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Existence of Three Solutions to the Discrete Fourth-order Boundary Value Problem with Four Parameters

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abstract: In this work, we will prove the existence of three solutions for the discrete nonlinear fourth order boundary value problems with four parameters. The methods used here are based on the critical point theory.

Key Words: Ddiscrete fourth order boundary value problems, Critical points theory, Three critical point.

Contents

1. Introduction

Let $T > 2$ be a positive integer and $[2, T]_{\mathbb{Z}}$ be the discrete interval given by $\{2, 3, 4....., T\}$. In this paper, we will examine a discrete nonlinear fourth order boundary value problems (BVP) with four parameters with intention of proving the existence of three solutions.The problem to be studied can be viewed as a discrete version of the generalized beam equation. Consider the fourth BVP :

$$\begin{aligned} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)) + \mu g(k, u(k)), k \in [2, T]_Z \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{aligned} \tag{1.1}$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^{i+1}u(k) = \Delta(\Delta^i u(k))$, $f, g : [2, T]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions, and $\alpha, \beta, \lambda, \mu$ are real parameters and satisfy : $\lambda > 0$, $\mu > 0$ and

$$1 + (T-1).T\alpha_- + T.(T-1)^3.\beta_- > 0, \tag{1.2}$$

where : $\alpha_- = min(\alpha, 0)$ and $\beta_- = min(\beta, 0)$.

The theory of nonlinear difference equations has been widely used to study the discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In recent years, a great deal of work has been

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done in the study of the existence and multiplicity of solutions for discrete boundary value problem. For the background and recent results, we refer the reader to the monographs [1-13] and the references therein. In this work , we will examine some applications of the variational methods to study the BVP (1).
Depending on the values of the parameters α, β, λ and μ, BVP (1.1) covers many problems . If $\lambda = 1$ and $\mu = 0$ the BVP (1) becomes

$$\Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = f(k, u(k)), \qquad k \in [2, T]_Z$$
$$\Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0,$$

has been recently investigated in [17], and existence results of sign-changing solutions are obtained using a topological degree theory and fixed point index theory. Also, If $\lambda > 0$ and $\mu = 0$ this problem has been studed by M.Ousbika and Z.El allali in [18], using the critical point theory and the direct method of calculus variational. Here, we will wish the existence of three solutions for BVP (1) by using some basic theorems in critical point theory and variational methods under some conditions imposed on the nonlinear functions f and g .
In this paper, we introduce in section 2 some preliminary theorems, the corresponding variational framwork of BVP (1) and we present some lemmas to prove our main results , in section 3 we obtain the existence of three solutions for BVP (1).

2. Preliminaries

Let us collect some theorems and lemmas that will be used below. One can refer to [14,19,20] for more details.

Proposition 2.1. *[see 14] Let E be a real reflexive Banach space and E^* be the dual space of E. Suppose that $T : E \to E^*$ is a continuous operator and there exists $\omega > 0$ such that*

$$(Tu - Tv, u - v) \geq \omega\|u - v\|^2; u, v \in E.$$

Then $T : E \to E^*$ is a homeomorphism between E and E^*

Theorem 2.2. *[see ,20,theorem 1] Let E be a real reflexive Banach space, E^* be the dual space of E , $\phi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional that is bounded on subsets of E and whose Gâteaux derivative admits a continuous inverse on E^*. $\psi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\phi(0) = \psi(0) = 0$.*
Assume that there exist $r > 0$ and $\bar{u} \in E$ with $r < \phi(\bar{u})$ such that

(i) $\dfrac{\sup\limits_{u\in\phi^{-1}(-\infty,r]} \psi(u)}{r} < \dfrac{\psi(\bar{u})}{\phi(\bar{u})}.$

(ii) for each $\lambda \in \Lambda =]\dfrac{\phi(\bar{u})}{\psi(\bar{u})}, \dfrac{r}{\sup\limits_{u\in\phi^{-1}(-\infty,r]} \psi(u)}[,$ *$\phi - \lambda\psi$ is coercive.*

Then , for each compact interval $[a, b] \subset \Lambda$, there exists $\gamma > 0$ with the following property : for each $\lambda \in [a, b]$ and every C^1 functional $\Gamma : E \to \mathbb{R}$ with compact derivative, there exists $\zeta > 0$ such that , for each $\mu \in (0, \zeta]$, the functional $\phi - \lambda\psi - \mu\Gamma$ has at least three distinct critical points in E whose norms are less than γ.

