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Almost Semicontinuous Multifunctions via Ideals

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abstract: The aim of this paper is to introduce and study upper and lower almost semi-$\mathcal{I}$-continuous multifunctions as a generalization of upper and lower semi-$\mathcal{I}$-continuous multifunctions, respectively.

Key Words: Semi-$\mathcal{I}$-open set, Almost semi-$\mathcal{I}$-continuous multifunction.

Contents

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [1,3,15,16,17,18,19]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [10] and Vaidyanathaswamy [21]. An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A\in\mathcal{I}$ and $B\subset A$ implies $B\in\mathcal{I}$ and (ii) $A\in\mathcal{I}$ and $B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X)\to\mathcal{P}(X)$, called the local function [21] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A\subset X$, $A^\star(\tau,\mathcal{I})=\{x\in X|U\cap A\notin\mathcal{I}$ for every $U\in\tau(x)\}$, where $\tau(x)=\{U\in\tau|x\in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau,\mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A)=A\cup A^\star(\tau,\mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [8] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 2002, Hatir and Noiri [5] further investigated semi-$\mathcal{I}$-open sets and semi-$\mathcal{I}$-continuous functions. Recently, Akdag and Canan [1] introduced and studied the concept of semi-$\mathcal{I}$-continuous multifunctions on ideal topological spaces. Also in [16], the theory of almost continuity for multifunctions is unified using certain minimal conditions. In this paper, we

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introduce and study upper (lower) almost-$\mathcal{I}$ continuous multifunctions and obtain several characterizations of upper (lower) almost semi-$\mathcal{I}$-continuous multifunctions and basic properties of such functions.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is semi-$\mathcal{I}$-open [8] if $S \subset \mathrm{Cl}^\star \mathrm{Int}(S)$. The complement of a semi-$\mathcal{I}$-closed set is said to be a semi-$\mathcal{I}$-open set. The semi-$\mathcal{I}$-closure and the semi-$\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $s\mathcal{I}\,\mathrm{Cl}(A)$ and $s\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all semi-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $S\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all semi-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $S\mathcal{I}O(X, x)$ (resp. $\mathcal{I}C(X, x)$). A subset A is said to be regular open [20] (resp. semiopen [11], preopen [12], semi-preopen [2]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, α-closed, semi pre-closed) set. The intersection (resp. union) of all semiclosed (resp. semiopen) set containing (resp. contained in) $A \subset X$ is called the semiclosure (resp. semiinterior) of A and is denoted by $s\,\mathrm{Cl}(A)$ (resp. $s\,\mathrm{Int}(A)$). The family of all regular open (resp. regular closed, semiopen, semiclosed, preopen, semi-preopen, semi-preclosed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $SPO(X)$, $SPC(X)$). By a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be lower semi-$\mathcal{I}$-continuous [1] (resp. upper semi-$\mathcal{I}$-continuous) multifunction if $F^-(V) \in S\mathcal{I}O(X, \tau)$ (resp. $F^+(V) \in S\mathcal{I}O(X, \tau)$) for every $V \in \sigma$. A subset N of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be semi-$\mathcal{I}$-neighborhood of a point $x \in X$, if there exists a semi-$\mathcal{I}$-open set V such that $x \in V \subset N$.

Lemma 2.1. *The following statements are true:*

1. *Let A be a subset of a space (X, τ). Then $A \in PO(X)$ if and only if $s\,\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$ [6].*
2. *A subset A of a space (X, τ) is semi-preopen if and only if $\mathrm{Cl}(A)$ is regular closed [2].*

Definition 2.2. *[19] A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:*

1. *lower weakly semi-$\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in S\mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$,*

2. *upper weakly semi-$\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in S\mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$,*

3. *weakly semi-$\mathcal{I}$-continuous if it is both upper weakly semi-$\mathcal{I}$-continuous and lower weakly semi-$\mathcal{I}$-continuous.*

3. On upper and lower almost semi-$\mathcal{I}$-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:*

1. *lower almost semi-$\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in S\mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$,*

2. *upper almost semi-$\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in S\mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$,*

3. *almost semi-$\mathcal{I}$-continuous if it is both upper almost semi-$\mathcal{I}$-continuous and lower almost semi-$\mathcal{I}$-continuous.*

It is clear that every upper (lower) semi-$\mathcal{I}$-continuous function is upper (lower) almost semi-$\mathcal{I}$-continuous. But the converse is not true as shown by the following example.

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the multifunction $F : (X, \tau, \mathcal{I}) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper almost semi-$\mathcal{I}$-continuous but is not upper semi-$\mathcal{I}$-continuous.*

Theorem 3.3. *The following statements are equivalent for a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:*

1. *F is upper almost semi-$\mathcal{I}$-continuous multifunction,*

2. *for each $x \in X$ and for each open set V such that $F(x) \subset V$, there exists $U \in S\mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V)) = s\,\mathrm{Cl}(V)$,*

3. *for each $x \in X$ and for each regular open set G of Y such that $F(x) \subset G$, there exists $U \in S\mathcal{I}O(X, x)$ such that $F(U) \subset G$,*

4. *for each $x \in X$ and for each closed set K such that $x \in F^+(Y \backslash K)$, there exists a semi-$\mathcal{I}$-closed set H such that $x \in X \backslash H$ and $F^-(\mathrm{Cl}(\mathrm{Int}(K))) \subset H$,*

5. *$F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in S\mathcal{I}O(X)$ for any open set $V \subset Y$,*

6. *$F^-(\mathrm{Cl}(\mathrm{Int}(K))) \in S\mathcal{I}C(X)$ for any closed set $K \subset Y$,*

7. *$F^+(G) \in S\mathcal{I}O(X)$ for any regular open set G of Y,*

8. *$F^-(K) \in S\mathcal{I}C(X)$ for any regular closed set K of Y,*

9. *for each point x of X and each neighbourhood V of $F(x)$, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is a semi-$\mathfrak{I}$-neighbourhood of x,*

10. *for each point x of X and each neighbourhood V of $F(x)$, there exists a semi-$\mathfrak{I}$-neighborhood U of x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.*

