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Generalized Contraction Mappings of Rational Type and Applications to Nonlinear Integral Equations

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ABSTRACT: The aim of the present paper is to introduce a new class of pair of contraction mappings, called $\psi-(\alpha,\beta,m)$-contraction pairs, and obtain common fixed point theorems for a pair of mappings in this class, satisfying a minimal commutativity condition. Further, we use mappings of this class to analyze the existence of solutions for a class of nonlinear integral equations on the space of continuous functions and in some of its subspaces. Concrete examples are also provided in order to illustrate the applicability of these results.

Key Words: Common fixed point, Altering distance, Integral equation, Weak compatible mappings.

Contents

1. Introduction and preliminaries

Generalizations of the Banach contraction principle have been extensively used to study common fixed points for contractive type pair of mappings, as well as in the existence of solutions of differential and integral equations, (see e.g., [3,4,6,7,10, 11,12,15,17,18,19,20]). In this paper, first, we establish some common fixed point theorems for a class of contractions of rational type wherein contractive inequality is controlled by a positive function satisfying a stability condition at 0. Then, we use the class of mappings in consideration (see, Definition 2.1), to establish the existence and uniqueness results for solutions of some nonlinear integral equations.

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A pair of self-mappings (S,T) on a metric space (M,d) is said to be *compatible* [8] if and only if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $(x_n)_n \subset M$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in M$. A pair of self-mappings (S,T) is said to be *noncompatible* [16] if there exists at least one sequence $(x_n)_n \subset M$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in M$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non-existent. A pair of self-mappings (S,T) is said to satisfy the *property* (E. A.) [1] if there exists a sequence $(x_n)_n \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in M$, and (S,T) is said to satisfy the *common limit in the range of T property* (CLR_T) [21] if there exists a sequence $(x_n)_n \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tt,$$

for some $t \in M$. Notice that the CLR_T property circumvents the requirement of the condition of the closedness of the ranges of the involved mappings.

A point $x \in M$ is called a *coincidence point* (CP) of S and T if $Sx = Tx$. The set of coincidence points of S and T will be denoted by $C(S,T)$. If $x \in C(S,T)$, then $w = Sx = Tx$ is called a *point of coincidence* (POC) of S and T.

Finally, a pair of mappings (S,T) is said to satisfy *non-trivially weakly compatible* (WC), condition [9], if they commute at their coincidence points, whenever the set of coincidences is nonempty.

Remark 1.1. *It may be observed that non-trivial weak compatibility is a necessary, hence minimal condition for the existence of common fixed points of contractive type mapping pairs. Commutativity at coincidence points is equivalent to the condition that Sx is a coincidence point of S and T whenever x is a coincidence point. Therefore, non-trivially weakly compatible mappings may equivalently be called as coincidence preserving mappings. Compatible mappings are necessarily coincidence preserving since compatible mappings commute at each coincidence points. However, the converse need not be true.*

To prove our results we will use the following lemma [2].

Lemma 1.2. *Let (M,d) be a metric space. Let (x_n) be a sequence in M such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If (x_n) is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $(m(k))$ and $(n(k))$ with

$$m(k) > n(k) > k$$

such that,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0, \quad d(x_{m(k)-1}, x_{n(k)}) < \varepsilon_0$$

and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon_0$,

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0$,

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon_0$.

2. $\psi - (\alpha, \beta, m)$-contraction pairs and their point of coincidence

As in [11], we will use functions $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ satisfying that $\alpha(t)+\beta(t) < 1$, for all $t \in \mathbb{R}_+ := [0, \infty)$, and

$$\begin{aligned} \limsup_{s\to 0^+} \beta(s) &< 1 \\ \limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} &< 1, \quad \forall t > 0. \end{aligned} \tag{2.1}$$

Now, we introduce the following class of pair of contraction mappings.