Theorem 2.3. *[see ,19,theorem 2] Let E be a real reflexive Banach space with the norm $\|.\|_E$, E^* be the dual space of E . Let $\phi : E \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional that is bounded on subsets of E and whose Gâteaux derivative admits a continuous inverse on E^* and $\psi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that ϕ has a strict local minimum u_0 with $\phi(u_0) = \psi(u_0) = 0$. Let*

$$\delta = max\{0, \limsup_{\|u\|_E\to\infty} \frac{\psi(u)}{\phi(u)}, \limsup_{\|u\|_E\to 0} \frac{\psi(u)}{\phi(u)}\},$$

and

$$\eta = \sup_{u\in\phi^{-1}(0,\infty)} \frac{\psi(u)}{\phi(u)},$$

and assume that $\delta < \eta$. Then , for each compact interval $[a, b] \subset (\frac{1}{\eta}, \frac{1}{\delta})$, there exists $K > 0$ with the following property : for each $\mu \in [a, b]$ and every C^1 functional $\Gamma : E \to \mathbb{R}$ with compact derivative, there exists $\zeta > 0$ such that, for each $\lambda \in (0, \zeta]$, the functional $\phi - \lambda\psi - \mu\Gamma$ has at least three distinct critical points in E whose norms are less than K.

We define the real vector space E

$$E = \{u : [0, T+2]_Z \to \mathbb{R} \quad , \quad u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0\},$$

which is a (T-1)-dimentional Hilbert space , see [17] with the inner product

$$(u, v) = \sum_{k=2}^{k=T} u(k)v(k).$$

The associated norm is defined by

$$\|u\| = (\sum_{k=2}^{k=T} |u(k)|^2)^{\frac{1}{2}}.$$

Definition 2.4. *We say that $u \in E$ is a weak solution of problem (1) if for any $v \in E$, we have*

$$\lambda \sum_{k=2}^{T} f(k, u(k))v(k) + \mu \sum_{k=2}^{T} g(k, u(k))v(k) = \sum_{k=2}^{T} \Delta^4 u(k-2)v(k) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) + \beta \sum_{k=2}^{T} u(k)v(k).$$

Lemma 2.5. *For any $u, v \in E$, we have*

$$\sum_{k=2}^{k=T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{k=T+1} \Delta^2 u(k-2)\Delta^2 v(k-2) \tag{2.1}$$

$$\sum_{k=2}^{k=T} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=2}^{k=T} \Delta^2 u(k-1)v(k) \tag{2.2}$$

Proof: We first prove (2.1). For any $u, v \in E$, by the summation by parts formula and the fact that $\Delta v(0) = \Delta v(T) = 0$, it follows that

$$\begin{aligned}
\sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2) &= \Delta^2 u(T)\Delta v(T) - \Delta^2 u(0)\Delta v(0) \\
&\quad - \sum_{k=2}^{T+1} \Delta^3 u(k-2)\Delta v(k-1) \\
&= -\sum_{k=2}^{T+1} \Delta^3 u(k-2)\Delta v(k-1) \\
&= -\sum_{k=2}^{T} \Delta^3 u(k-2)\Delta v(k-1),
\end{aligned}$$

in other hand , by the summation by parts formula and the fact that $\Delta^3 u(0) = \Delta^3 u(T-1) = 0$, we have

$$\sum_{k=2}^{T} \Delta^3 u(k-2)\Delta v(k-1) = \Delta^3 u(T-1)v(T) - \Delta^3 u(0)v(1) - \sum_{k=2}^{T} \Delta^4 u(k-2)v(k),$$

so

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2),$$

i.e.,(2.1) holds.
Next, we show (2.2). Again, by the summation by parts formula and the fact that $\Delta u(T) = 0$ and $v(1) = 0$, we have

$$\begin{aligned}
\sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) &= \Delta u(T)v(T) - \Delta u(1)v(1) - \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) \\
&= -\sum_{k=2}^{T} \Delta^2 u(k-1)v(k).
\end{aligned}$$

