One Sided Generalized $(\sigma, \tau)$-derivations on Rings

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ABSTRACT: Let $R$ be a prime ring with characteristic not 2 and $\sigma, \tau, \lambda, \mu, \alpha, \beta$ be automorphisms of $R$. Let $h$ be a nonzero left (resp. right)-generalized $(\sigma, \tau)$-derivation of $R$ and $I, J$ nonzero ideals of $R$ and $a \in R$. The main object in this article is to study the situations. (1) $h(I)a \subset C_{\lambda, \mu}(J)$ and $ah(I) \subset C_{\lambda, \mu}(J)$; (2) $h(I)C_{\lambda, \mu}(J)$, (3) $[h(I), a]_{\lambda, \mu} = 0$, (4) $h(I, a)_{\lambda, \mu} = 0$ (or $(h(I), a)_{\lambda, \mu} = 0$), (5) $[h(x), x]_{\lambda, \mu} = 0, \forall x \in I$, (6) $[h(x, a), x]_{\lambda, \tau} = 0, \forall x \in I$.

Key Words: $(\sigma, \tau)$—Lie ideal, Prime ring, Commutativity.

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1. Introduction

Let $R$ be an associative ring with center $Z$. Recall that $R$ is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $(x, y) = xy + yx$. Let $\sigma$ and $\tau$ be any two endomorphisms of $R$. For any $x, y \in R$ we set $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$ and $(x, y)_{\sigma, \tau} = x\sigma(y) + \tau(y)x$. Let $h$ and $d$ be additive mappings of $R$. If $d(xy) = d(x)y + xd(y), \forall x, y \in R$ then $d$ is called a derivation of $R$. If there exists a derivation $d$ such that $h(xy) = h(x)y + xd(y), \forall x, y \in R$ then $h$ is called generalized derivation of $R$ (see [3]). If $d(xy) = d(x)\sigma(y) + \tau(x)d(y), \forall x, y \in R$ then $d$ is called a $(\sigma, \tau)$—derivation of $R$. Obviously every derivation $d : R \rightarrow R$ is a $(1, 1)$—derivation of $R$, where $1 : R \rightarrow R$ is an identity mapping. If $h(xy) = h(x)\sigma(y) + \tau(x)h(y), \forall x, y \in R$ then $h$ is said to be a left-generalized $(\sigma, \tau)$—derivation with $d$ and if $h(xy) = h(x)\sigma(y) + \tau(x)d(y), \forall x, y \in R$ then $h$ is said to be a right-generalized $(\sigma, \tau)$—derivation associated with $(\sigma, \tau)$—derivation $d$, (see [4]). Every $(\sigma, \tau)$—derivation associated with $d$ is a right (and left)-generalized $(\sigma, \tau)$—derivation associated with $d$.

The mapping defined by $h(r) = [r, a]_{\sigma, \tau}, \forall r \in R$ is a right-generalized derivation associated with derivation $d(r) = [r, \sigma(a)], \forall r \in R$ and left-generalized derivation associated with derivation $d_1(r) = [r, \sigma(a)], \forall r \in R$. The mapping $h(r) = (a, r)_{\sigma, \tau}, \forall r \in R$ is a left-generalized $(\sigma, \tau)$—derivation associated with $(\sigma, \tau)$—derivation $d_2(r) = [a, r]_{\sigma, \tau}, \forall r \in R$ and right-generalized $(\sigma, \tau)$—derivation associated with $(\sigma, \tau)$—derivation $d_2$. 

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The following result is proved by Posner in (see [12]). Let $R$ be a prime ring and $d \neq 0$ derivation of $R$ such that $[d(x), x] = 0, \forall x \in R$. Then $R$ is commutative. Ashraf and Rehman (see [1]) generalized Posner’s result as follows. Let $R$ be a 2-torsion free prime ring. Suppose there exists a $(\sigma, \tau)$-derivation $d : R \rightarrow R$ such that $[d(x), x]_{\sigma, \tau} = 0, \forall x \in R$. Then either $d = 0$ or $R$ is commutative. Taking an ideal of $R$ instead of $R$, Marubayashi H. and Ashraf M., Rehman N., Ali Shakir, generalized Rehman’s result in (see [10]). On the other hand, Rehman (see [13]) gave another generalization of Posner’s Theorem as follows. Let $R$ be a prime ring. If $R$ admits a nonzero generalized derivation $h$ with $d$ such that $[h(x), x] = 0, \forall x \in R$, and if $d \neq 0$, then $R$ is commutative.

