Invariants of First Order Partial Differential Equations *

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ABSTRACT: In this paper we introduce the concepts of multiplicity and index of first order partial differential equations. In particular, the concept of multiplicity coincides with the multiplicity of implicit differential equations given by Bruce and Tari in [2]. We also show that these concepts are invariants by smooth equivalences. Following the work [10] on implicit differential equations with first integrals, we introduce a definition of multiplicity for this class of equations.

Key Words: Multiplicity, First order partial differential equations, Implicit differential equation.

Contents

1. Introduction

Let $F(x_1, ..., x_n, y, p_1, ..., p_n) = 0$ be a first order partial differential equation (first order PDE), where F is a smooth function on $\mathbb{R}^{2n+1}$ (here smooth means C^∞). A classical solution of this equation is a smooth function $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $y = f(x_1, ..., x_n)$ and $p_i = \frac{\partial f}{\partial x_i}$. If $\frac{\partial F}{\partial p_i}(q_0) \neq 0$ at $q_0 \in \mathbb{R}^{2n+1}$ for some $i \in \{1, ..., n\}$, the first order PDE defines a family of classical solutions near q_0 (see [15]). The locus points where $F = \frac{\partial F}{\partial p_1} = ... = \frac{\partial F}{\partial p_n} = 0$, denoted by $\Sigma(F)$, are called π-singular points. At such points, the notion of first order PDE with a singular solution was introduced in [12]. It was shown in [13] that the local normal form of such equation is $y = 0$, up to contactomorphism. In [11] Izumiya studied singularities of the first order PDEs, describing the singularities appearing in an open dense set in the space of all functions F with the Whitney C^∞-topology. At a π-singular point the first order PDE generically have no singular solution and the set $\Sigma(F)$ consists of isolated singular points.

When $n = 1$, the first order PDE is called implicit differential equation (IDE). A natural way to study IDEs is to lift the multi-valued direction field determined by the IDE to a single field on the surface $F^{-1}(0)$. In [6], Davydov classified (following the work of Dara [5]) generic IDE's when the discriminant is a regular curve and showed that the topological normal form of the IDE acquires moduli when the discriminant is a cusp.

Bruce and Tari introduced in [2] the multiplicity of an IDE, at a singular point, as the maximum number of singular points of the IDE which emerge when perturbing the equation F. In [3] the author defined the index of an IDE and showed that this index is invariant by smooth equivalences.

In this work we introduce the concepts of multiplicity and index of a first order PDE at an isolated singular point. We shall use results in [7], where the authors defined the multiplicity and index of a 1-form on an isolated complete intersection singularity (ICIS). This concept of multiplicity extends the definition of multiplicity of an IDE given in [2]. The invariance of the multiplicity and index by smooth equivalences is proven in Section 3. We also define an invariant of first order PDEs by contactomorphism. Following the work [10] on implicit differential equations with first integrals, we introduce a definition of multiplicity for this class of equations (see Section 4). In the last section we give examples to distinguish normal forms given in [10] and [17].

2. Multiplicity and index of first order PDE

As mentioned in the introduction, a first order partial differential equation is an equation of the form

$$F(x_1, ..., x_n, y, p_1, ..., p_n) = 0, \tag{2.1}$$

where F is a smooth function on $\mathbb{R}^{2n+1}$. Consider the projection $\pi : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{n+1}$ given by $\pi(x_1, ..., x_n, y, p_1, ..., p_n) = (x_1, ..., x_n, y)$. The set of critical points of the restriction of π to $F^{-1}(0)$ is called the criminant of the first order PDE and is given by the equations $F = F_{p_1} = ... = F_{p_n} = 0$, where $F_{p_i} = \frac{\partial F}{\partial p_i}$. These points are called π-singular points and their locus is denoted by $\Sigma(F)$. The image of $\Sigma(F)$ by the projection π is called discriminant of the first order PDE.

Let $\omega = dy - \sum_{i=1}^{n} p_i dx_i$ be the canonical contact 1-form on $\mathbb{R}^{2n+1}$. Since we will only study local properties, a solution of the first order PDE (2.1) is a submanifold germ $(L, q_0) \subset \mathbb{R}^{2n+1}$ such that $L \subset F^{-1}(0)$, $\dim(L) = n$ and $\omega_{|_L} = 0$, where $\omega_{|_L}$ is the restriction of the 1-form ω to L. Let (L, q_0) be a solution of the first order PDE, then (L, q_0) is said to be classical solution if there exist a function germ $f : (\mathbb{R}^n, x_0) \longrightarrow \mathbb{R}$ such that $(j^1 f(\mathbb{R}^n), j^1 f(x_0)) = (L, q_0)$, where $j^1 f : \mathbb{R}^n \longrightarrow \mathbb{R}^{2n+1}$ is the jet extension map. It is not difficult to show that (L, q_0) is a classical solution if and only if q_0 is a regular point of the map $\pi_{|_L}$.

The notion of a singular solution is defined as follows. If the set $\Sigma(F)$ is a solution of the first order PDE, then we call it a singular solution of the first order

PDE. The zeros of the 1-form $\omega_{|_{F^{-1}(0)}}$ correspond to zeros of the vector field

$$\xi = \sum_{i=1}^{n} F_{p_i} \frac{\partial}{\partial x_i} + (\sum_{i=1}^{n} p_i F_{p_i}) \frac{\partial}{\partial y} - \sum_{i=1}^{n} (F_{x_i} + p_i F_y) \frac{\partial}{\partial p_i}$$

on $F^{-1}(0)$. These zeros are called contact singular points. We shall denote by $\Sigma^2(F)$ the set of critical points of $\pi_{|_{\Sigma(F)}}$. This set is given by the equations $F = F_{p_1} = ... = F_{p_n} = \det(F_{p_i p_j}) = 0$.