This completes the proof of the lemma. □

We consider the functional as follows:

$$\Phi(u) = \frac{1}{2}(\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2 + \alpha\sum_{k=2}^{T}|\Delta u(k-1)|^2 + \beta\sum_{k=2}^{T}|u(k)|^2), \tag{2.3}$$

and

$$\rho = (1 + (T-1).T\alpha_- + T.(T-1)^3.\beta\_)T^{-1}(T-1)^{-3} \tag{2.4}$$

Lemma 2.6. *For any $u \in E$, we have*

$$\Phi(u) \geq 0 \quad \textit{and} \quad \Phi(u) \geq \frac{1}{2}\rho\|u\|^2.$$

Proof: Let $u \in E$ and $k \in [2,T]_{\mathbb{Z}}$, note that

$$\Delta u(k-1) = \Delta u(0) + \sum_{i=2}^{k}\Delta^2 u(i-2)$$

in fact that $\Delta u(0) = 0$, then by Hölder's inequality, we have

$$\begin{aligned} |\Delta u(k-1)| \leq \sum_{i=2}^{k}|\Delta^2 u(i-2)| \quad &\leq \quad \sum_{i=2}^{T+1}|\Delta^2 u(i-2)| \\ &\leq \sqrt{T}(\sum_{i=2}^{T+1}|\Delta^2 u(i-2)|^2)^{\frac{1}{2}}, \end{aligned}$$

so

$$\sum_{k=2}^{T}|\Delta u(k-1)|^2 \leq T(T-1)\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2.$$

Similarly, for any $u \in E$ and $k \in [2,T]_{\mathbb{Z}}$, note that

$$u(k) = u(1) + \sum_{i=2}^{k}\Delta u(i-1),$$

in fact that $u(1) = 0$, then by Hölder's inequality, we have

$$\begin{aligned} |u(k)| \quad &\leq \quad \sum_{i=2}^{k}|\Delta u(i-1)| \leq \sum_{i=2}^{T}|\Delta u(i-1)| \\ &\leq \sqrt{T-1}(\sum_{i=2}^{T}|\Delta u(i-1)|^2)^{\frac{1}{2}}, \end{aligned}$$

then

$$\sum_{k=2}^{T}|u(k)|^2 \leq (T-1)^2\sum_{k=2}^{T}|\Delta u(k-1)|^2,$$

so

$$\sum_{k=2}^{T}|u(k)|^2 \le T(T-1)^3\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2,$$

therefore, from (2.3) and by summation the parts inequalities, we deduce that

$$\begin{aligned}\Phi(u) \ &\ge \ \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2 \\ &\ge \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)T^{-1}(T-1)^{-3}\sum_{k=2}^{T}|u(k)|^2 \\ &\ge \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)T^{-1}(T-1)^{-3}\|u\|^2,\end{aligned}$$

then by (2.1), we deduce that

$$\Phi(u) \ge 0 \quad and \quad \Phi(u) \ge \frac{1}{2}\rho\|u\|^2$$

the proof of lemma is completed. □

Note that , for $u \in E$,

$$\Psi_1(u) = \sum_{k=2}^{T} F(k,u(k)), \tag{2.5}$$

and

$$\Psi_2(u) = \sum_{k=2}^{T} G(k,u(k)), \tag{2.6}$$

where

$$F(k,x) = \int_0^x f(k,t)dt \quad and \quad G(k,x) = \int_0^x g(k,t)dt \quad ,k\in[2,T]_{\mathbb{Z}}.$$

The functional corresponding of BVP(1) is given by

$$I := \Phi - \lambda\Psi_1 - \mu\Psi_2. \tag{2.7}$$

With any fixed $\lambda > 0$ and $\mu > 0$, the functionals Φ,Ψ_1 , Ψ_2 and I is of class $C^1(E,\mathbb{R})$, and for $u, v \in E$, we have

$$(\Phi'(u),v) = \sum_{k=2}^{T+1}\Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha\sum_{k=2}^{T}\Delta u(k-1)\Delta v(k-1) + \beta\sum_{k=2}^{T}u(k)v(k), \tag{2.8}$$

$$(\Psi_1'(u), v) = \sum_{k=2}^{T} f(k, u(k))v(k), \tag{2.9}$$

$$(\Psi_2'(u), v) = \sum_{k=2}^{T} g(k, u(k))v(k) \tag{2.10}$$

and

$$(I'(u), v) = (\Phi'(u), v) - \lambda(\Psi_1'(u), v) - \mu(\Psi_2'(u), v). \tag{2.11}$$

The search of solutions of BVP (1) reduce to finding critical points $u \in E$ of the functional I by the following,

Lemma 2.7. *If $u \in E$ is a critical point of the functional I then u is a solution of BVP (1).*