In this paper, using left-generalized $(\sigma, \tau)$-derivation of $R$, we have given another generalization of Ashraf and Rehman’s result (see [1]) as in Theorem 3. Also, we discuss the commutativity of prime rings admitting a left-generalized $(\sigma, \tau)$-derivation $h : R \rightarrow R$ satisfying several conditions on ideals.

Throughout the paper, $R$ will be a prime ring with characteristic not 2 and $\sigma, \tau, \lambda, \mu, \alpha, \beta$ be automorphisms of $R$. Let $J$ be an ideal of $R$ and $U$ be a prime ring admitting a nonzero derivation associated with a nonzero derivation $d : R \rightarrow R$.

We begin with the following known results which will be used to prove our theorems.

Lemma 2.1. [2, Lemma 1] Let $R$ be a prime ring and $d : R \rightarrow R$ be a $(\sigma, \tau)$-derivation. If $U$ is a nonzero right ideal of $R$ and $d(U) = 0$ then $d = 0$.

Lemma 2.2. [11, Lemma 2] If a prime ring contains a nonzero commutative right ideal then it is commutative.

Lemma 2.3. [6, Lemma 5] Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $[a, I]_{\alpha, \beta} \subset C_{\lambda, \mu}(R)$ or $(a, I)_{\alpha, \beta} \subset C_{\lambda, \mu}(R)$ then $a \in C_{\alpha, \beta}(R)$ or $R$ is commutative.

Lemma 2.4. [5, Corollary 1] If $I$ is a nonzero ideal of $R$ and $a \in R$ such that $[I, a]_{\alpha, \beta} \subset C_{\lambda, \mu}(R)$, then $a \in Z$.

Lemma 2.5. [7, Lemma 2.16] Let $R$ be a prime ring and $h : R \rightarrow R$ be a nonzero left-generalized $(\sigma, \tau)$-derivation associated with a nonzero $(\sigma, \tau)$-derivation $d$. If $I$ is a nonzero ideal of $R$ and $a \in R$ such that $(h(I), a)_{\lambda, \mu} = 0$ then $a \in Z$ or $d\tau^{-1}\mu(a) = 0$.

Lemma 2.6. [7, Theorem 2.7] Let $h : R \rightarrow R$ be a nonzero right-generalized $(\sigma, \tau)$-derivation associated with $(\sigma, \tau)$-derivation $d$ and $I, J$ be nonzero ideals of $R$. If $a \in R$ such that $ah(I) \subset C_{\lambda, \mu}(J)$ then $a \in Z$ or $d = 0$. 

2. Results

We begin with the following known results which will be used to prove our theorems.
Lemma 2.7. Let I be a nonzero ideal of R and a, b ∈ R. If h : R → R is a nonzero left-generalized (σ, τ)−derivation associated with (σ, τ)−derivation d such that [h(I)a, b] _λ, µ = 0 then a[a, λ(b)] = 0 or d(τ −1 µ(b)) = 0.

Proof. Using hypothesis we have,

\[0 = [h(τ⁻¹ µ(b)x)a, b] _λ, µ = [d(τ⁻¹ µ(b))σ(x)a + μ(b)h(x)a, b] _λ, µ = d(τ⁻¹ µ(b))[σ(x)a, λ(b)] + [d(τ⁻¹ µ(b)), b] _λ, µ σ(x)a + μ(b)[h(x)a, b] _λ, µ + [μ(b), h(x)a]
\]

That is

\[k[σ(x)a, λ(b)] + [k, b] _λ, µ σ(x)a = 0, ∀x ∈ I \text{ where } k = d(τ⁻¹ µ(b)). \tag{2.1}
\]

Replacing x by xσ⁻¹(a)y in (1) and using (1) we get,

\[0 = k[σ(x)aσ(y)a, λ(b)] + [k, b] _λ, µ σ(x)aσ(y)a = k[σ(x)aσ(y)a, λ(b)] + k[σ(x)a, λ(b)]σ(y)a + [k, b] _λ, µ σ(x)aσ(y)a = k[σ(x)aσ(y)a, λ(b)], ∀x, y ∈ I.
\]