Definition 2.1. *We say that $q_0 \in \mathbb{R}^{2n+1}$ is a singular point or a zero of the first order PDE (2.1) if q_0 is a contact singular point or a zero of the 1-form ω on $\Sigma^2(F)$.*

This definition coincides with Definition 2.3 given in [2] when the first order PDE is an implicit differential equation. Note that, by definition, the singular points of the first order PDE lie on the criminat of F. We denote by (F, q_0) the germ of the first order PDE (2.1) at an isolated singular point q_0. If F is a real analytic function we say that (F, q_0) is an analytic germ. The concept of multiplicity of an IDE given in [2] motivates the following definition.

Definition 2.2. *Let (F, q_0) be an analytic germ of first order PDE. The multiplicity $M(F, q_0)$ of (F, q_0) is the maximum number of zeros that can appear in a deformation of the equation $F = 0$ (including complex zeros).*

Note that the multiplicity is not defined if the 1-form ω vanishes identically on both $\Sigma^2(F)$ and $F^{-1}(0)$. The general problem of computing the multiplicity and index of the zero of a 1-form on an isolated complete intersection singularity (ICIS) has been considered by Ebeling and Gusein-Zade in [8]. They also give an algebraic formula for computing the multiplicity of a 1-form. We use this algebraic formula to define the following ideal.

Definition 2.3. *Let $\theta = \sum_{i=1}^{n} a_i dx_i$ be a smooth 1-form on $\mathbb{R}^n$ and let $f = (f_1, ..., f_k) : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^k$ be a smooth map germ, where $n \geq k+1$. We define $I(f^{-1}(0), \theta)$ as the ideal generated by $f_1, ..., f_k$ and the $(k+1) \times (k+1)$-minors of the matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\ a_1 & \cdots & a_n \end{pmatrix} \qquad (2.2)$$

Remark 2.4. *The following holds for a function germ $\lambda : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ with $\lambda(0) \neq 0$:*

$$I(f^{-1}(0), \theta) = I(f^{-1}(0), \lambda\theta).$$

The next lemma characterizes the zeros of the 1-form θ on $f^{-1}(0)$ that are regular points of $f^{-1}(0)$.

Lemma 2.5. *Let 0 be a regular value of f. Then, $x_0 \in \mathbb{R}^n$ is a zero of the 1-form θ on $f^{-1}(0)$ if and only if $g(x_0) = 0$ for all $g \in I(f^{-1}(0), \theta)$.*

Proof: We denote by A the matrix (2.2). Let $x_0 \in \mathbb{R}^n$ be a zero of the 1-form θ on $f^{-1}(0)$. Then, $\dim(\ker A) \geq n - k$. As $\dim(\ker A) + \operatorname{rank}(A) = n$, we obtain that $k \geq \operatorname{rank}(A)$. Then, $g(x_0) = 0$ for all $g \in I(f^{-1}(0), \theta)$. Conversely, if $g(x_0) = 0$ for all $g \in I(f^{-1}(0), \theta)$ then $k \geq \operatorname{rank}(A)$. Then, the vector $(a_1, ..., a_n)$ is a linear combination of $\nabla f_1, ..., \nabla f_k$. Therefore, $x_0 \in \mathbb{R}^n$ is a zero of the 1-form θ on $f^{-1}(0)$.
□

It is not difficult to show that if $x_0 \in f^{-1}(0)$ is a singular point of f, then $g(x_0) = 0$ for all $g \in I(f^{-1}(0), \theta)$. These points are also called the zeros of the 1-form θ on $f^{-1}(0)$ (see [8]). From Lemma 2.5, we obtain that $x_0 \in \mathbb{R}^n$ is a zero of the 1-form θ on $f^{-1}(0)$ if and only if $g(x_0) = 0$ for all $g \in I(f^{-1}(0), \theta)$. We denote by $\mathcal{E}_n$ the ring of function germs on $\mathbb{R}^n$ at 0. We set $I(f^{-1}(0), \theta) = \langle g_1, ..., g_l \rangle$ and denote by $g = (g_1, ..., g_l)$.

Lemma 2.6. *Let 0 be an isolated zero of the 1-form θ on $f^{-1}(0)$. Suppose that $l = n$. Then, $\dim_{\mathbb{R}} \mathcal{E}_n / (I(f^{-1}(0), \theta) = 1$ if and only if 0 is a regular value of g.*

Proof: Since $\dim_{\mathbb{R}} \mathcal{E}_n / (I(f^{-1}(0), \theta)$ coincides with the multiplicity of the ideal $I(f^{-1}(0), \theta)$ at 0, the result follows from Proposition 2.1 in [4]. □

Lemma 2.7. *Let $(F, 0)$ be a germ of first order PDE. Suppose that $I(\Sigma^2(F), \omega)$ is generated by $2n + 1$ elements. Then there exists a smooth family of functions $F_t : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$, $t \in \mathbb{R}$, with $F_0 = F$ such that*

$$\dim_{\mathbb{R}} \mathcal{E}_{2n+1}^{p_t} / I(\Sigma^2(F_t), \omega) = 1 \quad and \quad \dim_{\mathbb{R}} \mathcal{E}_{2n+1}^{q_t} / (I(F_t^{-1}(0), \omega) = 1,$$

for $t \neq 0$ sufficiently close to zero, where p_t and q_t are isolated zeros of the 1-form ω on $\Sigma^2(F_t)$ and $F_t^{-1}(0)$, respectively, and $\mathcal{E}_n^{p_t}$ (resp. $\mathcal{E}_n^{q_t}$) the ring of function germs on $\mathbb{R}^n$ at p_t (resp. q_t).