Proof: Let $u \in E$ is a critical point of the functional I then

$$(I'(u), v) = 0, \quad \forall v \in E,$$

so from (2.7) – (2.10) and lemma 2.5 , we deduce that

$$\begin{aligned} 0 \quad = \quad & \sum_{k=2}^{T}(\Delta^4 u(k-2) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1) + \beta \sum_{k=2}^{T} u(k))v(k) \\ & -\lambda \sum_{k=2}^{T}(f(k, u(k)) + g(k, u(k)))v(k), \qquad \text{forall} \;\; v \in E, \end{aligned}$$

thus by the arbitrarieness of $v \in E$, we have

$$\Delta^4 u(k-2) - \alpha \Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)) + \mu g(k, u(k)),$$

then $u \in E$ is a solution of BVP (1).This completes the proof. □

3. Main results

Theorem 3.1. *Assume that the following conditions holds :*

(H1) *There exists $c > 0$ and $d > 0$ such that $c < d\sqrt{T-1}$.*

(H2) *There exists $\delta > 0$ and $0 < s < 2$ such that for all $x \in \mathrm{R}$ and $k \in [2, T]_{\mathrm{Z}}$: $F(k, x) \leq \delta(1 + |x|^s)$.*

(H3) $\sum_{k=2}^{T} F(k, d) > 0.$

(H4) $\frac{\rho c^2}{d^2\gamma}\sum_{k=2}^{T} F(k,d) > (T-1) \max_{(k,x)\in[2,T]_{\mathbb{Z}}\times[-c,c]} F(k,x)$,
where : $\gamma := 10 + 3\alpha_+ + (T-1)\beta_+$.

Then for each compact interval

$$[a,b] \subset \Lambda =]\frac{\gamma d^2}{2\sum_{k=2}^{T} F(k,d)}, \frac{\rho c^2}{2(T-1) \max_{(k,x)\in[2,T]_{\mathbb{Z}}\times[-c,c]} F(k,x)}[,$$

there exist $\zeta > 0$ *such that , for each* $\lambda \in [a,b]$ *there exist* $\eta > 0$ *such that , for each* $\mu \in (0,\eta]$*, the BVP(1) has at least three distinct solutions in E whose norms are less than* ζ*.*

Proof: To prove the theorem 3.1, we will apply theorem 2.2 with $\psi = \Psi_1$ and $\Gamma = \Psi_2$.
Firsty, we show that the functionals Φ , Ψ_1 and Ψ_2 satisfy the regularity assumptions of theorem 2.2. By lemma 2.6, we prove that Φ is coercive, sequentielly weakly lower semicontinuous and is bounded on each bounded subset of E. From (2.3) and (2.8), we have

$$\forall u \in E \quad : \quad (\Phi'(u), u) = 2\Phi(u),$$

then

$$\forall u, v \in E \quad : \quad (\Phi'(u) - \Phi'(v), u - v) = 2\Phi(u - v) \geq \rho\|u - v\|^2.$$

Hence by proposition 2.1, $(\Phi')^{-1} : E^* \to E$ exist and is continuous.
Secondly, we show that Ψ_1' and $\Psi_2'$are compacts. Suppose that $u_n \to u \in E$ then since f and g are continuous and from (2.9), (2.10), we deduce that $\Psi_1'(u_n) \to \Psi_1'(u)$ and $\Psi_2'(u_n) \to \Psi_2'(u)$, thus Ψ_1' and Ψ_2' are compacts, also $\Phi(0) = \Psi_1(0) = 0$.
Next, put: $r = \frac{1}{2}\rho c^2$ and pick $\bar{u} \in E$ defined as for $k \in [2,T]_{\mathbb{Z}} \quad : \bar{u}(k) = d$.
using lemma 2.6 with $u = \bar{u}$ and $c < d\sqrt{T-1}$, we have

$$\Phi(\bar{u}) \geq \frac{1}{2}\rho\|\bar{u}\|^2 = \frac{1}{2}\rho(T-1)d^2 > \frac{1}{2}\rho c^2 = r.$$

Taking into the fact that , for any $k \in [2,T]_{\mathbb{Z}}$

$$|u(k)| \leq \|u\| \leq \sqrt{\frac{2.\Phi(u)}{\rho}},$$

we have

$$\Phi^{-1}((-\infty, r]) \subseteq \{u \in E : |u(k)| \leq c, k \in [2,T]_{\mathbb{Z}}\},$$

then

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi_1(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} \sum_{k=2}^{T} F(k,u(k)) \leq (T-1) \max_{(k,x)\in[2,T]_{\mathbb{Z}}\times[-c,c]} F(k,x),$$

therefore, it follows from (H_4) that

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi_1(u) < \frac{\rho c^2 \sum_{k=2}^{T} F(k,d)}{d^2\gamma} = r\frac{2\Psi(\bar{u})}{\gamma d^2}.$$

