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On a Nonlinear System Arising in a Theory of Thermal Explosion

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abstract: The purpose of this paper is to study the existence and multiplicity of positive solutions for a mathematical model of thermal explosion which is described by the system

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla v + b(v)v = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, Δ is the Laplacian operator, $\lambda > 0$ is a parameter, f, g belong to a class of non-negative functions that have a combined sublinear effect at ∞, and $a, b : [0, \infty) \to (0, \infty)$ are nondecreasing C^1 functions. We establish our existence and multiplicity results by the method of sub– and supersolutions.

Key Words: Nonlinear system; Thermal explosion; Sub-supersolutions.

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1. Introduction

A classical problem in combustion theory is a model of thermal explosion which occurs due to a spontaneous ignition in a rapid combustion process. In this paper, we consider a model involving a nonlinear boundary heat loss which is not a very typical one in classical combustion theory, but is relevant to some more applications (see [4,10,12,5] for details). The model reads as:

$$\begin{cases} \theta(t) - \Delta\theta = \lambda f(\eta), & (t, x) \in (0, \infty) \times \Omega, \\ \eta(t) - \Delta\eta = \lambda g(\theta), & (t, x) \in (0, \infty) \times \Omega, \\ \mathbf{n}.\nabla\theta + a(\theta)\theta = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ \mathbf{n}.\nabla\eta + b(\eta)\eta = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ \theta(0, x) = 0 = \eta(0, x). \end{cases} \tag{1.1}$$

Here θ, η are the appropriately scaled temperature in a bounded smooth domain $\Omega \subset \mathbb{R}^{N,}$ $N \geq 1$, and f, g are the normalized reaction rate. We assume that f, g satisfying the following assumptions:

(H1) $f, g \in C([0, \infty))$ are nondecreasing functions,

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and

(H2) $\lim_{s\to\infty} \frac{f(Ag(s))}{s} = 0$, for all $A > 0$.

On the C^2 boundary $\partial\Omega$, with the outward unit normal denoted by $\mathbf{n}$, the heat-loss parameters $a(\theta), b(\eta)$ are assumed to satisfy the following hypothesis:

(H3) $a, b : [0, \infty) \to (0, \infty)$ are nondecreasing bounded C^1 functions.

Physically this assumption means that a heat loss through the boundary always exists and increases linearly with the temperature even in the small temperature regime.

A bifurcation (or scaling) parameter $\lambda > 0$ can be associated with the size of domain Ω in (1) which grows linearly as the measure of Ω increases. It is well known that, after normalizing for the size of Ω, the long term behavior of solution of (1) is close to the solution of the time-independent system:

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla v + b(v)v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

The motivation for this study cames from the work in [7] where the authors established the existence, uniqueness and multiplicity of positive solutions for certain range of λ for the single equation of the form

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega. \end{cases}$$

Here we extend this study to Laplacian system of the form (1). In [1], Ali-Shivaji-Ramaswamy discussed the existence of multiple positive solutions to such systems with Dirichlet boundary conditions. One can refer to [3,8] for some recent existence and uniqueness results of elliptic problems with nonlinear boundary conditions.

2. Existence results

In this section, we shall establish our existence results via the method of sub - supersolution. A pair of nonnegative functions $(\psi_1, \psi_2) \in W^{1,2} \cap C(\overline{\Omega}) \times W^{1,2} \cap C(\overline{\Omega})$ and $(z_1, z_2) \in W^{1,2} \cap C(\overline{\Omega}) \times W^{1,2} \cap C(\overline{\Omega})$ are called a subsolution and supersolution of (1) if they satisfy

$$\begin{cases} -\Delta \psi_1 \le \lambda f(\psi_2), & x \in \Omega, \\ -\Delta \psi_2 \le \lambda g(\psi_1), & x \in \Omega, \\ \mathbf{n}.\nabla \psi_1 + a(\psi_1)\psi_1 \le 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla \psi_2 + b(\psi_2)\psi_2 \le 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\Delta z_1 \geq \lambda f(z_2), & x \in \Omega, \\ -\Delta z_2 \geq \lambda g(z_1), & x \in \Omega, \\ \mathbf{n}.\nabla z_1 + a(z_1)z_1 \geq 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla z_2 + b(z_2)z_2 \geq 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

respectively. It is well known that if there exist sub and supersolutions (ψ_1, ψ_2) and (z_1, z_2) respectively of (1) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then (1) has a solution (u, v) such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$ (see [2,6]).