Proof: Note that $I(F^{-1}(0), \omega) = \langle F, F_{p_1}, ..., F_{p_n}, F_{x_1} + p_1 F_y, ..., F_{x_n} + p_n F_y \rangle$ is generated by $2n + 1$ elements. Using Thom's transversality Theorem, we obtain a smooth family of mapping F_t with $F_0 = F$ such that 0 is a regular value of $(F_t, F_{t_{p_1}}, ..., F_{t_{p_n}}, F_{t_{x_1}} + p_1 F_{t_y}, ..., F_{t_{x_n}} + p_n F_{t_y})$, for $t \neq 0$ sufficiently close to zero. From Lemma 2.6, we get $\dim_{\mathbb{R}} \mathcal{E}_{2n+1}^{q_t} / (I(F_t^{-1}(0), \omega) = 1$. Analogously one proves the other equality. □

We can state the following consequences from previous results.

Proposition 2.1. *Let $(F, 0)$ be an analytic germ of first order PDE. Suppose that the ideal $I(\Sigma^2(F), \omega)$ is generated by $2n + 1$ elements. Then:*

(a) *If $\det(F_{p_i p_j})(0) \neq 0$, then the multiplicity of the germ of first order PDE $(F, 0)$ is given by $M_1(F, 0) = \dim_{\mathbb{R}} \mathcal{E}_{2n+1} / I(F^{-1}(0), \omega)$.*

(b) *If 0 is not a contact singular point, then the multiplicity of $(F, 0)$ is given by*
$$M_2(F, 0) = \dim_{\mathbb{R}} \mathcal{E}_{2n+1}/I(\Sigma^2(F), \omega).$$

(c) *If 0 is a contact singular point and $\det(F_{p_i p_j})(0) = 0$, then the multiplicity of $(F, 0)$ is the sum of the numbers holding in (a) and (b).*

Proof: Lemmas 2.6 and 2.7 remain valid in the complex analytic case. The result follows by complexifying the algebras $\mathcal{E}_{2n+1}/I(F^{-1}(0), \omega)$ and $\mathcal{E}_{2n+1}/I(\Sigma^2(F), \omega)$.
□

From Proposition 2.1, the multiplicity of the germ of first order PDE is invariant by deformations of the complexification of F. It is not true that $M_1(F, 0)$ and $M_2(F, 0)$ are invariants by real deformations of F. We denote by $\deg_0(f)$ the degree of $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ at 0 (see [9] for more details).

Definition 2.8. *Let $(F, 0)$ be a germ of first order PDE. We define the index of $(F, 0)$ as the integer*

$$\text{ind}_0(F) = \deg_0(FF_y, F_{p_1}, ..., F_{p_n}, F_{x_1} + p_1 F_y, ..., F_{x_n} + p_n F_y).$$

Proposition 2.2. *Let 0 be a regular value of F. If $F_t : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ is a smooth family of functions with $F_0 = F$, then*

$$\text{ind}_0(F) = \sum_{i=1}^{s} \text{ind}_{q_i}(F_t),$$

for $t \neq 0$ sufficiently close to zero, where q_i are the contact singular points of the first order partial differential equation F_t.

Proof: The proof follows from Proposition 2.2 in [4].   □

Let $h : (\mathbb{R}^{n+1}, (x_0, y_0)) \longrightarrow (\mathbb{R}^{n+1}, (x_1, y_1))$ be a germ of diffeomorphism. The germ of diffeomorphism $\hat{h} : (\mathbb{R}^{2n+1}, q_0) \longrightarrow (\mathbb{R}^{2n+1}, q_1)$ is said to be the canonical contact lift of h if

$$\hat{h}^*(\omega) \wedge \omega = 0 \quad \text{and} \quad h \circ \pi = \pi \circ \hat{h},$$

where $\omega = dy - \sum_{i=1}^{n} p_i dx_i$ is the canonical contact 1-form on $\mathbb{R}^{2n+1}$ and $\hat{h}^*$ is the pull-back of $\hat{h}$.

Remark 2.9. *If $\hat{h} : (\mathbb{R}^{2n+1}, q_0) \longrightarrow (\mathbb{R}^{2n+1}, q_1)$ is the canonical contact lift of h, then there exists a function germ $\lambda : (\mathbb{R}^{2n+1}, q_0) \longrightarrow \mathbb{R}$ such that $\lambda(q_0) \neq 0$ and $\hat{h}^*(\omega) = \lambda \omega$. Using this equality and $h \circ \pi = \pi \circ \hat{h}$, we get*

$$\frac{\partial \hat{h}_{n+1}}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial \hat{h}_j}{\partial x_i} \hat{h}_{n+1+j} = -p_i \lambda \tag{2.3}$$

$$\frac{\partial \hat{h}_{n+1}}{\partial y} - \sum_{j=1}^{n} \frac{\partial \hat{h}_j}{\partial y} \hat{h}_{n+1+j} = \lambda \tag{2.4}$$

$$\frac{\partial \hat{h}_s}{\partial p_i} = 0, \tag{2.5}$$

where $i \in \{1, ..., n\}$, $s \in \{1, ..., n+1\}$ and $\hat{h} = (\hat{h}_1, \hat{h}_2, ..., \hat{h}_{2n+1})$.