Proof. (1)⇔(2): The proof follows from Definition 3.1 and lemma 2.1.
(2)⇒(3): Let $x \in X$ and G be a regular open set of Y such that $F(x) \subset G$. By (2), there exists $U \in S\mathfrak{I}O(X, x)$ such that if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(G)) = G$. We obtain $F(U) \subset G$.
(3)⇒(2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (3), there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.
(2)⇒(4): Let $x \in X$ and K be a closed set of Y such that $x \in F^+(Y\backslash K)$. By (2), there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(Y\backslash K))$. We have $\mathrm{Int}(\mathrm{Cl}(Y\backslash K)) = Y\backslash \mathrm{Cl}(\mathrm{Int}(K))$ and $U \subset F^+(Y\backslash \mathrm{Cl}(\mathrm{Int}(K))) = X\backslash F^-(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain $F^-(\mathrm{Cl}(\mathrm{Int}(K))) \subset X\backslash U$. Take $H = X\backslash U$. Then, $x \in X\backslash H$ and H is semi-$\mathfrak{I}$-closed set.
(4)⇒(2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then $Y\backslash V$ is closed in Y and $x \in F^+(V) = F^+(Y\backslash(Y\backslash V))$. By (4), there exits a semi-$\mathfrak{I}$-closed set L such that $x \in X\backslash L$ and $F^-(\mathrm{Cl}(\mathrm{Int}(Y\backslash V))) \subset L$. This implies that $X\backslash L \subseteq F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Put $U = X\backslash L$. Then $U \in S\mathfrak{I}O(X)$ and if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V))$.
(1)⇒(5): Let V be any open set of Y and $x \in F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. By (1), there exists $U_x \in S\mathfrak{I}O(X, x)$ such that $U_x \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Therefore, we obtain $F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \bigcup_{x\in F^+(\mathrm{Int}(\mathrm{Cl}(V)))} U_x$. Hence, $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in S\mathfrak{I}O(X)$.
(5)⇒(1): Let V be any open set of Y and $x \in F^+(V)$. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in S\mathfrak{I}O(X)$. Take $U = F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence, F is upper almost semi-$\mathfrak{I}$-continuous.
(5)⇒(6): Let K be any closed set of Y. Then, $Y\backslash K$ is an open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(Y\backslash K))) \in S\mathfrak{I}O(X)$. Since $\mathrm{Int}(\mathrm{Cl}(Y\backslash K)) = Y\backslash \mathrm{Cl}(\mathrm{Int}(K))$, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(Y\backslash K))) = F^+(Y\backslash \mathrm{Cl}(\mathrm{Int}(K))) = X\backslash F^-(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain that $F^-(\mathrm{Cl}(\mathrm{Int}(K)))$ is semi-$\mathfrak{I}$-closed in X.
(6)⇒(5): It can be obtained similarly as (5)⇒(6).
(5)⇒(7): Let G be any regular open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(G))) = F^+(G) \in S\mathfrak{I}O(X)$.
(7)⇒ (5): Let V be any open set of Y. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (7), $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in S\mathfrak{I}O(X)$.
(6)⇒(8): It can be obtained similarly as (5)⇒(7).
(8)⇒(6): It can be obtained similarly as (7)⇒(5).
(5)⇒(9): Let $x \in X$ and V be a neighbourhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. Then we have $x \in F^+(G) \subset F^+(V)$. Since $F^+(\mathrm{Int}(\mathrm{Cl}(G))) \in S\mathfrak{I}O(X)$, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is a semi-$\mathfrak{I}$-neighbourhood of x.
(9)⇒(10): Let $x \in X$ and V be a neighbourhood of $F(x)$. By (9), $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is a semi-$\mathfrak{I}$-neighbourhood of x. Take $U = F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset$

$\text{Int}(\text{Cl}(V))$.
(10)⇒(1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighbourhood of $F(x)$. By (10), there exists a semi-$\mathcal{I}$-neighbourhood U of x such that $F(U) \subset \text{Int}(\text{Cl}(V))$. Therefore, there exists $G \in S\mathcal{I}O(X)$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset \text{Int}(\text{Cl}(V))$. We obtain that F is upper almost semi-$\mathcal{I}$-continuous. □

Theorem 3.4. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is lower almost semi-$\mathcal{I}$-continuous multifunction,*
2. *for each $x \in X$ and for each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in S\mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap \text{Int}(\text{Cl}(V)) \neq \emptyset$,*
3. *for each $x \in X$ and for each regular open set G of Y such that $F(x) \cap G \neq \emptyset$, there exists $U \in S\mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap G \neq \emptyset$,*
4. *for each $x \in X$ and for each closed set K such that $x \in F^-(Y \backslash K)$, there exists a semi-$\mathcal{I}$-closed set H such that $x \in X \backslash H$ and $F^+(\text{Cl}(\text{Int}(K))) \subset H$,*
5. *$F^-(\text{Int}(\text{Cl}(V))) \in S\mathcal{I}O(X)$ for any open set $V \subset Y$,*
6. *$F^+(\text{Cl}(\text{Int}(K))) \in S\mathcal{I}C(X)$ for any closed set $K \subset Y$,*
7. *$F^-(G) \in S\mathcal{I}O(X)$ for any regular open set G of Y,*
8. *$F^+(K) \in S\mathcal{I}C(X)$ for any regular closed set K of Y.*

Proof. We Prove only (1)⇒(2), (2)⇒(3), (3)⇒(4). The other proofs can be obtained similarly as Theorem 3.3.
(1)⇒(2): Let $x \in X$ and V be an open subset of Y such that $F(x) \cap V \neq \emptyset$. Since F is lower almost semi-$\mathcal{I}$-continuous, there exists $U \in S\mathcal{I}O(X, x)$ such that $U \subset F^-(\text{Int}(\text{Cl}(V)))$. This implies that if $y \in U$, then $F(y) \cap \text{Int}(\text{Cl}(V)) \neq \emptyset$.
(2)⇒(3): Let $x \in x$ and G be a regular open subset of Y such that $F(x) \cap G \neq \emptyset$. Then $G = \text{Int}(\text{Cl}(G))$ is open in Y. By (2), there exits $U \in S\mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap \text{Int}(\text{Cl}(G)) \neq \emptyset$. That is, if $y \in U$, then $F(y) \cap G \neq \emptyset$.
(3)⇒(4): Let $x \in X$ and K be a closed subset of Y such that $x \in F^-(Y \backslash K)$. Then $\text{Int}(\text{Cl}(Y \backslash K))$ is regular open in Y such that $x \in F^-(\text{Int}(\text{Cl}(Y \backslash K)))$. Thus $F(x) \cap \text{Int}(\text{Cl}(Y \backslash K)) \neq \emptyset$. By (3), there exits $U \in S\mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap \text{Int}(\text{Cl}(Y \backslash K)) \neq \emptyset$. Hence $U \subset F^-(\text{Int}(\text{Cl}(Y \backslash K)))$, and so $U \subset X \backslash F^+(\text{Cl}(\text{Int}(K)))$. Set $L = X \backslash U$. Then L is a semi-$\mathcal{I}$-closed set such that $x \in X \backslash L$ and $F^+(\text{Cl}(\text{Int}(K))) \subset L$.
(4)⇒(1): Let $x \in x$ and V be an open subset of Y such that $x \in F^-(V)$. Then $Y \backslash V$ is closed in Y such that $x \in F^-(Y \backslash (Y \backslash V))$. By (4), there exits a semi-$\mathcal{I}$-closed set L such that $x \in X \backslash L$ and $F^+(\text{Cl}(\text{Int}(Y \backslash V))) \subset L$. Set $U = X \backslash L$. Thus U is semi-$\mathcal{I}$-open in X such that $x \in U$ and $U \subset F^-(\text{Int}(\text{Cl}(V)))$. Therefore, F is lower almost semi-$\mathcal{I}$-continuous. □

Theorem 3.5. *The following are equivalent for a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$:*

1. *F is upper almost semi-$\mathfrak{I}$-continuous;*
2. *$s\mathfrak{I}\operatorname{Cl}(F^-(V)) \subset F^-(\operatorname{Cl}(V))$ for every $V \in SPO(Y)$;*
3. *$s\mathfrak{I}\operatorname{Cl}(F^-(V)) \subset F^-(\operatorname{Cl}(V))$ for every $V \in SO(Y)$;*
4. *$F^+(V) \subset s\mathfrak{I}\operatorname{Int}(F^+(\operatorname{Int}(\operatorname{Cl}(V))))$ for every $V \in PO(Y)$.*