By strict sub and super-solutions we understand functions (ψ_1, ψ_2) and (z_1, z_2) for which strict inequalities (3) and (4) hold.

Our multiplicity results are obtained by constructing sub and super-solution pairs that satisfy the following Lemma:

Lemma 2.1. (See [6,9,11]). Suppose the system (1) has a sub-solution (ψ_1, ψ_2), a strict super-solution (ζ_1, ζ_2), a strict sub-solution (w_1, w_2), and a super-solution (z_1, z_2) for (1) such that

$$(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2),$$

$$(\psi_1, \psi_2) \leq (w_1, w_2) \leq (z_1, z_2),$$

and $(w_1, w_2) \nleq (\zeta_1, \zeta_2)$. Then (1) has at least three distinct solutions (u_i, v_i), $i = 1, 2, 3$ such that

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], \quad (u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$$

and

$$(u_3, v_3) \in \Big[(\psi_1, \psi_2), (z_1, z_2)\Big] \backslash \Big(\Big[(\psi_1, \psi_2), (\zeta_1, \zeta_2)\Big] \cup \Big[(w_1, w_2), (z_1, z_2)\Big]\Big).$$

To precisely state our existence result we consider the unique classical solution e_r of the following linear elliptic problem

$$\begin{cases} -\Delta e_r = 1, & x \in \Omega, \\ \mathbf{n}.\nabla e_r + r_0 e_r = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

for $r = a, b$, where $r_0 = r(0)$. Then we establish:

Theorem 2.2. Let $(H1) - (H3)$ hold and $f(0)$ or $g(0)$ be strictly positive. Then (1) has a positive solution (u, v) for all $\lambda > 0$.

Proof. It is easy to see that $(\psi_1, \psi_2) = (0, 0)$ is a subsolution of (1). We now construct the supersolution (z_1, z_2). Let $(z_1, z_2) = \Big(C_\lambda e_a, \lambda g(C_\lambda \|e_b\|_\infty) e_b\Big)$, where

C_λ is a large number to be chosen later. We shall verify that (z_1, z_2) is a supersolution of (1) for all $\lambda > 0$. By $(H2)$ we can choose C_λ large enough so that

$$C_\lambda \geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty)\|e_b\|_\infty\Big),$$

and therefore

$$\begin{aligned} -\Delta z_1 = C_\lambda &\geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty)\|e_b\|_\infty\Big) \\ &\geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty) e_b\Big) \\ &= \lambda f(z_2) \ \text{ in}\, \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla z_1 + a(z_1) z_1 &\geq C_\lambda \mathbf{n}.\nabla e_a + C_\lambda e_a a_0 \\ &= C_\lambda(\mathbf{n}.\nabla e_a + e_a a_0) \\ &= 0 \ \text{ on}\, \partial\Omega. \end{aligned}$$

Next,

$$\begin{aligned} -\Delta z_2 &= \lambda g\Big(C_\lambda \|e_b\|_\infty\Big) \\ &\geq \lambda g\Big(C_\lambda e_b\Big) \\ &= \lambda g(z_1) \ \text{ in}\, \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla z_2 + b(z_2) z_2 &\geq \lambda g\Big(C_\lambda \|e_b\|_\infty\Big)\mathbf{n}.\nabla e_b + \lambda g\Big(C_\lambda \|e_b\|_\infty\Big) e_b b_0 \\ &= \lambda g\Big(C_\lambda \|e_b\|_\infty\Big)(\mathbf{n}.\nabla e_b + b_0 e_b) \\ &= 0 \ \text{ on}\, \partial\Omega, \end{aligned}$$

which implies that (z_1, z_2) is indeed a positive supersolution of (1). Therefore (1) has a positive solution for all $\lambda > 0$. $\square$

Our second result concerns with multiplicity of solution for the system (1) and gives an estimate on the parameter λ when such a situation occurs. For positive constants a_i, b_i; $i = 1, 2$, define

$$Q_1(a_1, b_1) = \min\{\frac{a_1}{f(b_1)}, \frac{b_1}{g(a_1)}\}$$

and

$$Q_2(a_2, b_2) = \max\{\frac{a_2}{f(b_2)}, \frac{b_2}{g(a_2)}\}.$$

Then we establish:

Theorem 2.3. Assume $f(0)$ or $g(0)$ be strictly positive. Let B_R be the largest ball of radius R inscribed in Ω, for $0 < \epsilon < R$, we define

$$C_1(\Omega) = \inf_{\epsilon} \frac{N}{\epsilon^N} \frac{R^{N-1}}{R-\epsilon},$$

and $C(\Omega) = C_1(\Omega)\|e_r\|_\infty$, for $r = a, b$. Let $(H1) - (H3)$ hold and $\frac{Q_1}{Q_2} > C(\Omega)$ for some a_i, b_i, $i = 1, 2$. Then (1) has at least three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$, where $\lambda_* = CQ_2$ and $\lambda^* = \frac{Q_1}{\|e_r\|_\infty}$, for $r = a, b$.

Proof. We will establish a pair of subsolutions (ψ_1, ψ_2), (w_1, w_2) and a pair of supersolutions (ζ_1, ζ_2), (z_1, z_2), satisfying Lemma 2.1. Clearly $(\psi_1, \psi_2) = (0, 0)$ is a subsolution of (1).

We next construct a positive supersolution (ζ_1, ζ_2), of (1) when $\lambda < \frac{Q_1}{\|e_r\|_\infty}$, for $r = a, b$. Since $\lambda < \frac{a_1}{f(b_1)\|e_a\|_\infty}$, we can choose $\epsilon > 0$ so small that $\lambda f(b_1) < \frac{a_1}{\epsilon + \|e_a\|_\infty}$. Let $(\zeta_1, \zeta_2) = (a_1 \frac{e_a+\epsilon}{\|e_a\|_\infty+\epsilon}, b_1 \frac{e_b+\epsilon}{\|e_b\|_\infty+\epsilon})$. Then, we have

$$\begin{aligned} -\Delta\zeta_1 = \frac{a_1}{\epsilon + \|e_a\|_\infty} &> \lambda f(b_1) \\ &\geq \lambda f\Big(b_1 \frac{e_b + \epsilon}{\|e_b\|_\infty + \epsilon}\Big) \\ &= \lambda f(\zeta_2) \ \text{ in } \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla\zeta_1 + a(\zeta_1)\zeta_1 &\geq \frac{a_1}{\epsilon + \|e_a\|_\infty}\Big(\mathbf{n}.\nabla e_a + (e_a + \epsilon)a_0\Big) \\ &= \frac{a_1}{\epsilon + \|e_a\|_\infty}(\mathbf{n}.\nabla e_a + a_0 e_a + a_0\epsilon) \\ &= \frac{a_1 a_0 \epsilon}{\epsilon + \|e_a\|_\infty} \\ &> 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Similar argument shows that ζ_2 satisfies $-\Delta\zeta_2 > \lambda g(\zeta_1)$ in Ω, and $\mathbf{n}.\nabla\zeta_2 + b(\zeta_2)\zeta_2 > 0$.

Next let us construct a strict sub-solution (w_1, w_2) of (1). First note that a system

$$\begin{cases} -\Delta u_D = \lambda f(v_D), & x \in \Omega, \\ -\Delta v_D = \lambda g(u_D), & x \in \Omega, \\ u_D = 0 = v_D, & x \in \partial\Omega, \end{cases}$$

admits a strict sub-solution (w_{1D}, w_{2D}) with $\|w_{1D}\|_\infty \geq a_2$ and $\|w_{2D}\|_\infty \geq b_2$ provided $\lambda < \lambda^*$ (see [1]). Then we have $(w_1, w_2) \nleq (\zeta_1, \zeta_2)$. By the Hopf's lemma we have that $\mathbf{n}.\nabla w_{iD} < 0$ for $i = 1, 2$. Therefore, setting $w_1 = w_{1D}$ and $w_2 = w_{2D}$ we obtain a strict sub-solution for (1) for $\lambda > \lambda_*$.

Let (z_1, z_2) be the super solution as in the proof of Theorem 2.2 Further $w_i, \zeta_i \leq z_i$, $i = 1, 2$ for C_λ large. Hence there exist positive solutions (u_i, v_i), $i = 1, 2, 3$ such that

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], \quad (u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$$

and

$$(u_3, v_3) \in \Big[(\psi_1, \psi_2), (z_1, z_2)\Big] \backslash \Big(\Big[(\psi_1, \psi_2), (\zeta_1, \zeta_2)\Big] \cup \Big[(w_1, w_2), (z_1, z_2)\Big]\Big). \qquad \square$$

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