Definition 2.10. *We say that (F, q_0) and (G, q_1) are equivalent if there exist a germ of diffeomorphism $h : (\mathbb{R}^{n+1}, \pi(q_0)) \longrightarrow (\mathbb{R}^{n+1}, \pi(q_1))$ and a function germ $\gamma : (\mathbb{R}^{2n+1}, q_0) \longrightarrow \mathbb{R}$, $\gamma(q_0) \neq 0$ such that $F = \gamma(G \circ \hat{h})$.*

A diffeomorphism $H : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}$ is said to be a contactomorphism (or contact diffeomorphism) if $H^*(\omega) = \lambda \omega$ for some nowhere zero function λ. Another equivalence relation of germs of first order PDEs is introduced in [15]. This relation is defined as follows.

Definition 2.11. *We say that (F, q_0) and (G, q_1) are contact equivalent if there exists a germ of contactomorphism $H : (\mathbb{R}^{2n+1}, q_0) \longrightarrow (\mathbb{R}^{2n+1}, q_1)$ and a function germ $\gamma : (\mathbb{R}^{2n+1}, q_0) \longrightarrow \mathbb{R}$, $\gamma(q_0) \neq 0$ such that $F = \gamma(G \circ H)$.*

It is clear that equivalence of germs of first order PDEs implies contact equivalent. The converse is not true in general.

Theorem 2.12. *Let $f, g : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^k$ be smooth map germs. Let $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be a germ of diffeomorphism and let $C : (\mathbb{R}^n, 0) \longrightarrow GL(\mathbb{R}^k)$ be a map germ with $C(g \circ h^{-1}) = f$. If θ is a 1-form on $\mathbb{R}^n$, then*

$$h^*(I(f^{-1}(0), \theta)) = I(g^{-1}(0), h^*(\theta)).$$

Proof: We set $g \circ h^{-1} = \tilde{g}$ and write $\theta = \sum_{i=1}^n a_i dx_i$. Since $C\tilde{g} = f$, then the ideals $\langle f_1, ..., f_n \rangle$ and $\langle \tilde{g}_1, ..., \tilde{g}_n \rangle$ are equal. Also,

$$\left(\begin{array}{ccc} \frac{\partial C}{\partial x_{i_1}}\tilde{g} & \cdots & \frac{\partial C}{\partial x_{i_k}}\tilde{g} \end{array} \right) + C \left(\begin{array}{ccc} \frac{\partial \tilde{g}}{\partial x_{i_1}} & \cdots & \frac{\partial \tilde{g}}{\partial x_{i_k}} \end{array} \right) = \left(\begin{array}{ccc} \frac{\partial f}{\partial x_{i_1}} & \cdots & \frac{\partial f}{\partial x_{i_k}} \end{array} \right) \tag{2.6}$$

where $i_1, ..., i_k \in \{1, ..., n\}$. Note that every $(k+1) \times (k+1)$-minor of the matrix (2.2) is of the form

$$\sum_{s=1}^{k+1} (-1)^{k+1+i_s} a_{i_s} \det \left(\begin{array}{ccccc} \frac{\partial f}{\partial x_{i_1}} & \cdots & \frac{\hat{\partial} f}{\partial x_{i_s}} & \cdots & \frac{\partial f}{\partial x_{i_{k+1}}} \end{array} \right),$$

where the above matrix is obtained from $\left(\begin{array}{ccc} \frac{\partial f}{\partial x_{i_1}} & \cdots & \frac{\partial f}{\partial x_{i_{k+1}}} \end{array} \right)$ by cancelling the s-th column. Using (2.6), we deduce $I(f^{-1}(0), \theta) \subset I(\tilde{g}^{-1}(0), \theta)$. The opposite inclusion follows by applying the same argument to $C^{-1}f = \tilde{g}$. Therefore,

$$I(f^{-1}(0), \theta) = I(\tilde{g}^{-1}(0), \theta). \tag{2.7}$$

We write $h^*(\theta) = \sum_{i=1}^n b_i dy_i$. Using $g = \tilde{g} \circ h$, we get

$$\left(\begin{array}{ccc} \frac{\partial g}{\partial y_{i_1}} & \cdots & \frac{\partial g}{\partial y_{i_{k+1}}} \\ b_{i_1} & \cdots & b_{i_{k+1}} \end{array} \right) = \left(\begin{array}{ccc} \sum_{j=1}^n (\frac{\partial \tilde{g}}{\partial x_j} \circ h)\frac{\partial h_j}{\partial y_{i_1}} & \cdots & \sum_{j=1}^n (\frac{\partial \tilde{g}}{\partial x_j} \circ h)\frac{\partial h_j}{\partial y_{i_{k+1}}} \\ \sum_{j=1}^n (a_j \circ h)\frac{\partial h_j}{\partial y_{i_1}} & \cdots & \sum_{j=1}^n (a_j \circ h)\frac{\partial h_j}{\partial y_{i_{k+1}}} \end{array} \right).$$

Since the determinant is linear in each row, we deduce from the above equality that $I(g^{-1}(0), h^*(\theta)) \subset h^*(I(\tilde{g}^{-1}(0), \theta))$. The opposite inclusion follows from $\tilde{g} = g \circ h^{-1}$ and the result follows from Equation (2.7).   □

Let $f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^k$ be a map germ. We denote by $\operatorname{ind}_0(\theta_{|f})$ the index of the 1-form θ on $f^{-1}(0)$ at 0, introduced by Ebeling and Gusein-Zade in [8]. We also denote by $B(r) \subset \mathbb{R}^n$ the open ball of radius r centered at 0.