Proof. (1)→(2). Let V be any semi-$\mathfrak{I}$-open set of Y. Since $\operatorname{Cl}(V) \in RC(Y)$, by Theorem 3.3 $F^-(\operatorname{Cl}(V))$ is semi-$\mathfrak{I}$-closed in X and $F^-(V) \subset F^-(\operatorname{Cl}(V))$. Therefore, we obtain $s\mathfrak{I}\operatorname{Cl}(F^-(V)) \subset F^-(\operatorname{Cl}(V))$.
(2)→(3). This is obvious since $SO(Y) \subset SPO(Y)$.
(3)→(4). Let $V \in PO(Y)$. Then, we have $V \subset \operatorname{Int}(\operatorname{Cl}(V))$ and $Y\backslash V \supset \operatorname{Cl}(\operatorname{Int}(Y\backslash V))$. Since $\operatorname{Cl}(\operatorname{Int}(Y\backslash V)) \in SO(Y)$, $X\backslash F^+(V) = F^-(Y\backslash V) \supset F^-(\operatorname{Cl}(\operatorname{Int}(Y\backslash V))) \supset s\mathfrak{I}\operatorname{Cl}(F^-(\operatorname{Cl}(\operatorname{Int}(Y\backslash V)))) =$
$s\mathfrak{I}\operatorname{Cl}(F^-(Y\backslash \operatorname{Int}(\operatorname{Cl}(V)))) = s\mathfrak{I}\operatorname{Cl}\operatorname{Cl}(X\backslash F^+(\operatorname{Int}(\operatorname{Cl}(V))))$
$= X\backslash s\mathfrak{I}\operatorname{Int}(F^+(\operatorname{Int}(\operatorname{Cl}(V))))$. Therefore, $F^+(V) \subset s\mathfrak{I}\operatorname{Int}(F^+(\operatorname{Int}(\operatorname{Cl}(V))))$.
(4)→(1). Let V be any regular open set of Y. Since $V \in PO(Y)$, we have $F^+(V) \subset s\mathfrak{I}\operatorname{Int}(F^+(\operatorname{Int}(\operatorname{Cl}(V)))) = s\mathfrak{I}\operatorname{Int}(F^+(V))$ and hence $F^+(V) \in S\mathfrak{I}O(X)$. It follows from Theorem 3.3, that F is upper almost semi-$\mathfrak{I}$-continuous. □

Theorem 3.6. *The following are equivalent for a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$:*

1. *F is lower almost semi-$\mathfrak{I}$-continuous;*
2. *$s\mathfrak{I}\operatorname{Cl}(F^+(V)) \subset F^+(\operatorname{Cl}(V))$ for every $V \in SPO(Y)$;*
3. *$s\mathfrak{I}\operatorname{Cl}(F^+(V)) \subset F^+(\operatorname{Cl}(V))$ for every $V \in SO(Y)$;*
4. *$F^-(V) \subset s\mathfrak{I}\operatorname{Int}(F^-(\operatorname{Int}(\operatorname{Cl}(V))))$ for every $V \in PO(Y)$.*

Proof. The proof is similar to that of Theorem 3.5 and is thus omitted. □

Definition 3.7. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space and let (x_α) be a net in X. It is said that the net (x_α) semi-$\mathfrak{I}$-converges to x if for each semi-$\mathfrak{I}$-open set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.*

Theorem 3.8. *If $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ is a lower (upper) almost semi-$\mathfrak{I}$-continuous multifunction, then for each $x \in X$ and for each net (x_α) which semi-$\mathfrak{I}$-converges to x in X and for each open set $V \subset Y$ such that $x \in F^-(V)$ (resp. $x \in F^+(V)$), the net (x_α) is eventually in $F^-(\operatorname{Int}(\operatorname{Cl}(V)))$ (resp. $F^+(\operatorname{Int}(\operatorname{Cl}(V)))$).*

Proof. Let (x_α) be a net semi-$\mathcal{I}$-converges to x in X and let V be any open set in Y such that $x \in F^-(V)$. Since F is lower almost semi-$\mathcal{I}$-continuous multifunction, there exists a semi-$\mathcal{I}$-open set U in X containing x such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. Since (x_α) semi-$\mathcal{I}$-converges to x, there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(\mathrm{Int}(\mathrm{Cl}(V)))$.
The proof of the upper almost semi-$\mathcal{I}$-continuity of F is similar to the above. □

Definition 3.9. *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization. In case when $\tau = \tau^*$, the space (X, τ) is called semiregular [20].*

Theorem 3.10. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction from a topological space (X, τ) to a semiregular topological space (Y, σ). Then F is lower almost semi-$\mathcal{I}$-continuous multifunction if and only if F is lower semi-$\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y, σ) is a semiregular space, there exist regular open sets U_i for $i \in I$ such that $V = \underset{i\in I}{\cup} U_i$. We have $F^-(V) = F^-(\underset{i\in I}{\cup} U_i) = \underset{i\in I}{\cup} F^-(U_i)$. By Theorem 3.3, $F^-(U_i) \in S\mathcal{I}O(X)$ for $i \in I$. We obtain $F^-(V) \in S\mathcal{I}O(X)$. Hence F is lower semi-$\mathcal{I}$-continuous. The converse is obvious. □

Corollary 3.11. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost semi-$\mathcal{I}$-continuous multifunction if and only if $F : (X, \tau, \mathcal{I}) \to (Y, \sigma^*)$ is lower semi-$\mathcal{I}$-continuous.*

Suppose that (X, τ), (Y, σ) and (Z, η) are topological spaces. It is known that if $F_1 : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ and $F_2 : (Y, \sigma) \to (Z, \eta)$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : (X, \tau) \to (Z, \eta)$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 3.12. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper (lower) semicontinuous multifunction and $G : (Y, \sigma) \to (Z, \eta)$ is an upper (lower) semicontinuous multifunction, then $G \circ F : (X, \tau) \to (Z, \eta)$ is an upper (lower) almost semi-$\mathcal{I}$-continuous multifunction.*

Proof. Let $V \subset Z$ be any regular open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is upper (lower) semicontinuous multifunction, $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper (lower) semi-$\mathcal{I}$-continuous multifunction, $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is a semi-$\mathcal{I}$-open set. It shows that $G \circ F$ is an upper (resp. lower) almost semi-$\mathcal{I}$-continuous multifunction. □

Theorem 3.13. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous if and only if $s\,\mathrm{Cl}\,F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous, where $s\,\mathrm{Cl}\,F(x) = s\,\mathrm{Cl}(F(x))$ for each point $x \in X$.*

Proof. Suppose that F is upper almost semi-$\mathcal{I}$-continuous. Let V be any open set of Y such that $s\,\mathrm{Cl}\,F(x) \subset V$. Then $F(x) \subset V$ and by Theorem 3.3, there exists $U \in S\mathcal{I}O(X, x)$ such that $F(U) \subset s\,\mathrm{Cl}(V)$. For each $u \in U$, $F(U) \subset s\,\mathrm{Cl}(V)$ and hence $(s\,\mathrm{Cl}\,F)^+(V) \subset s\mathcal{I}\,\mathrm{Int}(s\,\mathrm{Cl}\,F)^+(s\,\mathrm{Cl}(V))$. It follows from Theorem 3.3, that $s\,\mathrm{Cl}\,F$ is upper almost semi-$\mathcal{I}$-continuous. Conversely, suppose that $s\,\mathrm{Cl}\,F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous. Let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and $s\,\mathrm{Cl}\,F(x) \subset s\,\mathrm{Cl}(V)$. There exists $U \in S\mathcal{I}O(X, x)$ such that $s\,\mathrm{Cl}\,F(U) \subset s\,\mathrm{Cl}(V)$. Therefore, we have $U \subset (s\,\mathrm{Cl}\,F)^+(s\,\mathrm{Cl}(V)) \subset F^+(s\,\mathrm{Cl}(V))$ and hence $x \in U \subset s\mathcal{I}\,\mathrm{Int}(F^+(s\,\mathrm{Cl}(V)))$. Thus, we obtain $F^+(V) \subset s\mathcal{I}\,\mathrm{Int}(F^+(s\,\mathrm{Cl}(V)))$ and by Theorem 3.3, F is upper almost semi-$\mathcal{I}$-continuous. □