That is kσ(I)a[σ(I)a, λ(b)] = 0. Since σ(I) is a nonzero ideal of R then we have

\[d(τ⁻¹ µ(b)) = 0 \text{ or } a[σ(I)a, λ(b)] = 0. \tag{2.2}
\]

If a[σ(I)a, λ(b)] = 0 in (2) then we get,

\[0 = a[σ(σ⁻¹(a)x)a, λ(b)] = a[aσ(x)a, λ(b)] = aa[σ(x)a, λ(b)] + a[a, λ(b)]σ(x)a = a[a, λ(b)]σ(x)a, ∀x ∈ I.
\]

From the last relation we obtain that a[a, λ(b)] = 0 for two case. \(\square\)

Remark 2.8. Let J be a nonzero ideal of R. If b ∈ C _λ, µ(J) then b ∈ C _λ, µ(R).

Proof. If b ∈ C _λ, µ(J) then we have 0 = [b, xr] _λ, µ = μ(x)[b, r] _λ, µ + [b, x] _λ, µ λ(r) = μ(x)[b, r] _λ, µ, ∀x ∈ J, r ∈ R. That is μ(J)J, R] _λ, µ = 0. This gives that b ∈ C _λ, µ(R). \(\square\)

Theorem 2.9. Let h : R → R be a nonzero left-generalized (σ, τ)−derivation associated with nonzero (σ, τ)−derivation d and a, b ∈ R. Let I, J be nonzero ideals of R.

(i) If h(I)a ⊆ C _λ, µ(J) then a ∈ Z ,

(ii) If ah(I) ⊆ C _λ, µ(J) then a ∈ Z or adτ⁻¹(a) = 0.
Proof. (i) If \( h(I)a \subset C_{\lambda,\mu}(J) \) then we have \([h(I)a, x]_{\lambda,\mu} = 0, \forall x \in J\). Using this relation and Lemma 7 we get, for any \( a \in J \),
\[
a[a, \lambda(x)] = 0 \text{ or } d\tau^{-1}\mu(x) = 0
\]

Let \( K = \{ x \in J \mid a[a, \lambda(x)] = 0 \} \) and \( L = \{ x \in J \mid d\tau^{-1}\mu(x) = 0 \} \). Then \( K \) and \( L \) are subgroups of \( J \) and \( J = K \cup L \). A group can not write the union of its proper subgroups. Hence we have \( K = J \) or \( L = J \). That is,
\[
a[a, \lambda(J)] = 0 \text{ or } d(\tau^{-1}\mu(J)) = 0
\]

Since \( d \neq 0 \) then \( d(\tau^{-1}\mu(J)) \neq 0 \) by Lemma 1. If \( a[a, \lambda(J)] = 0 \) then we get
\[
0 = a[a, \lambda(\sigma(x))] = a\lambda(x)[a, \lambda(\sigma(r))] + a[a, \lambda(x)]\lambda(r)
= a\lambda(x)[a, \lambda(r)], \forall x \in J, r \in R
\]

and so \( a\lambda(J)[a, R] = 0 \). From this relation we obtain that \( a \in Z \).

(ii) If \( ah(I) \subset C_{\lambda,\mu}(J) \) then we have \( ah(I) \subset C_{\lambda,\mu}(R) \) by Remark 1. Using this relation we get
\[
0 = [ah(\tau^{-1}(a)y), \mu^{-1}(a)]_{\lambda,\mu} = [ad(\tau^{-1}(a))\sigma(y) + aah(y), \mu^{-1}(a)]_{\lambda,\mu}
= ad(\tau^{-1}(a))\sigma(y, \lambda\mu^{-1}(a)) + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y)
+ a[ah(y), \mu^{-1}(a)]_{\lambda,\mu} + [a, a]ah(y)
= ad(\tau^{-1}(a))\sigma(y, \lambda\mu^{-1}(a)) + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y), \forall y \in I,
\]

and so
\[
k[\sigma(y), p] + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y) = 0, \forall y \in I, \text{ where } k = ad(\tau^{-1}(a)) \text{ and } p = \lambda\mu^{-1}(a).
\] (2.3)
Replacing \( y \) by \( yx, x \in I \) in (3) we obtain that
\[
0 = k\sigma(y)[\sigma(x), p] + k[\sigma(y), p]\sigma(x) + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y)\sigma(x)
= k\sigma(y)[\sigma(x), p], \forall x, y \in I.
\]
That is,
\[
k\sigma(I)[\sigma(I), p] = 0 \quad \text{(2.4)}
\]