It is easy to verify that

$$\Phi(\bar{u}) = \frac{1}{2}(10+3\alpha+(T-1)\beta)d^2 \leq \frac{1}{2}\gamma d^2$$

then

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi_1(u)}{r} < \frac{\Psi_1(\bar{u})}{\Phi(\bar{u})},$$

this imply that the assumption (i) of Theorem 2.2 is verified.
From (H_2) and by lemma 2.6, we deduce that for $u \in E$, we obtain

$$I(u) \geq \frac{1}{2}\rho\|u\|^2 - \lambda\sum_{k=2}^{T}\delta(1+|u(k)|^s) \geq \frac{1}{2}\rho\|u\|^2 - \lambda\delta(T-1) - \lambda\delta(T-1)\|u\|^s,$$

sinse $s < 2$, then I is coercive, this imply that the assumption (ii) of Theorem 2.2 is verified, therfore according to theorem 2.2, the proof of theorem 3.1 is completed. □

Theorem 3.2. *Assume that the following conditions holds :*

(H5) There exists $C > 0$ such that $max(g^0; g^\infty) < C$, where

$$g^0 = \max_{k\in[2,T]_{\mathbb{Z}}} \limsup_{|x|\to 0} \frac{g(k,x)}{x} \quad \textit{and} \quad g^\infty = \max_{k\in[2,T]_{\mathbb{Z}}} \limsup_{|x|\to\infty} \frac{g(k,x)}{x}$$

(H6) There exists $d > 0$ such that $\sum_{k=2}^{T} G(k,d) > 0$ and

$$2\rho\sum_{k=2}^{T} G(k,d) > Cd^2\gamma, \quad \textit{where} : \gamma := 10 + 3\alpha_+ + (T-1)\beta_+$$

Then for each compact interval $[a,b] \subset \Lambda =]\frac{\gamma d^2}{2\sum_{k=2}^{T} F(k,d)}, \frac{\rho}{C}[$, *there exist* $\zeta > 0$ *such that, for each* $\mu \in [a,b]$, *there exists* $\eta > 0$ *such that for each* $\lambda \in (0,\eta]$, *the BVP(1) has at least three distinct solutions in* E *whose norms are less than* ζ.

Proof: As in proof of theorem 3.1, we see that Φ, Ψ_1 and Ψ_2 satisfy the regularity assumptions of theorem 2.3. By lemma 2.6 , for $u_0 = 0$, Φ has a strict local minimum and it is clearly that $\Phi(0) = \Psi_2(0) = 0$.
From $(H6)$, there exists $r > 0$ and $R > 0$ with $r < R$, such that

$$\frac{g(k,x)}{x} \le C \quad \forall \quad |(x| < r \quad or \quad |x| > R),$$

then for $k \in [2,T]_{\mathrm{Z}}$, we have

$$g(k,x) \le Cx : \forall x \in [0,r) \cup (R,+\infty) \quad and \quad g(k,x) \ge Cx : \forall x \in (-\infty,-R) \cup (-r,0].$$

Since g is continuous, $x \mapsto \frac{g(k,x) - Cx}{x^2}$, is continuous on $[-R,r] \cup [r,R]$,then there exists $C' > 0$ such that

$$g(k,x) \le Cx + C'x^2 : \forall x \in [0,+\infty) \quad and \quad g(k,x) \ge Cx - C'x^2 : \forall x \in (-\infty,0].$$

Therfore, for any $k \in [2,T]_{\mathrm{Z}}$, we have

$$G(k,x) \le \frac{1}{2}Cx^2 + \frac{1}{3}C'|x|^3 : \forall x \in \mathrm{R},$$

so, for $u \in E$, we have

$$G(k,u(k)) \le \frac{1}{2}C\|u\|^2 + \frac{1}{3}C'(T-1)\|u\|^3.$$

Then from (2.7) and by lemma 2.6 , we deduce that for $u \in E$

$$\frac{\Psi_2(u)}{\Phi(u)} \le \frac{C}{\rho} + \frac{2}{3\rho}C'(T-1)\|u\|,$$

then

$$\limsup_{\|u\| \to 0} \frac{\Psi_2(u)}{\Phi(u)} \le \frac{C}{\rho}. \tag{3.1}$$