Lemma 2.13. *Let $f, g : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^k$ be smooth map germs. Let $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be a germ of diffeomorphism with $g \circ h^{-1} = f$. If θ is a 1-form on $\mathbb{R}^n$, then*

$$\operatorname{ind}_0(\theta_{|f}) = \operatorname{ind}_0(h^*(\theta)_{|g}).$$

Proof: By Definition 1 in [8] we have that $\operatorname{ind}_0(\theta_{|f}) = 1 + \sum_{q \in B(r)} \operatorname{ind}_q(\tilde{\theta}_{|f})$, where $\tilde{\theta}$ is a smooth 1-form on $B(r)$ which coincides with θ near the boundary $\partial B(r)$ and with a radial 1-form θ_{rad} on $B(s)$, $s < r$, and has only isolated singular points. It is not difficult to show that $h^*(\theta_{rad})$ is a radial 1-form on $h^{-1}(B(s))$ and $\operatorname{ind}_{h^{-1}(q)}(h^*(\tilde{\theta})_{|g}) = \operatorname{ind}_q(\tilde{\theta}_{|f})$. As $h^*(\tilde{\theta})$ coincides with $h^*(\theta)$ near the boundary $\partial h^{-1}(B(r))$ and with $h^*(\theta_{rad})$ on $h^{-1}(B(s))$, the result follows.   □

Let $A \subset \mathbb{R}^n$. We denote by $\chi(A)$ the Euler characteristic of A.

Lemma 2.14. *Let $f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^k$ be a smooth map germ and let $\lambda : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ be a function germ with $\lambda(0) \neq 0$. If θ is a 1-form on $\mathbb{R}^n$, then*

(a) $\operatorname{ind}_0(\theta_{|\lambda f}) = \operatorname{ind}_0(\theta_{|f})$.

(b) If $\lambda(0) > 0$, then $\operatorname{ind}_0(\lambda\theta)_{|f} = \operatorname{ind}_0(\theta_{|f})$.

(c) If $\lambda(0) < 0$, then

$$\operatorname{ind}_0(\lambda\theta)_{|f} = (-1)^{n-k}(\operatorname{ind}_0(\theta_{|f}) - 1) - \chi(f^{-1}(0) \cap \partial B(r))) + 1.$$

Proof: The proof follows from Definition 1 in [8].   □

3. Invariance of the index and multiplicity

Let $(F, 0)$ and $(G, 0)$ be two equivalent germs of first order PDEs as in Definition 2.10. From Remark 2.9 we get

$$\begin{pmatrix} F_{p_1} \\ \vdots \\ F_{p_n} \end{pmatrix} = (G \circ \hat{h}) \begin{pmatrix} \gamma_{p_1} \\ \vdots \\ \gamma_{p_n} \end{pmatrix} + \gamma \begin{pmatrix} \frac{\partial \hat{h}_{n+2}}{\partial p_1} & \cdots & \frac{\partial \hat{h}_{2n+1}}{\partial p_1} \\ \vdots & & \vdots \\ \frac{\partial \hat{h}_{n+2}}{\partial p_n} & \cdots & \frac{\partial \hat{h}_{2n+1}}{\partial p_n} \end{pmatrix} \begin{pmatrix} G_{p_1} \circ \hat{h} \\ \vdots \\ G_{p_n} \circ \hat{h} \end{pmatrix} \quad (3.1)$$

and the determinant of the Jacobian matrix of $\hat{h}$ is

$$J(\hat{h}) = J(h) \ \det(\frac{\partial \hat{h}_{n+1+i}}{\partial p_j}). \quad (3.2)$$

Proposition 3.1. *Let $(F, 0)$ and $(G, 0)$ be the germs of first order PDEs. If $(F, 0)$ and $(G, 0)$ are equivalent, then $M_2(F, 0) = M_2(G, 0)$.*

Proof: We denote by $N = \det(\frac{\partial \hat{h}_{n+1+i}}{\partial p_j})$. Using Equations (3.1), we can write

$$\det(F_{p_i p_j}) = k_0(G \circ \hat{h}) + \sum_{s=1}^{n} k_s(G_{p_s} \circ \hat{h}) + \gamma N^2[\det(G_{p_i p_j}) \circ \hat{h}], \qquad (3.3)$$

where $k_0, k_1, ..., k_n$ are smooth functions on $\mathbb{R}^{2n+1}$. From Equations (3.1) and (3.3), we obtain the following system of equations

$$\begin{pmatrix} \gamma & 0 & \dots & 0 & 0 \\ \gamma_{p_1} & \gamma\frac{\partial \hat{h}_{n+2}}{\partial p_1} & \dots & \gamma\frac{\partial \hat{h}_{2n+1}}{\partial p_1} & 0 \\ \vdots & \vdots & & \vdots & 0 \\ \gamma_{p_n} & \gamma\frac{\partial \hat{h}_{n+2}}{\partial p_n} & \dots & \gamma\frac{\partial \hat{h}_{2n+1}}{\partial p_n} & 0 \\ k_0 & k_1 & \dots & k_n & \gamma N^2 \end{pmatrix} \begin{pmatrix} G \circ \hat{h} \\ G_{p_1} \circ \hat{h} \\ \vdots \\ G_{p_n} \circ \hat{h} \\ \det(G_{p_i p_j}) \circ \hat{h} \end{pmatrix} = \begin{pmatrix} F \\ F_{p_1} \\ \vdots \\ F_{p_n} \\ \det(F_{p_i p_j}) \end{pmatrix}$$

By Remark 2.9 we have $(\hat{h}^{-1})^*(\omega) = \frac{1}{\lambda}\omega$. From Theorem 2.12 and Remark 2.4, we obtain that $(\hat{h}^{-1})^*(I(\Sigma^2(F), \omega)) = I(\Sigma^2(G), \omega)$. Since $\hat{h}$ is a diffeomorphism, the result follows. $\qquad \square$

Proposition 3.2. *Let $(F, 0)$ and $(G, 0)$ be the germs of first order PDEs. If $(F, 0)$ and $(G, 0)$ are equivalent, then $M_1(F, 0) = M_1(G, 0)$*