Theorem 3.14. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost semi-$\mathcal{I}$-continuous if and only if $s\,\mathrm{Cl}\,F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost semi-$\mathcal{I}$-continuous.*

Proof. Suppose that F is lower almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $s\,\mathrm{Cl}(F)(x) \cap V \neq \emptyset$. Then we have $x \in (s\,\mathrm{Cl}\,F)^-(V) = F^-(V)$ and $F(x) \cap V \neq \emptyset$. By Theorem 3.4, there exists $U \in S\mathcal{I}O(X, x)$ such that $F(U) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Therefore, we obtain that $(s\,\mathrm{Cl}\,F)(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. It follows from Theorem 3.4, that $s\,\mathrm{Cl}\,F$ is lower almost semi-$\mathcal{I}$-continuous. Conversely, suppose that $s\,\mathrm{Cl}\,F$ is lower almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F^-(V) = (s\,\mathrm{Cl}\,F)^-(V)$ and hence $(s\,\mathrm{Cl}\,F)(x) \cap V \neq \emptyset$. Since $s\,\mathrm{Cl}\,F$ is lower almost semi-$\mathcal{I}$-continuous, by Theorem 3.4, there exists $U \in S\mathcal{I}O(X, x)$ such that $(s\,\mathrm{Cl}\,F)(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Therefore, we obtain that $F(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. It follows from Theorem 3.4, F is lower almost semi-$\mathcal{I}$-continuous. □

Definition 3.15. *A subset A of a topological space (X, τ) is said to be:*

1. *α-regular [9] if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;*
2. *α-paracompact [9] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.*

Lemma 3.16. *[9] If A is an α-paracompact and α-regular set of a topological space (X, τ) and U an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 3.17. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then we have the following*

1. *$G^+(V) = F^+(V)$ for each open set V of Y,*
2. *$G^-(V) = F^-(V)$ for each closed set V of Y, where G denotes $\mathrm{Cl}\,F$ or $s\mathcal{I}\,\mathrm{Cl}\,F$.*

Proof. (1). Let V be any open set of Y and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 3.16, there exists an open set H of V such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $F^+(V) \subset G^+(V)$. Therefore, $G^+(V) = F^+(V)$.
(2). This follows from (1). □

Theorem 3.18. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*

1. *F is upper almost semi-$\mathfrak{I}$-continuous;*
2. *$s\mathfrak{I}\,\mathrm{Cl}\,F$ is upper almost semi-$\mathfrak{I}$-continuous;*
3. *$\mathrm{Cl}\,F$ is upper almost semi-$\mathfrak{I}$-continuous.*

Proof. We put $G = s\mathfrak{I}\,\mathrm{Cl}\,F$ or $\mathrm{Cl}\,F$. Suppose that F is upper almost semi-$\mathfrak{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $G(x)$. By Lemma 3.17, $x \in G^+(V) = F^+(V)$ and there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset s\,\mathrm{Cl}(V)$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset s\,\mathrm{Cl}(V)$ hence $G(u) \subset \mathrm{Cl}(H) \subset s\,\mathrm{Cl}(V)$ for each $u \in U$. Therefore, we obtain $G(U) \subset s\,\mathrm{Cl}(V)$. This shows that G is upper almost semi-$\mathfrak{I}$-continuous. Conversely, suppose that G is upper almost semi-$\mathfrak{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 3.17, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in S\mathfrak{I}O(X, x)$ such that $G(U) \subset s\,\mathrm{Cl}(V)$. Therefore, we obtain $F(u) \subset s\,\mathrm{Cl}(V)$. This shows that F is upper almost semi-$\mathfrak{I}$-continuous. □

Theorem 3.19. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*

1. *F is lower almost semi-$\mathfrak{I}$-continuous;*
2. *$s\mathfrak{I}\,\mathrm{Cl}\,F$ is lower almost semi-$\mathfrak{I}$-continuous;*
3. *$\mathrm{Cl}\,F$ is lower almost semi-$\mathfrak{I}$-continuous.*

Proof. We put $G = s\mathfrak{I}\,\mathrm{Cl}\,F$ or $\mathrm{Cl}\,F$. Suppose that F is lower almost semi-$\mathfrak{I}$-continuous. Let $x \in x$ and V be any open set of Y such that $G(x) \cap V \neq \emptyset$. Since V is open, $F(x) \cap V \neq \emptyset$ and there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. Therefore, we obtain $G(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. This shows that G is lower almost semi-$\mathfrak{I}$-continuous. Conversely, suppose that G is lower almost semi-$\mathfrak{I}$-continuous. Let $x \in X$ and V any open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x) \subset G(x)$, $G(x) \cap V \neq \emptyset$ and there exists $U \in S\mathfrak{I}O(X, x)$ such that $G(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. Therefore, by Theorem 3.4, F is lower almost semi-$\mathfrak{I}$-continuous. □

Theorem 3.20. *For a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ such that $F(x)$ is an α-regular and α-paracompact set for each $x \in X$, the following are equivalent:*

1. *F is upper weakly semi-$\mathfrak{I}$-continuous,*
2. *F is upper almost semi-$\mathfrak{I}$-continuous,*
3. *F is upper semi-$\mathfrak{I}$-continuous.*

Proof. (1)⇒(3). Suppose that F is upper weakly semi-$\mathfrak{I}$-continuous. Let $x \in X$ and G and open set of Y such that $F(x) \subset G$. Since $F(x)$ is α-regular α-paracompact, by Lemma 3.16, there exists an open set V such that $F(x) \subset V \subset \text{Cl}(V) \subset G$. Since F is upper weakly semi-$\mathfrak{I}$-continuous at x and $F(x) \subset V$, there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset \text{Cl}(V)$ and hence $F(U) \subset \text{Cl}(V) \subset G$. Therefore, F is upper semi-$\mathfrak{I}$-continuous. □

Corollary 3.21. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$ and Y is regular. Then, the following are equivalent:*

1. *F is upper weakly semi-$\mathfrak{I}$-continuous;*
2. *F is upper almost semi-$\mathfrak{I}$-continuous;*
3. *F is upper semi-$\mathfrak{I}$-continuous.*

Lemma 3.22. *[17] If A is an α-regular set of X, then for every open set G which intersects A, there exists an open set D such that $A \cap D \neq \emptyset$ and $\text{Cl}(D) \subset G$.*

Theorem 3.23. *For a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ such that $F(x)$ is an α-regular set of Y for each $x \in X$, the following are equivalent:*

1. *F is lower weakly semi-$\mathfrak{I}$-continuous,*
2. *F is lower almost semi-$\mathfrak{I}$-continuous,*
3. *F is lower semi-$\mathfrak{I}$-continuous.*

Proof. (1)⇒(3): Suppose that F is lower weakly semi-$\mathfrak{I}$-continuous. Let $x \in X$ and G an open set of Y such that $F(x) \cap G \neq \emptyset$. Since $F(x)$ is α-regular, by Lemma 3.22, there exists an open set D of Y such that $F(x) \cap D \neq \emptyset$ and $\text{Cl}(D) \subset G$. Since F is lower weakly semi-$\mathfrak{I}$-continuous at x, there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(u) \cap \text{Cl}(D) \neq \emptyset$ for each $u \in U$. Since $\text{Cl}(D) \subset G$, we have $F(u) \cap G \neq \emptyset$ for each $u \in U$. Therefore, F is lower semi-$\mathfrak{I}$-continuous. □