Since \( \sigma(I) \) is a nonzero ideal of \( R \) then \( k = 0 \) or \([\sigma(I), p] = 0 \) is obtained by the (4). This gives that \( ad(\tau^{-1}(a)) = 0 \) or \( a \in Z \). \( \square \)

Corollary 2.10. Let \( I, J \) be nonzero ideals of \( R \) and \( a, b \in R \).

(i) If \([I, b]_{\sigma,\tau}a \subset C_{\lambda,\mu}(J) \) then \( a \in Z \) or \( b \in Z \).
(ii) If \([b, I]_{\sigma,\tau}a \subset C_{\lambda,\mu}(J) \) then \( a \in Z \) or \( b \in C_{\sigma,\tau}(R) \).
(iii) If \( a(b, I)_{\sigma,\tau} \subset C_{\lambda,\mu}(J) \) then \( a \in Z \) or \( b \in C_{\sigma,\tau}(R) \) or \( a[b, \tau^{-1}(a)]_{\sigma,\tau} = 0 \).
Proof. (i) Let \( h(r) = [r, b]_{\sigma, \tau}, \forall r \in R \) and \( d(r) = [r, \tau(b)], \forall r \in R \). Since,

\[
    h(rs) = [rs, b]_{\sigma, \tau} = [r, \sigma(b)] + [r, \tau(b)]s = d(rs) + rh(s), \forall r, s \in R, \tag{2.5}
\]

then \( h \) is a left-generalized derivation associated with derivation \( d \). If \( h = 0 \) then \( d = 0 \) (and so \( b \in Z \)) is obtained by the relation (5).

If \([I, b]_{\sigma, \tau}a \in C_{\lambda, \mu}(J)\) then we can write \( h(I)a \subset C_{\lambda, \mu}(J) \). If \( h \neq 0 \) and \( d \neq 0 \) then we have \( a \in Z \) by Theorem 1(ii).

(ii) The mapping defined by \( d_1(r) = [b, r]_{\sigma, \tau}, \forall r \in R \) is a \((\sigma, \tau)\)-derivation and so, left (and right)-generalized \((\sigma, \tau)\)-derivation with \( d_1 \). If \( d_1 = 0 \) then we have \( b \in C_{\sigma, \tau}(R) \).

Let \( d_1 \neq 0 \). If \([b, I]_{\sigma, \tau}a \subset C_{\lambda, \mu}(J)\) then we can write \( d_1(I)a \subset C_{\lambda, \mu}(J) \). This gives that \( a \in Z \) by Theorem 1(i). Finally we obtain that \( a \in Z \) or \( b \in C_{\sigma, \tau}(R) \).

(iii) The mapping defined by \( g(r) = (b, r)_{\sigma, \tau}, \forall r \in R \) is a left-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d_2(r) = [b, r]_{\sigma, \tau}, \forall r \in R \). If \( g = 0 \) then \( d_1 = 0 \) and so \( b \in C_{\sigma, \tau}(R) \) is obtained. Let \( g \neq 0 \) and \( d_1 \neq 0 \). If \( a(b, I)_{\sigma, \tau} \subset C_{\lambda, \mu}(J) \) then we have \( ag(I) \subset C_{\lambda, \mu}(J) \). This implies that \( a \in Z \) or \( ad_1(\tau^{-1}(a)) = 0 \) by Theorem 1(ii). That is \( a \in Z \) or \( a[b, \tau^{-1}(a)]_{\sigma, \tau} = 0 \). \( \square \)

**Lemma 2.11.** Let \( I \) be a nonzero ideal of \( R \) and \( h : R \to R \) be a nonzero left-generalized \((\sigma, \tau)\)-derivation associated with a nonzero \((\sigma, \tau)\)-derivation \( d \). If \( a \in R \) such that \([h(I), a]_{\lambda, \mu} = 0 \) then \( a \in Z \) or \( d(\tau^{-1}(\mu(a))) = 0 \).