Proof: Note that $\gamma(G \circ \hat{h}) = F$ and $(\hat{h}^{-1})^*(\omega) = \frac{1}{\lambda}\omega$. By Theorem 2.12 and Remark 2.4 this implies that $(\hat{h}^{-1})^*(I(F^{-1}(0), \omega)) = I(G^{-1}(0), \omega)$ and result follows. $\qquad \square$

Lemma 3.1. *Let $(F, 0)$ be the germ of first order PDE. Then*

$$\mathrm{ind}_0(\omega_{|_F}) = \deg_0(FF_y, F_{p_1}, ..., F_{p_n}, F_{x_1} + p_1 F_y, ..., F_{x_n} + p_n F_y).$$

Proof: We denote by $F_x + pF_y = (F_{x_1} + p_1 F_y, ..., F_{x_n} + p_n F_y)$ and $F_p = (F_{p_1}, ..., F_{p_n})$. Let $(FF_y - tF_y, F_p, F_x + pF_y)$ be a perturbation of $T = (FF_y, F_p, F_x + pF_y)$. Then, by Proposition 2.2 in [4],

$$\deg_0 T = \sum_{q_i \in V_t} \mathrm{sign}\,[F_y(q_i)]\deg_{q_i}(F - t, F_p, F_x + pF_y) - \mathrm{sign}\,[t]\deg_0 \nabla F.$$

It is not difficult to show that $\mathrm{ind}_{q_i}(\omega_{|_{V_t}}) = \mathrm{sign}\,[F_y(q_i)]\deg_{q_i}(F - t, F_p, F_x + pF_y)$, for all $q_i \in V_t$, where $V_t = F^{-1}(t) \cap B(r)$. From [14] it follows that $\chi(V_t) - 1 = \mathrm{sign}\,[t]\deg_0 \nabla F$. The result follows by Theorem 3 in [8]. $\qquad \square$

Theorem 3.2. *Let $(F, 0)$ and $(G, 0)$ be the germs of first order PDEs. If $(F, 0)$ and $(G, 0)$ are equivalent, then $\operatorname{ind}_0(F) = \operatorname{ind}_0(G)$.*

Proof: Since $(F, 0)$ and $(G, 0)$ are equivalent, $\gamma(G \circ \hat{h}) = F$ and $(\hat{h}^{-1})^*(\omega) = \frac{1}{\lambda}\omega$. Using Lemma 2.13, we get $\operatorname{ind}_0(\omega_{|_F}) = \operatorname{ind}_0((\frac{1}{\lambda}\omega)_{|_{\tilde{\gamma} G}})$, where $\tilde{\gamma} = \gamma \circ \hat{h}^{-1}$. Note that the dimension of $F^{-1}(0)$ is $2n$. By Lemma 2.14, we have that $\operatorname{ind}_0(\omega_{|_F}) = \operatorname{ind}_0(\omega_{|_G})$ and the result follows from Lemma 3.1. $\square$

Proposition 3.3. *Let $(F, 0)$ and $(G, 0)$ be the germs of first order PDEs. If $(F, 0)$ and $(G, 0)$ are contact equivalent, then*

$$M_1(F, 0) = M_1(G, 0) \ \text{ and } \ \operatorname{ind}_0(F) = \operatorname{ind}_0(G).$$

Proof: It follows from the hypothesis that there exist a germ of contact diffeomorphism $H : (\mathbb{R}^{2n+1}, 0) \longrightarrow (\mathbb{R}^{2n+1}, 0)$ and a germ of function $\gamma : (\mathbb{R}^{2n+1}, 0) \longrightarrow \mathbb{R}$, $\gamma(0) \neq 0$ such that $\gamma(G \circ H) = F$. Note that $(H^{-1})^*(\omega) = \lambda\omega$ for some smooth function λ with $\lambda(0) \neq 0$. By Theorem 2.12 and Remark 2.4 this implies that $(H^{-1})^*(I(F^{-1}(0), \omega)) = I(G^{-1}(0), \omega)$. So, $M_1(F, 0) = M_1(G, 0)$. Following the same argument in the proof of Theorem 2.12, we obtain the equality of the indices.

$\square$

Theorem 3.3. *Let $(F, 0)$ be a germ of first order PDE. Then,*

$$|\operatorname{ind}_0(F)| \leq M_1(F) + \dim_{\mathbb{R}} \mathcal{E}_{2n+1}/\nabla F.$$

Proof: The proof follows by using the formula of Eisenbud and Levine [9]. $\square$

4. First order PDEs with first integral

When $n = 1$, the first order PDE is called implicit differential equation (IDE). At points where the partial derivative $F_y \neq 0$, the IDE is locally the image of a germ of an immersion $(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$. Conversely, the image of every germ of an immersion $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ define an germ of IDE and is denoted by $(R_f, 0)$.

Definition 4.1. *Let $(R_f, 0)$ be a germ of IDE. We say that $(R_f, 0)$ is a differential equation germ with first integral if there exists a germ of a submersion $\mu : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ such that $d\mu \wedge f^*(\omega) = 0$.*

We call μ a first integral of f and the pair $(f, \mu) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$ is called a germ of IDE with first integral. Note that the solutions of the IDE with first integrals in the plane are the images under $\pi \circ f$ of the level sets of μ.

Definition 4.2. *Let (g, μ) be a pair of a map germ $g : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and a germ of a submersion $\mu : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$. Then the diagram*

$$(\mathbb{R}, 0) \xleftarrow{\ \mu\ } (\mathbb{R}^2, 0) \xrightarrow{\ g\ } (\mathbb{R}^2, 0),$$

or briefly (g, μ), is called an integral diagram if there exists a germ of an immersion $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ such that $d\mu \wedge f^*(\omega) = 0$ and $g = \pi \circ f$.