Theorem 3.24. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following are equivalent:*

1. *F is upper weakly semi-$\mathfrak{I}$-continuous,*
2. *F is upper almost semi-$\mathfrak{I}$-continuous,*

3. F is upper semi-$\mathcal{I}$-continuous.

Proof. (1)⇒(3): Suppose that F is upper weakly semi-$\mathcal{I}$-continuous. Let $x \in X$ and G an open set of Y containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y there exists an open set V of Y such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly semi-$\mathcal{I}$-continuous, there exists $U \in S\mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper semi-$\mathcal{I}$-continuous. □

Definition 3.25. *A topologial space (X, τ) is said to be rimcompact if each point of X has a base of neighborhoods with compact frontiers.*

Theorem 3.26. *If (Y, σ) is a rimcompact space and $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a compact valued multifunction with the closed graph, then the following are equivalent:*

1. *F is upper weakly semi-$\mathcal{I}$-continuous;*
2. *F is upper almost semi-$\mathcal{I}$-continuous;*
3. *F is upper semi-$\mathcal{I}$-continuous.*

Proof. Suppose that F is upper weakly α-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since Y is rimcompact, for each $z \in F(x)$. Since Y is rimcompact, for each $z \in F(x)$ there exists an open set $W(z)$ such that $z \in W(z) \subset V$ and the frontier $Fr(W(z))$ is compact. The family $\{W(z) : z \in F(x)\}$ is a cover of $F(x)$ by open sets of Y. Since $F(x)$ is compact, there exists a finite number of points, say, $z_1, z_2, ..z_n$ in $F(x)$ such that $F(x) \subset \cup\{W(z_j) : 1 \leq j \leq n\}$. Let $W = \cup\{W(z_j) : 1 \leq j \leq n\}$, then we have $Fr(W)$ is compact, $F(x) \subset W \subset V$ and $F(x) \cap Fr(W) = F(x) \cap \mathrm{Cl}(W) \cap \mathrm{Cl}(Y \backslash W) \subset F(x) \cap Y \backslash W = \emptyset$. For each $y \in Fr(W)$, $(x, y) \in X \times Y \backslash G(F)$. Since $G(F)$ is closed, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $F(U(y)) \cap V(y) = \emptyset$. The family $\{V(y) : y \in Fr(W)\}$ is a cover of $Fr(W)$ by open sets of Y. Since $Fr(W)$ is compact, there exists a finite subset K of $Fr(W)$ such that $Fr(W) \subset \cup\{V(y) : y \in K\}$. Since F is upper weakly semi-$\mathcal{I}$-continuous, there exits $U_0 \in S\mathcal{I}O(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0 \cap (\cap\{U(y) : y \in K\})$. Then we obtain $U \in S\mathcal{I}O(X, x)$, $F(U) \subset \mathrm{Cl}(W)$ and $F(U) \cap Fr(W) = \emptyset$. Therefore, we obtain $F(U) \subset W \subset V$. This shows that F is upper semi-$\mathcal{I}$-continuous. □

Corollary 3.27. *If (Y, σ) is a rimcompact space and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an almost semi-$\mathcal{I}$-continuous function with closed graph, then f is semi-$\mathcal{I}$-continuous.*

For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the graph multifunction $G_F : X \Rightarrow X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 3.28. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following hold:*

1. *$G^+F(A \times B) = A \cap F^+(B)$,*
2. *$G^-F(A \times B) = A \cap F^-(B)$*

for any subsets $A \subset X$ and $B \subset Y$ [15].

Theorem 3.29. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper almost semi-$\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is upper almost semi-$\mathcal{I}$-continuous.*

Proof. Suppose that $G_F : X \to X \times Y$ is upper almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in S\mathcal{I}O(X, x)$ such that $G_F(U) \subset \text{Int}(\text{Cl}(X \times V)) = X \times \text{Int}(\text{Cl}(V))$. By Lemma 3.28, we have $U \subset G_F^+(X \times \text{Int}(\text{Cl}(V))) = F^+(\text{Int}(\text{Cl}(V)))$ and $F(U) \subset \text{Int}(\text{Cl}(V))$. This shows that F is upper almost semi-$\mathcal{I}$-continuous. Conversely, suppose that $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family of $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, say $y_1, y_2, y_3, ..., y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : i = 1, 2, ..., n\}$. Take $U = \cap\{U(y_i) : i = 1, 2, ..., n\}$ and $V = \cup\{V(y_i) : i = 1, 2, ..., n\}$. Then U and V are open sets in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is upper almost semi-$\mathcal{I}$-continuous, there exists $U_0 \in S\mathcal{I}O(X, x)$ such that $F(U_0) \subset \text{Int}(\text{Cl}(V))$. By Lemma 3.28, we have $U \cap U_0 \subset U \cap F^+(\text{Int}(\text{Cl}(V))) = G_F^+(U \times \text{Int}(\text{Cl}(V))) \subset G_F^+(\text{Int}(\text{Cl}(U \times V))) \subset G_F^+(\text{Int}(\text{Cl}(W)))$. Therefore, we obtain $U \cap U_0 \in S\mathcal{I}O(X, x)$ and $G_F(U \cap U_0) \subset \text{Int}(\text{Cl}(W))$. This shows that G_F is upper almost semi-$\mathcal{I}$-continuous. □

Theorem 3.30. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost semi-$\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is lower almost semi-$\mathcal{I}$-continuous.*

Proof. Suppose that F is lower almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in S\mathcal{I}O(X, x)$ such that $G \subset F^-(\text{Int}(\text{Cl}(V)))$. By Lemma 3.28, $U \cap G \subset U \cap F^-(\text{Int}(\text{Cl}(V))) = G_F^-(U \times \text{Int}(\text{Cl}(V))) \subset G_F^-(\text{Int}(\text{Cl}(W)))$. Furthermore, $x \in U \cap G \in S\mathcal{I}O(X)$ and hence G_F is lower almost semi-$\mathcal{I}$-continuous. Conversely, suppose that G_F is lower almost semi-$\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost semi-$\mathcal{I}$-continuous, there exists a semi-$\mathcal{I}$-open set U containing x such that $U \subset G_F^-(\text{Int}(\text{Cl}(X \times V)))$. Since $G_F^-(\text{Int}(\text{Cl}(X \times V))) = G_F^-(X \times \text{Int}(\text{Cl}(V)))$, by Lemma 3.28, we have $U \subset F^-(\text{Int}(\text{Cl}(V)))$. This shows that F is lower almost semi-$\mathcal{I}$-continuous. □

Definition 3.31. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The multigraph $G(F)$ is said to be semi-$\mathcal{I}$-closed graph in $X \times Y$ if for each $(x, y) \in X \times Y \backslash G(F)$, there exist semi-$\mathcal{I}$-open set U and an open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.*

Theorem 3.32. *Let $F : (X,\tau) \to (Y,\sigma)$ be an upper almost semi-$\mathfrak{I}$-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y,σ). Then the multigraph $G(F)$ of F is a semi-$\mathfrak{I}$-closed graph in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y,σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighbourhood of y_0 such that W_0 intersects only finitely many members $U_{\beta_1}, U_{\beta_2}, ..., U_{\beta_n}$ of Φ. Choose $y_1, y_2, ..., y_n$ in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, ...n$ and set $W = W_0 \cap (\bigcap_{i=1}^{n} W(y_i))$. Then W is an open neighbourhood of y_0 with $W \cap (\bigcup_{\beta \in I} U_\beta) = \emptyset$, which implies that $W \cap \text{Int}(\text{Cl}(\bigcup_{\beta \in I} U_\beta)) = \emptyset$. By the upper almost $\mathfrak{I}$ continuity of F, there exists $U \in S\mathfrak{I}O(X, x_0)$ such that $F(U) \subset \text{Int}(\text{Cl}(\bigcup_{\beta \in I} U_\beta))$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, the graph $G(F)$ is a semi-$\mathfrak{I}$-closed graph in $X \times Y$. □

Let $\{X_\alpha : \alpha \in \nabla\}$ and $\{Y_\alpha : \alpha \in \nabla\}$ be any two families of topological spaces with same index set ∇. For each $\alpha \in \nabla$, let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction. The product space $\Pi\{X_\alpha : \alpha \in \nabla\}$ will be denoted by ΠX_α and the product multifunction $\Pi F_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$, defined by $F(x) = \Pi\{F_\alpha(x_\alpha) : \alpha \in \nabla\}$ for each $x = \{x_\alpha\} \in \Pi X_\alpha$, is simply denoted by $F : \Pi X_\alpha \to Y_\alpha$.