Proof. Using hypothesis we get,

\[
    0 = [h(\tau^{-1}(\mu(a)x), a]_{\lambda, \mu} = [d(\tau^{-1}(\mu(a)))\sigma(x) + \mu(a)h(x), a]_{\lambda, \mu}
    = d(\tau^{-1}(\mu(a)))\sigma(x), \lambda(a)] + [d(\tau^{-1}(\mu(a))), a]_{\lambda, \mu}\sigma(x)
    + \mu(a)[h(x), a]_{\lambda, \mu} + [\mu(a), \mu(a)]h(x)
    = d(\tau^{-1}(\mu(a)))\sigma(x), \lambda(a)] + [d(\tau^{-1}(\mu(a))), a]_{\lambda, \mu}\sigma(x), \forall x \in I.
\]

That is,

\[
k[\sigma(x), \lambda(a)] + [k, a]_{\lambda, \mu}\sigma(x) = 0, \forall x \in I, \quad \text{where } k = d(\tau^{-1}(\mu(a))). \tag{2.6}
\]

Replacing \( x \) by \( xr, r \in R \) in (6) and using (6) we get

\[
    0 = k\sigma(x)[\sigma(r), \lambda(a)] + k[\sigma(x), \lambda(a)]\sigma(r) + [k, a]_{\lambda, \mu}\sigma(x)\sigma(r)
    = k\sigma(x)[\sigma(r), \lambda(a)], \forall x \in I, r \in R.
\]

and so \( k\sigma(I)[R, \lambda(a)] = 0 \). Since \( \sigma(I) \neq 0 \) is an ideal and \( R \) is prime then we have \( a \in Z \) or \( d(\tau^{-1}(\mu(a))) = 0 \). \( \square \)

**Theorem 2.12.** Let \( h \) be a nonzero left-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d \neq 0 \) and \( I, J \) be nonzero ideals of \( R \).

(i) If \( h(I) \subset C_{\lambda, \mu}(J) \) then \( R \) is commutative.
(ii) If \([h(I), J]_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \) or \([h(I), J]_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \) then \( R \) is commutative.
(iii) If \([J, h(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \) then \( R \) is commutative.
Proof. (i) If \( h(I) \subset C_{\lambda,\mu}(J) \) then we have \([h(I), x]_{\lambda,\mu} = 0, \forall x \in J\). This means that, for any \( x \in J \),
\[
x \in Z \quad \text{or} \quad d(\tau^{-1}\mu(x)) = 0
\]  
(2.7)

by Lemma 8. Using (7), let us consider the following sets, \( K = \{ x \in J \mid x \in Z \} \) and \( L = \{ x \in J \mid d\tau^{-1}\mu(x) = 0 \} \). Considering as in the proof of Theorem 1 we obtain that \( J \subset Z \) or \( d(\tau^{-1}\mu(J)) = 0 \). Since \( d \neq 0 \) then we have \( d(\tau^{-1}\mu(J)) \neq 0 \) by Lemma 1. Hence, we obtain that \( K = J \) and so \( J \subset Z \). This means that \( R \) is commutative by Lemma 2.

(ii) If \([h(I), J]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)\) or \((h(I), J)_{\alpha,\beta} \subset C_{\lambda,\mu}(R)\) then we have \( h(I) \subset C_{\alpha,\beta}(R) \) or \( R \) is commutative by Lemma 3. On the other hand \( h(I) \subset C_{\alpha,\beta}(R) \) means that \( R \) is commutative by (i).

(iii) If \([J, h(I)]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)\) then we have \( h(I) \subset Z \) by Lemma 4 and so \( R \) is commutative by (i).

\[ \square \]

Corollary 2.13. [8, Lemma 2] Let \( U \) be a nonzero ideal of \( R \). If \( d : R \rightarrow R \) is a nonzero \((\sigma, \tau)\)–derivation such that \( d(U) \subset C_{\lambda,\mu}(R) \). Then \( R \) is commutative.