It is not hard to see that if the critical set of g is nowhere dense, then f is uniquely determined by (g, μ).

Definition 4.3. *Let (g, μ) and (g', μ') be integral diagrams. Then (g, μ) is called equivalent to (g', μ') if the diagram*

$$
\begin{array}{ccccc}
(\mathbb{R}, 0) & \xleftarrow{\ \mu\ } & (\mathbb{R}^2, 0) & \xrightarrow{\ g\ } & (\mathbb{R}^2, 0) \\
{\scriptstyle k} \downarrow & & {\scriptstyle \psi} \downarrow & & {\scriptstyle \phi} \downarrow \\
(\mathbb{R}, 0) & \xleftarrow{\ \mu'\ } & (\mathbb{R}^2, 0) & \xrightarrow{\ g'\ } & (\mathbb{R}^2, 0)
\end{array}
$$

commutes for some germs of diffeomorphisms k, ψ and ϕ.

The following proposition reduces the equivalence problem for IDEs, which admit independent first integral, to that for the corresponding induced integral diagrams.

Theorem 4.4. ([10], Proposition 2.8) *Let (f, μ) and $(f', \mu') : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$ be germs of IDEs with first integral. Assume that the sets of critical points of $\pi \circ f$ and $\pi \circ f'$ are nowhere dense. Then $(R_f, 0)$ and $(R_{f'}, 0)$ are equivalent if and only if $(\pi \circ f, \mu)$ and $(\pi \circ f', \mu')$ are equivalent as integral diagrams.*

Let (g, μ) be an integral diagram. We denote by

$$
m(g, \mu) = \dim_{\mathbb{R}} \mathcal{E}_2 / I(J(g)^{-1}(0), d\mu),
$$

where $J(g)$ is the determinant of the jacobian matrix of g.

Theorem 4.5. *Let (g, μ) and (g', μ') be integral diagrams. If (g, μ) and (g', μ') are equivalent then*

$$
m(g, \mu) = m(g', \mu').
$$

Proof: It follows from the hypothesis that there exist germs of diffeomorphims k, ψ and ϕ such that $\phi \circ g = g' \circ \psi$ and $k \circ \mu = \mu' \circ \psi$. Differentiating this equations, we obtain

$$
\frac{1}{J(\psi)}[J(\phi) \circ g]J(g) = J(g') \circ \psi \quad \text{and} \quad \psi^*(d\mu') = (\frac{dk}{dt} \circ \mu)d\mu.
$$

From Theorem 2.12 and Remark 2.4, we obtain that $\psi^*(I(J(g')^{-1}(0), d\mu')) = I(J(g)^{-1}(0), d\mu)$. Thus, $m(g, \mu) = m(g', \mu')$. $\qquad\square$

Let $(f, \mu) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$ be a germ of IDE with first integral. We denote by $M(f, \mu) = \dim_{\mathbb{R}} \mathcal{E}_2 / I(J(\pi \circ f)^{-1}(0), d\mu)$.

Corollary 4.6. *Let (f, μ) and $(f', \mu') : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$ be germs of IDEs with first integral. If $(R_f, 0)$ and $(R_{f'}, 0)$ are equivalent then*

$$
M(f, \mu) = M(f', \mu').
$$

Proof: The proof follows from Theorems 4.4 and 4.5. $\qquad\square$

5. The cases n=1 and n=2

In this section we give examples to distinguish normal forms given in [10] and [17]. When n=1, we have $M_1(F,0) = \dim_{\mathbb{R}} \mathcal{E}_3/\langle F, F_p, F_x + pF_y \rangle$ and $M_2(F,0) = \dim_{\mathbb{R}} \mathcal{E}_3/\langle F, F_p, F_{pp} \rangle$. This shows that the multiplicity of $(F,0)$ coincides with the multiplicity introduced by Bruce and Tari in [2]. A particular class of implicit differential equations that have been most intensively studied are the IDEs that define at most two directions in the plane. This class of equations are called binary differential equations and are of the form

$$F(x, y, p) = p^2 - \delta(x, y) = 0. \tag{5.1}$$

In this case, $M_1(F,0) = \dim_{\mathbb{R}} \mathcal{E}_2/(\delta, \delta_x)$, $M_2(F,0) = 0$, $\mathrm{ind}(F,0) = \deg_0(\delta\delta_y, \delta_x)$. In [17], Tari studied the singularities of codimension 2 of binary differential equations. He also obtained the topological normal forms of these singularities. We calculate in Table 1 the index and multiplicity of this class of equations.

Table 1: Normal forms for codimension 2 singularities.

Normal forms of δ	$\mathrm{ind}(F,0)$	$M_1(F,0)$
$-y + x^4$	-1	3
$-y - x^4$	1	3
$xy + x^3$	0	3
$x^2 + y^3$	-1	3
$-x^2 + y^3$	1	3

When $n = 2$, we have $M_2(F,0) = \dim_{\mathbb{R}} \mathcal{E}_5/\langle F, F_{p_1}, F_{p_2}, R, \det(B) \rangle$ and $M_1(F,0) = \dim_{\mathbb{R}} \mathcal{E}_5/\langle F, F_{p_1}, F_{p_2}, F_{x_1} + p_1F_y, F_{x_2} + p_2F_y \rangle$, where $R = \det(F_{p_ip_j})$ and

$$B = \begin{pmatrix} F_{x_1} + p_1F_y & F_{x_2} + p_2F_y & 0 & 0 \\ F_{p_1x_1} + p_1F_{p_1y} & F_{p_1x_2} + p_2F_{p_1y} & F_{p_1p_1} & F_{p_1p_2} \\ F_{p_2x_1} + p_1F_{p_2y} & F_{p_2x_2} + p_2F_{p_2y} & F_{p_2p_1} & F_{p_2p_2} \\ 0 & 0 & R_{p_1} & R_{p_2} \end{pmatrix}. \tag{5.2}$$

Generically $F^{-1}(0) \subset \mathbb{R}^5$ is a 4-manifold and $\pi_{|_{F^{-1}(0)}}$ has only fold, cusp and swallowtail singularities (see [1]).

