta = 0$ and then $z = 0$ we have $\frac{D^2 f(z, \zeta)}{D f(z, \zeta)} = 1$, hence the condition (3.7) may be write as

$$i\alpha \left( (z - \zeta) h'(z) - (z - \zeta) g'(z) \right) \neq (z - \zeta)^2 h''(z) + (z - \zeta)^2 g''(z) + (z - \zeta) h'(z) + (z - \zeta) g'(z)$$
where $\alpha \in \mathbb{R}$. By the minimum principle for harmonic functions, it is sufficient to verify this condition for $|z| = |\zeta|$ and so, we may assume that $\zeta = wz$ with $|w| = 1$, then from the definition of the dual set for harmonic functions (3.6), with straightforward calculation we conclude that $\frac{h*z}{z} + \frac{g*z}{z} \neq 0$, so the first assertion follows.

For obtaining coefficients condition, let $f(z) = h(z) + g(z)$ is of the form (3.1), and $\varphi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be the series expansion of analytic function $\varphi(z)$, then $|\phi_n| \leq n(2n - 1)$ for all $n \geq 2$, by Lemma 3.1. From previous part we see that

$$\left| \frac{h*z}{z} + \frac{g*z}{z} \right| = \left| 1 + \sum_{n=2}^{\infty} a_n \phi_n z^{n-1} + \sigma \left( b_1 + \sum_{n=2}^{\infty} b_n \phi_n z^{n-1} \right) \right|$$

$$\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} |a_n||\phi_n||z|^{n-1} - |\sigma| \sum_{n=2}^{\infty} |b_n||\phi_n||z|^{n-1}$$

$$\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} n(2n - 1)|a_n| - \sum_{n=2}^{\infty} n(2n - 1)|b_n|$$

$$> 0$$

when $\sum_{n=2}^{\infty} n(2n - 1)|a_n| + n(2n - 1)|b_n| < 1 - |b_1|$. \hfill \Box

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