Theorem 2.14. Let \( h : R \rightarrow R \) be a nonzero left-generalized \((\sigma, \tau)\)–derivation associated with a nonzero \((\sigma, \tau)\)–derivation \( d \). If \( I \neq 0 \) is an ideal of \( R \) such that \([h(x), x]_{\lambda,\tau} = 0, \forall x \in I \) then \( R \) is commutative.

Proof. Linearizing the hypothesis, we get
\[
[h(x), y]_{\lambda,\tau} + [h(y), x]_{\lambda,\tau} = 0, \forall x, y \in I.
\]  
(2.8)

Replacing \( x \) by \( xy \) in (8) and using (8) we have
\[
0 = [h(yx), y]_{\lambda,\tau} + [h(y), yx]_{\lambda,\tau}
= [d(y)\sigma(x) + \tau(y)h(x), y]_{\lambda,\tau} + [h(y), yx]_{\lambda,\tau}
= d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x) + [\tau(y), h(x)]_{\lambda,\tau} + [h(y), y]_{\lambda,\tau}h(x)
\]
\[
+ [\tau(y), \sigma(x)]h(x) + [\tau(y), h(y)]_{\lambda,\tau} + [h(y), y]_{\lambda,\tau}h(x)
= d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x), \forall x, y \in I.
\]

That is
\[
d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x) = 0, \forall x, y \in I.
\]  
(2.9)

Taking \( xy, r \in R \) instead of \( x \) in (9) and using (9) then we arrive
\[
0 = d(y)[\sigma(x)[\sigma(r), \lambda(y)] + [d(y), x]_{\lambda,\tau}\sigma(r) + [d(y), y]_{\lambda,\tau}\sigma(x)\sigma(r)
= d(y)[\sigma(x)[\sigma(r), \lambda(y)], \forall x, y \in I, r \in R
\]

which leads to
\[
d(y)[\sigma(I), R, \lambda(y)] = 0, \forall y \in I.
\]  
(2.10)

Since \( \sigma(I) \neq 0 \) an ideal then, for any \( y \in I \), we have \([R, \lambda(y)] = 0 \) or \( d(y) = 0 \) by (10) and so \( y \in Z \) or \( d(y) = 0 \).
Let $K = \{y \in I \mid y \in Z\}$ and $L = \{y \in I \mid d(y) = 0\}$. Considering as in the proof of Theorem 1 we have, $I \subset Z$ or $d(I) = 0$. Since $I \neq 0$ an ideal and $d \neq 0$ then we obtain that $K = I$ by Lemma 1 and so $I \subset Z$. This means that $R$ is commutative by Lemma 2.

\[\square\]

**Corollary 2.15.** [1, Theorem 1] Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a nonzero $(\alpha, \beta)$–derivation $d$ such that $[d(x), x]_{\alpha, \beta} = 0, \forall x \in I$, then $R$ is commutative.

**Theorem 2.16.** Let $R$ be a prime ring and $0 \neq a \in R$. If $h : R \to R$ is a nonzero left-generalized $(\sigma, \tau)$–derivation associated with a nonzero $(\sigma, \tau)$–derivation $d$ and $I \neq 0$ an ideal of $R$ such that $[h(x)a, x]_{\lambda, \tau} = 0, \forall x \in I$ then $R$ is commutative.

**Proof.** Replacing $x$ by $x + y$ in hypothesis we have

$$[h(x)a, y]_{\lambda, \tau} + [h(y)a, x]_{\lambda, \tau} = 0, \forall x, y \in I. \quad (2.11)$$

If we take $yx$ instead of $x$ in (11) and using (11) we get

$$0 = [h(yx)a, y]_{\lambda, \tau} + [h(y)a, yx]_{\lambda, \tau}$$
$$= [d(y)\sigma(x)a + \tau(y)h(x)a, y]_{\lambda, \tau} + [h(y)a, yx]_{\lambda, \tau}$$
$$= d(y)[\sigma(x)a, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)a + \tau(y)[h(x)a, y]_{\lambda, \tau}$$
$$+ [\tau(y), \tau(y)]h(x)a + \tau(y)[h(y)a, x]_{\lambda, \tau} + [h(y)a, y]_{\lambda, \tau}\lambda(x), \forall x, y \in I.$$ 