Theorem 3.33. *Let $F_\alpha : (X,\tau,\mathfrak{I}) \to (Y,\sigma)_\alpha$ be a multifunction for each $\alpha \in \nabla$ and $F : X \to \Pi Y_\alpha$ a multifunction defined by $F(x) = \Pi\{F_\alpha(x) : \alpha \in \nabla\}$ for each $x \in X$. If F is upper almost semi-$\mathfrak{I}$-continuous (resp. lower almost semi-$\mathfrak{I}$-continuous), then F_α is upper almost semi-$\mathfrak{I}$-continuous (resp. lower almost semi-$\mathfrak{I}$-continuous) for each $\alpha \in \nabla$.*

Proof. Let $x \in X$, $\alpha \in \nabla$ and V_α any regular open set of Y_α containing $F_\alpha(x)$. Then $P_\alpha^{-1}(V_\alpha) = V_\alpha \times \Pi\{Y_\beta : \beta \in \nabla \text{ and } \beta \neq \alpha\}$ is a regular open set of ΠY_α containing $F(x)$, where P_α is the natural projection of ΠY_α onto Y_α. Since F is upper almost semi-$\mathfrak{I}$-continuous, there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset p_\alpha^{-1}(V_\alpha)$. Therefore, we obtain $F_\alpha(U) \subset P_\alpha(F(U)) \subset P_\alpha(P_\alpha^{-1}(V_\alpha)) = V_\alpha$. This shows that $F_\alpha : (X,\tau,\mathfrak{I}) \to (Y,\sigma)_\alpha$ is upper almost semi-$\mathfrak{I}$-continuous for each $\alpha \in \nabla$. The proof for lower almost semi-$\mathfrak{I}$-continuous is similar and is thus omitted. □

Theorem 3.34. *If (Y,σ) is a Hausdorff space and $F, G : (X,\tau,\mathfrak{I}) \to (Y,\sigma)$ are multifunctions such that*

1. *$F(x)$ and $G(x)$ are compact for each $x \in X$,*
2. *G is upper weakly semi-$\mathfrak{I}$-continuous,*
3. *F is upper almost semi-$\mathfrak{I}$-continuous,*

then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ *is semi-$\mathfrak{I}$-closed in* X*.*

Proof. Let $x \in X \backslash A$. Then $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are compact in a Hausdorff space (Y, σ), there exist disjoint regular open sets V and W of Y such that $F(x) \subset V$, $G(x) \subset W$ and $V \cap \mathrm{Cl}(W) = \emptyset$. Since F is upper almost semi-$\mathfrak{I}$-continuous, there exists $U_1 \in S\mathfrak{I}O(X, x)$ such that $F(U_1) \subset V$. Since G is upper weakly semi-$\mathfrak{I}$-continuous, there exists $U_2 \in S\mathfrak{I}O(X, x)$ such that $G(U_2) \subset \mathrm{Cl}(W)$. Put $U = U_1 \cap U_2$, then $U \in S\mathfrak{I}O(X, x)$ and $F(U) \cap G(U) = \emptyset$. Therefore, we obtain $U \cap A = \emptyset$ and $x \in X \backslash s\mathfrak{I}\,\mathrm{Cl}(A)$. Hence A is semi-$\mathfrak{I}$-closed in X. □

Theorem 3.35. *If* $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ *is an upper almost semi-$\mathfrak{I}$-continuous multifunction such that* $F(x)$ *is α-nearly paracompact for each* $x \in X$ *and* Y *is Hausdorff, then for each* $(x, y) \in X \times Y \backslash G(F)$*, there exist* $U \in S\mathfrak{I}O(X, x)$ *and an open set* V *containing* y *such that* $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$*.*

Proof. Let $(x, y) \in X \times Y \backslash G(F)$, then $y \in Y \backslash F(x)$. Since Y is Hausdorff, for each $a \in F(X)$ there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$, hence $\mathrm{Int}(\mathrm{Cl}(V(a))) \cap W(a) = \emptyset$. The family $V = \{\mathrm{Int}(\mathrm{Cl}(V(a))) : a \in F(x)\}$ is a cover of $F(x)$ by regular open sets of Y and $F(x)$ is α-nearly paracompact. There exists a locally finite open refinement $H = \{H_\alpha : \alpha \in \triangledown\}$ of V such that $F(x) \subset \cup\{H_\alpha : \alpha \in \triangledown\}$. Since H is locally finite, there exists an open neighborhood W_0 of Y and a finite subset $\triangledown_0$ of $\triangledown$ such that $W_0 \cap H_\alpha = \emptyset$ for every $\alpha \in \triangledown \backslash \triangledown_0$. For each $\alpha \in \triangledown_0$, there exists $a(\alpha) \in F(x)$ such that $H_\alpha \subset V(a(\alpha))$. Now, put $W = W_0 \cap (\cap\{W(a(\alpha)) : \alpha \in \triangledown_0\})$ and $H = \cup\{H_\alpha : \alpha \in \triangledown\}$. Then W is an open neighborhood of y, H is open in Y and $W \cap H = \emptyset$. Therefore, we obtain $F(x) \subset H$ and $\mathrm{Cl}(W) \cap H = \emptyset$ an hence $F(x) \subset Y \backslash \mathrm{Cl}(W)$. Since W is open, $Y \backslash \mathrm{Cl}(W)$ is regular open in Y. Since F is upper almost semi-$\mathfrak{I}$-continuous, there exists $U \in S\mathfrak{I}O(X, x)$ such that $F(U) \subset Y \backslash \mathrm{Cl}(W)$, hence $F(U) \cap \mathrm{Cl}(W) = \emptyset$. Therefore, we obtain $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$. □

Corollary 3.36. *If* $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ *is an upper almost semi-$\mathfrak{I}$-continuous multifunction such that* $F(x)$ *is compact for each* $x \in X$ *and* Y *is Hausdorff, then for each* $(x, y) \in X \times Y \backslash G(F)$*, there exist* $U \in S\mathfrak{I}O(X, x)$ *and an open set* V *containing* y *such that* $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$*.*

Corollary 3.37. *If* $f : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ *is a semi-$\mathfrak{I}$-continuous function into a Hausdorff space* Y*, then* $G(f)$ *is semi-$\mathfrak{I}$-closed.*

Theorem 3.38. *Suppose that* (X, τ) *and* (X_α, τ_α) *are topological spaces, where* $\alpha \in J$*. Let* $F : X \to \prod_{\alpha \in J} X_\alpha$ *be a multifunction from* X *to the product space* $\prod_{\alpha \in J} X_\alpha$ *and let* $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ *be the projection for each* $\alpha \in J$*. If* F *is upper (lower) almost semi-$\mathfrak{I}$-continuous multifunction, then* $P_\alpha \circ F$ *is upper (resp. lower) almost semi-$\mathfrak{I}$-continuous multifunction for each* $\alpha \in J$*.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^+(P^+_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) = F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) \times \prod_{\alpha \neq \alpha_0} X_\alpha$ (resp.