Theorem 5.1. *Let $(F,0)$ be an analytic germ of first order PDE. If $n = 2$ and 0 is not a zero of the 1-form ω on $\Sigma(F)$, then the multiplicity of $(F,0)$ is the maximum number of swallowtail points that appear in a generic deformation of $\pi_{|_{F^{-1}(0)}}$.*

Proof: By Lemma 2.5 we have that 0 is a zero of the 1-form ω on $\Sigma(F)$ if and only if $F(0) = F_{p_1}(0) = F_{p_2}(0) = \det(R_1)(0) = \det(R_2)(0) = 0$, where

$$R_1 = \begin{pmatrix} F_{x_1} & F_{x_2} & 0 \\ F_{p_1x_1} & F_{p_1x_2} & F_{p_1p_1} \\ F_{p_2x_1} & F_{p_2x_2} & F_{p_2p_1} \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} F_{x_1} & F_{x_2} & 0 \\ F_{p_1x_1} & F_{p_1x_2} & F_{p_1p_2} \\ F_{p_2x_1} & F_{p_2x_2} & F_{p_2p_2} \end{pmatrix}.$$

Using (5.2) we get $M_1(F,0) = 0$ and $M_2(F,0) = \dim_{\mathbb{R}} \mathcal{E}_5 / \langle F, F_{p_1}, F_{p_2}, R, C_1, C_2 \rangle$, where $C_1 = R_{p_2} F_{p_2 p_1} - R_{p_1} F_{p_2 p_2}$ and $C_2 = R_{p_2} F_{p_1 p_1} - R_{p_1} F_{p_1 p_2}$. The result follows from Theorem 5.1 in [16]. $\square$

In [10] is studied the classification of generic implicit differential equation with first integral. This problem is reduced to the classification of generic integral diagrams. Normal forms of generic integral diagram also are given in [10]. We calculate the multiplicity this normal forms and present them in Table 2 .

Table 2: Normal forms for integral diagram (g, μ).

g	μ	$m(g, \mu)$
(u^2, v)	$v - \frac{1}{3}u^3$	0
(u, v^2)	$v - \frac{1}{2}u$	0
$(u^3 + uv, v)$	$\frac{3}{4}u^4 + \frac{1}{2}u^2 v + v$	1
$(u, v^3 + uv)$	v	0
$(u, v^3 + uv^2)$	$\frac{1}{2}v^2 + u$	2

Acknowledgments

References

1. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differential maps,* Birkhauser, Boston-Besel-Stuttgart (1985).

2. J. W. Bruce and F. Tari, *On the multiplicity of implicit differential equations,* J. Differential Equations 148, 122-147, (1998).

3. L. S. Challapa, *Invariants of binary differential equations,* Journal of Dynamical and Control Systems 15, 157-176, (2009).

4. A. Cima, A. Gasull and J. Torregrosa, *On the relation between index and multiplicity.* J. Londo Math. Soc. 57(2), 757-768, (1998).

5. L. Dara, *Singularités génériques des équations differentielles multiformes,* Bol. Soc. Brasil. Math 6, 95-128, (1975).

6. A. A. Davydov, *Normal forms of differential equations unresolved with respect to derivatives in a neighbourhood of its singular point,* Funct. Anal. Appl. 19, 1-10, (1985).

7. W. Ebeling and S. M. Gusein-Zade, *On the index of a vector field at an isolated singularity.* The Arnoldfest (Toronto, ON, 1997), 141152, Fields Inst. Commun. 24, Amer. Math. Soc. Providence, RI, (1999).

8. W. Ebeling and S. M. Gusein-Zade, *Indices of 1-forms on an isolated complete intersection singularity.* Mosc. Math. J. 3, 439-455, (2003).

9. D. Eisenbud, H.I. Levine, *An algebraic formula for the degree of a C^∞ map-germs*, Ann. Math. 106, 19-44, (1977).

10. A. Hayakawa, G. Ishikawa, S. Izumiya, K. Yamaguchi, *Classification of generic integral diagrams and first order ordinary differential equations,* International Journal of Mathematics 5, 567-586, (1994).

11. S. Izumiya, *Generic properties of first order partial differential equations,* Topology Hawaii (ed. K. H. Dovermann, World Scientific, Singapore), 91-100, (1991).

12. S. Izumiya, *Singular solutions of first-order differential equations,* Bull. London Math. Soc. 26, 69-74, (1994).

13. S. Izumiya, *A normal form of first order partial differential equations with singular solution,* Tokyo J. Math. 18, no. 2, 485-488, (1995).

14. G. M. Khimshiashvili, *On the local degree of a smoooth mapping,* Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze 64, 105-124, (1980).

15. V. V. Lychagin, *Local classification of non-linear first order partial differential equations,* Russian Math. Surveys 30, 105-175, (1975).

16. J. J. Nuño-Ballesteros and M. J. Saia, *Multiplicity of Boardman strata and deformation of map germs,* Glassgow Math. J. 40, no. 1, 21-32, (1998).

17. F. Tari, *Two-parameter families of implicit differential equations,* Discrete Contin. Dyn. Syst. 13, 139-162, (2005).

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