That is

$$d(y)[\sigma(x)a, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)a = 0, \forall x, y \in I. \quad (2.12)$$

Replacing $x$ by $x\sigma^{-1}(a)$ in (12) and using (12) we have

$$0 = d(y)[\sigma(x)a, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)aa$$
$$= d(y)\sigma(x)a[a, \lambda(y)] + d(y)[\sigma(x)a, \lambda(y)aa + [d(y), y]_{\lambda, \tau}\sigma(x)aa$$
$$= d(y)\sigma(x)a[a, \lambda(y)], \forall x, y \in I.$$ 

That is

$$d(y)\sigma(I)a[a, \lambda(y)] = 0, \forall y \in I. \quad (2.13)$$

Since $\sigma(I)$ a nonzero ideal of $R$ then, for any $y \in I$, we obtain that

$\sigma(I) = 0$ or $d(y) = 0$

by (13). Hence, the additive group $I$ is a union of subgroups $K = \{y \in I \mid a[a, \lambda(y)] = 0\}$ and $L = \{y \in I \mid d(y) = 0\}$. Considering as in the proof of the Theorem 1, we obtain that $K = I$ and so $a[a, \lambda(I)] = 0$. Using this result we get,

$$0 = a[a, \lambda(yr)] = a\lambda(y)[a, \lambda(r)] + a[a, \lambda(y)]\lambda(r)$$
$$= a\lambda(y)[a, \lambda(r)], \forall r \in R, y \in I.$$
That is $a \lambda(I)[a, R] = 0$. This means that $a \in Z$. On the other hand, considering that $a \in Z$ and hypothesis, we get

$$0 = [h(x)a, x]_{\lambda, \tau} = h(x)[a, \lambda(x)] + [h(x), x]_{\lambda, \tau}a$$

$$= [h(x), x]_{\lambda, \tau}a \text{ for all } x \in I.$$  

That is $[h(x), x]_{\lambda, \tau}a = 0, \forall x \in I$. Since $a \in Z$ and $a \neq 0$ we have $[h(x), x]_{\lambda, \tau} = 0$ for all $x \in I$. This gives that $R$ is commutative by Theorem 3. □

**Remark 2.17.** Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $(I, a)_{\lambda, \mu}b = 0$ or $(b(I, a)_{\lambda, \mu} = 0$ then $a \in Z$ or $b = 0$.

**Proof.** If $(I, a)_{\lambda, \mu}b = 0$ then we have

$$0 = (rx, a)_{\lambda, \mu}b = r(x, a)_{\lambda, \mu}b - [r, \mu(a)]xb = -[r, \mu(a)]xb, \forall r \in R, x \in I. \text{ That is } [R, \mu(a)]Ib = 0. \text{ This gives that } a \in Z \text{ or } b = 0.$$

Let $b(I, a)_{\lambda, \mu} = 0$. Then $0 = b(xr, a)_{\lambda, \mu} = bx[r, \lambda(a)] + b(x, a)_{\lambda, \mu}r = bx[r, \lambda(a)], \forall r \in R, x \in I.$

This gives that $bI[R, \lambda(a)] = 0$ and so $a \in Z$ or $b = 0$. □

**Lemma 2.18.** Let $I$ be a nonzero ideal of $R$ and $a$ be a noncentral element of $R$. Let $h : R \to R$ be a nonzero right-generalized derivation associated with $d$. If $h(I, a)_{\lambda, \mu} = 0$ or $(h(I), a)_{\lambda, \mu} = 0$ then $d\lambda(a) = 0$.

**Proof.** If $(I, a)_{\lambda, \mu} = 0$ then using that $h$ is a right generalized derivation we get

$$0 = h(x\lambda(a), a)_{\lambda, \mu}h\{x[\lambda(a), \lambda(a)] + (x, a)_{\lambda, \mu}\lambda(a)\} = h\{(x, a)_{\lambda, \mu}\lambda(a)\}$$

$$= h(x, a)_{\lambda, \mu}\lambda(a) + (x, a)_{\lambda, \mu}d\lambda(a) = (x, a)_{\lambda, \mu}d\lambda(a), \forall x \in I,$$

which leads to

$$(I, a)_{\lambda, \mu}d\lambda(a) = 0. \quad (2.14)$$

Using Remark 2 and (14) we have $a \in Z$ or $d\lambda(a) = 0$. Since $a$ be a noncentral then $d\lambda(a) = 0$ is obtained.