$(P_{\alpha_0}\circ F)^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^-(P^-_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) = F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))\times \prod_{\alpha\neq\alpha_0} X_\alpha))$. Since F is upper (resp. lower) almost semi-$\mathcal{I}$-continuous multifunction and since $\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha$ is a regular open set, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha))$ is semi-$\mathcal{I}$-open in (X,τ). It shows that $P_{\alpha_0}\circ F$ is upper (lower) almost semi-$\mathcal{I}$-continuous multifunction. Hence, we obtain that $P_\alpha\circ F$ is upper (lower) almost semi-$\mathcal{I}$-continuous multifunction for each $\alpha\in J$. □

Theorem 3.39. *Suppose that for each $\alpha\in J$, $(X_\alpha,\tau_\alpha),(Y_\alpha,\sigma_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha\to Y_\alpha$ be a multifunction for each $\alpha\in J$ and let $F:\prod_{\alpha\in J} X_\alpha\to\prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha))=\prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is upper (lower) almost semi-$\mathcal{I}$-continuous multifunction, then each F_α is upper (resp. lower) almost semi-$\mathcal{I}$-continuous multifunction for each $\alpha\in J$.*

Proof. Let $V_\alpha\subseteq Y_\alpha$ be an open set. Then $\mathrm{Int}(\mathrm{Cl}(V_\alpha))\times\prod_{\alpha\neq\beta} Y_\beta$ is a regular open set. Since F is upper (lower) almost semi-$\mathcal{I}$-continuous multifunction, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_\alpha))\times\prod_{\alpha\neq\beta} Y_\beta) = F^+_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))\times\prod_{\alpha\neq\beta} X_\beta$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_\alpha))\times\prod_{\alpha\neq\beta} Y_\beta) = F^-_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))\times\prod_{\alpha\neq\beta} X_\beta$) is a semi-$\mathcal{I}$-open set. Consequently, we obtain that $F^+_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))$ (resp. $F^-_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha))))$ is a semi-$\mathcal{I}$-open set. Thus, we show that F_α is upper (resp. lower) almost semi-$\mathcal{I}$-continuous multifunction. □

Theorem 3.40. *Suppose that (X,τ), (Y,σ), (Z,η) are topological spaces and $F_1 : (X,\tau,\mathcal{I})\to(Y,\sigma)$, $F_2:(X,\tau)\to(Z,\eta)$ are multifunctions. Let $F_1\times F_2 : (X,\tau,\mathcal{I})\to(Y,\sigma)\times Z$ be a multifunction which is defined by $(F_1\times F_2)(x)=F_1(x)\times F_2(x)$ for each $x\in X$. If $F_1\times F_2$ is upper (lower) almost semi-$\mathcal{I}$-continuous multifunction, then F_1 and F_2 are upper (resp. lower) almost semi-$\mathcal{I}$-continuous multifunctions.*

Proof. Let $x\in X$ and let $K\subset Y$, $H\subset Z$ be open sets such that $x\in F_1^+(K)$ and $x\in F_2^+(H)$. Then we obtain that $F_1(x)\subset K$ and $F_2(x)\subset H$ and so $F_1(x)\times F_2(x)=(F_1\times F_2)(x)\subset K\times H$. We have $x\in(F_1\times F_2)^+(K\times H)$.Since $F_1\times F_2$ is upper almost semi-$\mathcal{I}$-continuous multifunction, there exists a semi-$\mathcal{I}$-open set U containing x such that $U\subset(F_1\times F_2)^+(\mathrm{Int}(\mathrm{Cl}(K\times H)))$. We obtain that $U\subset F_1^+(\mathrm{Int}(\mathrm{Cl}(K)))$ and $U\subset F_2^+(\mathrm{Int}(\mathrm{Cl}(H)))$. Thus, we obtain that F_1 and F_2 are upper almost semi-$\mathcal{I}$-continuous multifunctions. The proof of the lower almost $\mathcal{I}$ continuity of F_1 and F_2 is similar to the above. □

Lemma 3.41. *[5] Let A and B be subsets of an ideal topological space $(X,\tau,\mathcal{I})$. Then*

1. *If $A\in S\mathcal{I}O(X)$ and $B\in\tau$, then $A\cap B\in S\mathcal{I}O(B)$;*

2. *If $A \in S\mathcal{I}O(B)$ and $B \in S\mathcal{I}O(X)$, then $A \in S\mathcal{I}O(X)$.*

Lemma 3.42. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper almost semi-$\mathcal{I}$-continuous (lower almost semi-$\mathcal{I}$-continuous) multifunction and $U \in \tau$, then $F_{|U} : (U, \tau_U, \mathcal{I}_U) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous (lower almost semi-$\mathcal{I}$-continuous).*

Proof. Suppose that V is an open subset of Y. Let $x \in U$ and let $x \in (F_{|U})^-(V)$. Since F is lower almost semi-$\mathcal{I}$-continuous multifunction, there exists a semi-$\mathcal{I}$-open set G such that $x \in G \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. By Lemma 3.41, we obtain that $x \in G \cap U \in S\mathcal{I}O(U)$ and $G \cap U \subset (F_{|U})^-(\mathrm{Int}(\mathrm{Cl}(V)))$. Hence $F_{|U}$ is lower almost semi-$\mathcal{I}$-continuous. The proof of the upper almost semi-$\mathcal{I}$-continuity of $F_{|U}$ is similar to the above. □

Theorem 3.43. *Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a space (X, τ). Then a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous (resp. lower almost semi-$\mathcal{I}$-continuous) if and only if the restriction $F_{|U_\alpha} : (U_\alpha, \tau_\alpha) \to (Y, \sigma)$ is upper almost semi-$\mathcal{I}$-continuous (resp. lower almost semi-$\mathcal{I}$-continuous) for each $\alpha \in \Lambda$.*

Proof. We prove only the case for F upper almost semi-$\mathcal{I}$-continuous, the proof for F lower almost semi-$\mathcal{I}$-continuous being analogous. Let $\alpha \in \Lambda$ and V be any open set of Y. Since F is upper almost semi-$\mathcal{I}$-continuous, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is semi-$\mathcal{I}$-open in X. By Lemma 3.41, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V))) = F^+(\mathrm{Int}(\mathrm{Cl}(V))) \cap U_\alpha$ is semi-$\mathcal{I}$-open in U_α and hence $F_{|U_\alpha}$ is upper almost semi-$\mathcal{I}$-continuous. Conversely, let V be any open set of Y. Since $F_{|U_\alpha}$ is upper almost semi-$\mathcal{I}$-continuous for each $\alpha \in \Lambda$, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V))) = F^+(\mathrm{Int}(\mathrm{Cl}(V))) \cap U_\alpha$ is semi-$\mathcal{I}$-open in U_λ. By Lemma 3.41, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is semi-$\mathcal{I}$-open in X for each $\alpha \in \Lambda$. We obtain that $F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \bigcup_{\alpha \in \Lambda} (F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is semi-$\mathcal{I}$-open in X. Hence F is upper almost semi-$\mathcal{I}$-continuous. □

Recall that a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be punctually connected if for each $x \in X$, $F(x)$ is connected.