By definition of g^∞, there exists $A > 0$ such that

$$\frac{g(k,x)}{x} \le C \quad ,\forall \quad |x| > A.$$

then

$$g(k,x) \le Cx : \forall x \in (R,+\infty) \quad and \quad g(k,x) \ge Cx, \forall x \in (-\infty,-R).$$

Since g is continuous then there exists $C'' > 0$ such that

$$g(k,x) \leq Cx + C'' : \forall x \in [0,+\infty) \quad and \quad g(k,x) \geq Cx - C'', \forall x \in (-\infty, 0].$$

Therfor, for any $k \in [2,T]_Z$, we have

$$G(k,x) \leq \frac{1}{2}Cx^2 + C''|x|, \forall x \in R.$$

So, for $u \in E$, we have

$$G(k,u(k)) \leq \frac{1}{2}C\|u\|^2 + C''(T-1)\|u\|.$$

Then from (2.7) and by lemma 2.6 , we deduce that for $u \in E$

$$\limsup_{\|u\|\to\infty} \frac{\Psi_2(u)}{\Phi(u)} \leq \frac{C}{\rho}. \tag{3.2}$$

Hence, from (3.1) and (3.2), we see that

$$\delta = max\{0; \limsup_{\|u\|\to\infty} \frac{\Psi_2(u)}{\Phi(u)}; \limsup_{\|u\|\to 0} \frac{\Psi_2(u)}{\Phi(u)}\} \leq \frac{C}{\rho}.$$

Next, for $d > 0$, we pick $\bar{u} \in E$ defined as for $k \in [2,T]_Z$: $\bar{u}(k) = d$, then by lemma 2.6, we have

$$\Phi(\bar{u}) \geq \frac{\rho(T-1)d^2}{2} > 0, \qquad so \qquad \bar{u} \in \Phi^{-1}(0,+\infty).$$

Then

$$\eta = \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{\Psi_2(u)}{\Phi(u)} \geq \frac{\Psi_2(\bar{u})}{\Phi(\bar{u})} = \frac{2\sum_{k=2}^{T} G(k,d)}{(10+3\alpha+(T-1)\beta)d^2} \geq \frac{2\sum_{k=2}^{T} G(k,d)}{\gamma d^2},$$

where $\gamma := 10 + 3\alpha_+ + (T-1)\beta_+$.
Then from $(H6)$, we infer that, $\delta < \eta$. Hence , all the assumptions of Theorem 2.3 are satisfied. This completes the proof of the theorem.□ □

We conclude this work with the example as follows to illustrate our results.

Example 3.1. .
We consider the BVP (1) with $\alpha = 1$, $\beta = 1$ and $T = 5$.
Then $\rho = \frac{25}{8}10^{-3}$ and $\gamma = 17$.
For $k \in [2,5]_Z$ and $x \in R$, let $f : [2,5]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$, and

$$g(k,x) = \begin{cases} k & \text{si } |x| > 1 \\ kx^2 & \text{si } |x| \leq 1 \end{cases}$$

It is clear that, $g^0 = 0$ and $g^\infty = 0$.
Then $(H5)$ holds for any $C > 0$, (for example $C = 10^{-5}$).
It is easy to see that

$$G(k,x) = \int_0^x g(k,t)dt = \begin{cases} k(x-\frac{2}{3}) & \text{si } x > 1 \\ \frac{1}{3}kx^3 & \text{si } |x| \le 1 \\ k(x+\frac{2}{3}) & \text{si } x < -1 \end{cases}$$

we choose $d = 1$, then we have

$$\sum_{k=2}^{5} G(k,1) = \frac{14}{3} > 0 \quad , \quad Cd^2\gamma = 17.10^{-5} \quad \text{and} \quad 2\rho\sum_{k=2}^{5} G(k,1) = \frac{175}{6}10^{-3}$$

therfore $H(6)$ is satisfied.
We deduce that for each compact interval $[a,b]$ satisfying $[a,b] \subset]\frac{51}{28}, \frac{25}{8}10^2[$, there exist $\zeta > 0$ such that, for each $\mu \in [a,b]$, there exists $\eta > 0$ such that for each $\lambda \in (0,\eta]$, the BVP(1) has at least three distinct solutions in E whose norms are less than ζ.

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