If $(h(I), a)_{\lambda, \mu} = 0$ then we have

$$0 = (h(x\lambda(a)), a)_{\lambda, \mu} = (h(x)\lambda(a) + xd\lambda(a), a)_{\lambda, \mu}$$

$$= h(x)[\lambda(a), \lambda(a)] + (h(x), a)_{\lambda, \mu}\lambda(a) + x(d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)]d\lambda(a)$$

$$= x(d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)]d\lambda(a), \forall x \in I.$$  

That is,

$$x(d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)]d\lambda(a) = 0, \forall x \in I. \quad (2.15)$$

Replacing $x$ by $xy, y \in I$ in (15) and using (15) we get

$$0 = xy[d\lambda(a), a)_{\lambda, \mu} - [y, \mu(a)]d\lambda(a) - [x, \mu(a)]yd\lambda(a)$$

$$= -[x, \mu(a)]yd\lambda(a), \forall x, y \in I.$$

and so $[I, \mu(a)]Id\lambda(a) = 0$. Since $R$ is prime and $a$ be a noncentral element then we obtain that $d\lambda(a) = 0$. □
Lemma 2.19. Let $I$ be a nonzero ideal of $R$ and $a$ be a noncentral element of $R$. Let $h : R \rightarrow R$ be a nonzero left generalized derivation associated with derivation $d_1 : R \rightarrow R$. If $h((I, a)_{\lambda, \mu}) = 0$ or $(h(I), a)_{\lambda, \mu} = 0$ then $d_1 \mu(a) = 0$.

Proof. If $(h(I), a)_{\lambda, \mu} = 0$ then using that $h$ is a left-generalized derivation we get

$$
0 = h(\mu(a)x, a)_{\lambda, \mu} = h(\mu(a)(x, a)_{\lambda, \mu} - [\mu(a), \mu(a)]x)
$$

$$
= h(\mu(a)(x, a)_{\lambda, \mu}) = d_1(\mu(a))(x, a)_{\lambda, \mu} + \mu(a)h((x, a)_{\lambda, \mu})
$$

$$
= d_1(\mu(a))(x, a)_{\lambda, \mu}, \forall x \in I.
$$

That is,

$$
d_1(\mu(a))(I, a)_{\lambda, \mu} = 0. \quad (2.16)
$$

Since $a$ be noncentral then using Remark 2 and (16) we obtain that $d_1(\mu(a)) = 0$.

On the other hand, If $(h(I), a)_{\lambda, \mu} = 0$ then we have $d_1(\mu(a)) = 0$ by Lemma 5.

\[ \square \]

Theorem 2.20. Let $I$ be a nonzero ideal of $R$ and $a$ is a noncentral element of $R$. Let $h : R \rightarrow R$ be a nonzero right-generalized derivation associated with $d$ and left-generalized derivation associated with $d_1$. Then $h((I, a)_{\lambda, \mu}) = 0$ if and only if $(h(I), a)_{\lambda, \mu} = 0$.

Proof. If $h((I, a)_{\lambda, \mu}) = 0$ or $(h(I), a)_{\lambda, \mu} = 0$ then $d(\lambda(a)) = 0$ and $d_1(\mu(a)) = 0$ are obtained by Lemma 9 and Lemma 10.

Using these results we get

$$
h((I, a)_{\lambda, \mu}) = 0 \iff h(x\lambda(a) + \mu(a)x) = 0, \forall x \in I.
$$

$$
\iff h(x)\lambda(a) + xd(\lambda(a)) + d_1(\mu(a))x + \mu(a)h(x) = 0, \forall x \in I.
$$

$$
\iff h(x)\lambda(a) + \mu(a)h(x) = 0, \forall x \in I.
$$

$$
\iff (h(I), a)_{\lambda, \mu} = 0.
$$

\[ \square \]

Corollary 2.21. [9, Theorem 7] Let $R$ be a prime ring of characteristic different from two, $d : R \rightarrow R$ be a nonzero derivation and $a \in R$. Then $(d(R), a) = 0$ if and only if $d(R, a) = 0$.

References


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