Definition 3.44. *[13] An ideal topological space $(X, \tau, \mathcal{I})$ is called semi-$\mathcal{I}$-connected provided that X is not the union of two nonempty disjoint semi-$\mathcal{I}$-open sets.*

Theorem 3.45. *Let F be a multifunction from a semi-$\mathcal{I}$-connected topological space (X, τ) onto a topological space (Y, σ) such that F is punctually connected. If F is an upper almost semi-$\mathcal{I}$-continuous multifunction, then Y is a connected space.*

Proof. Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an upper almost semi-$\mathcal{I}$-continuous multifunction from an $\mathcal{I}$ connected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is upper almost semi-$\mathcal{I}$-continuous, $F^+(H)$ and $F^+(K)$ are semi-$\mathcal{I}$-open subsets of X. In view of the fact that $F^+(H), F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the semi-$\mathcal{I}$-connectedness of X. Hence, it is obtained that Y is a connected space. □

Recall that a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be punctually closed if for each $x \in X, F(x)$ is closed.

Theorem 3.46. *Let F be an upper almost semi-$\mathfrak{I}$-continuous punctually closed multifunction and G be an upper almost continuous punctually closed multifunction from a space (X, τ) to a normal space (Y, σ). Then the set $K = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is semi-$\mathfrak{I}$-closed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a normal space, there exists disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper almost semi-$\mathfrak{I}$-continuous and upper almost continuous, respectively the sets $F^+(\mathrm{Int}(\mathrm{Cl}(U)))$ and $G^+(\mathrm{Int}(\mathrm{Cl}(V)))$ are semi-$\mathfrak{I}$-open and open sets, respectively containing x. Let $H = F^+(\mathrm{Int}(\mathrm{Cl}(U))) \cap G^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then H is a semi-$\mathfrak{I}$-open set containing x and $H \cap K = \emptyset$. Hence, K is $\mathfrak{I}$ closed in X. □

Definition 3.47. *An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be semi-$\mathfrak{I}$-T_2 [13] if for each pair of distinct points x and y in X, there exist disjoint semi-$\mathfrak{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 3.48. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be an upper almost semi-$\mathfrak{I}$-continuous multifunction and punctually closed from a topological space (X, τ) to a normal topological space (Y, σ) and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a semi-$\mathfrak{I}$-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since (Y, σ) is a normal space, it follows that there exist disjoints open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(\mathrm{Int}(\mathrm{Cl}(U)))$ and $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ are disjoint semi-$\mathfrak{I}$-open sets containing x and y, respectively. Thus, it is obtained that (X, τ) is semi-$\mathfrak{I}$-T_2. □

Definition 3.49. *The semi-$\mathfrak{I}$-frontier of a subset A of a space (X, τ), denoted by $s\mathfrak{I}Fr(A)$, is defined by $s\mathfrak{I}Fr(A) = s\mathfrak{I}\,\mathrm{Cl}(A) \cap s\mathfrak{I}\,\mathrm{Cl}(X \backslash A) = s\mathfrak{I}\,\mathrm{Cl}(A) \backslash \mathfrak{I}\,\mathrm{Int}(A)$.*

Theorem 3.50. *The set all points of X at which a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ is not upper almost semi-$\mathfrak{I}$-continuous (lower almost semi-$\mathfrak{I}$-continuous) is identical with the union of the semi-$\mathfrak{I}$-frontier of the upper (lower) inverse images of regular open sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper almost semi-$\mathfrak{I}$-continuous. Then there exists a regular open set V of Y containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in S\mathfrak{I}O(X, x)$. Therefore, we have $x \in s\mathfrak{I}\,\mathrm{Cl}(X \backslash F^+(V)) = X \backslash \mathfrak{I}\,\mathrm{Int}(F^+(V))$ and $x \in F^+(V)$. Thus, we obtain $x \in s\mathfrak{I}Fr(F^+(V))$. Conversely, suppose that V is a regular open set of Y containing $F(x)$ such that $x \in \mathfrak{I}Fr(F^+(V))$. If F is upper almost semi-$\mathfrak{I}$-continuous at x, then there exists $U \in S\mathfrak{I}O(X, x)$ such that $U \subset F^+(V)$; hence $x \in s\mathfrak{I}\,\mathrm{Int}(F^+(V))$. This is a contradiction and hence F is not upper almost semi-$\mathfrak{I}$-continuous at x.
The case for lower almost semi-$\mathfrak{I}$-continuous is similarly shown. □

In the following $(D, >)$ is a directed set, (F_λ) is a net of multifunction $F_\lambda : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ for every $\lambda \in D$ and F is a multifunction from X into Y .

Definition 3.51. *[3] Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from X to Y. A multifunction $F^\star : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is defined as follows: for each $x \in X$, $F^\star(x) = \{y \in Y$: for each open neighborhood V of y and each $\mu \in D$, there exists $\lambda \in D$ such that $\lambda > \mu$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit of the net $(F_\lambda)_{\lambda \in D}$.*

Definition 3.52. *A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper almost semi-$\mathcal{I}$-continuous at $x_0 \in X$ if for every open set V containing $F_\lambda(x_0)$, there exists a semi-$\mathcal{I}$-open set U containing x_0 such that $F_\lambda(U) \subset \operatorname{Int}(\operatorname{Cl}(V_\lambda))$ for all $\lambda \in D$.*

Theorem 3.53. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a topological space (X, τ) into a compact space (Y, σ). If the following are satisfied:*

1. *$\cup\{F_\mu(x) : \mu > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$;*
2. *$(F_\lambda)_{\lambda \in D}$ is equally upper almost semi-$\mathcal{I}$-continuous on X, then $F^\star$ is upper almost semi-$\mathcal{I}$-continuous on X, then $F^\star$ is upper almost semi-$\mathcal{I}$-continuous on X.*

Proof. We have $F^\star(x) = \cap\{(\cup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\}$. Since the net $(\cup\{F_\mu(x) : \mu > \lambda\})_{\lambda \in D}$ is a family of closed sets having the finite intersection property and Y is compact, $F^\star(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper open subset of Y such that $F^\star(x_0) \subset V$. Since $F^\star(x_0) \cap (Y \backslash V) = \emptyset$, $F^\star(x_0) \neq \emptyset$ and $Y \backslash V \neq \emptyset$, $\cap\{(\cup\{F_\mu(x_0) : \mu > \lambda\}) : \lambda \in D\} \cap (Y \backslash V) = \emptyset$ and hence $\cap\{(\cup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\cup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that $F_\mu(x_0) \cap (Y \backslash V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda \in D}$ is equally upper almost semi-$\mathcal{I}$-continuous on X, there exists a semi-$\mathcal{I}$-open set U containing x_0 such that $F_\mu(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$ for each $\mu > \lambda$, that is, $F_\mu(x) \cap (Y \backslash \operatorname{Int}(\operatorname{Cl}(V))) = \emptyset$ for each $x \in U$. Then we have $\cup\{F_\mu(x) \cap (Y \backslash \operatorname{Int}(\operatorname{Cl}(V))) : \mu > \lambda\} = \emptyset$ and hence $\cap\{\cup\{F_\mu(x) : \mu > \lambda\} : \lambda \in D\} \cap (Y \backslash \operatorname{Int}(\operatorname{Cl}(V))) = \emptyset$. This implies that $F^*(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$. If $V = Y$, then it is clear that for each semi-$\mathcal{I}$-open set U containing x_0 we have $F^\star(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$. Hence $F^\star$ is upper almost semi-$\mathcal{I}$-continuous at x_0. Since x_0 is arbitrary, the proof completes. □

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