Definition 2.1. *Let (M, d) be a metric space and let $S, T : M \longrightarrow M$ be mappings. The pair (S, T) is called a $\psi - (\alpha, \beta, m)$-contraction pair if for all $x, y \in M$*

$$\psi\left(d(Sx, Sy)\right) \leq \alpha(d(Tx, Ty))\psi(d(Tx, Ty)) + \beta(d(Tx, Ty))\psi(m(x, y))$$

where $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function satisfying that

$$\psi(t_n) \to 0 \quad \textit{implies} \quad t_n \to 0, \tag{2.2}$$

$\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ are functions satisfying (2.1) and

$$m(x, y) := \max\left\{d(Sy, Ty)\frac{1 + d(Sx, Tx)}{1 + d(Tx, Ty)}, d(Tx, Ty)\right\}. \tag{2.3}$$

Remark 2.2. *Due to the symmetry of the distance, the $\psi - (\alpha, \beta, m)$-contraction pair implicitly includes the following dual one*

$$\begin{aligned} \psi\left(d(Sx, Sy)\right) \leq \alpha(d(Tx, Ty))\psi(d(Tx, Ty)) + \\ \beta(d(Tx, Ty))\psi(\max\left\{d(Sx, Tx)\tfrac{1+d(Sy,Ty)}{1+d(Tx,Ty)}, d(Tx, Ty)\right\}), \end{aligned}$$

obtained by interchanging x and y.

Example 2.1. *Let (M, d) be a metric space. If we consider $S \equiv a$, a constant map, and T any selfmapping on M, we can check that the pair (S, T) is a $\psi - (\alpha, \beta, m)$-contraction pair for all functions $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ such that $\alpha(t) + \beta(t) < 1$, for all $t \in \mathbb{R}_+$ and satisfying (2.1).*

Proposition 2.3. *Let (M, d) be a metric space and let $S, T : M \longrightarrow M$ be mappings with $S(M) \subset T(M)$. If the pair (S, T) is a $\psi - (\alpha, \beta, m)$-contraction pair, then for any $x_0 \in M$, a sequence (y_n) defined by*

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \ldots$$

satisfies:

(1) $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

(2) $(y_n) \subset M$ *is a Cauchy sequence in* $T(M)$.

Proof: To prove (1), let $x_0 \in M$ be an arbitrary point. Since $S(M) \subset T(M)$, then there exists $x_1 \in M$ such that $Sx_0 = Tx_1$. By continuing this process inductively we get a sequence (x_n) in M such that

$$y_n = Sx_n = Tx_{n+1}.$$

Now,

$$\psi(d(Tx_{n+1}, Tx_{n+2})) = \psi(d(Sx_n, Sx_{n+1})) \leq \alpha(d(Tx_n, Tx_{n+1}))\psi(d(Tx_n, Tx_{n+1})) + \beta(d(Tx_n, Tx_{n+1}))\psi(m(x_{n+1}, x_{n+1})) \quad (2.4)$$

where

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\left\{ d(Sx_{n+1}, Tx_{n+1})\frac{1 + d(Sx_n, Tx_n)}{1 + d(Tx_n, Tx_{n+1})}, d(Tx_n, Tx_{n+1}) \right\} \\ &= \max\left\{ d(Tx_{n+2}, Tx_{n+1})\frac{1 + d(Tx_{n+1}, Tx_n)}{1 + d(Tx_n, Tx_{n+1})}, d(Tx_n, Tx_{n+1}) \right\} \\ &= \max\{d(Tx_{n+1}, Tx_{n+2}), d(Tx_n, Tx_{n+1})\}. \end{aligned}$$

If $m(x_n, x_{n+1}) = d(Tx_{n+1}, Tx_{n+2})$, then from (2.4) we obtain

$$\begin{aligned} \psi(d(Tx_{n+1}, Tx_{n+2})) \leq& \alpha(d(Tx_n, Tx_{n+1}))\psi(d(Tx_n, Tx_{n+1})) \\ &+ \beta(d(Tx_n, Tx_{n+1}))\psi(d(Tx_{n+1}, Tx_{n+2})). \end{aligned}$$

Thus, it follows that

$$\psi(d(Tx_{n+1}, Tx_{n+2})) \leq \frac{\alpha(d(Tx_n, Tx_{n+1}))}{1 - \beta(d(Tx_n, Tx_{n+1}))}\psi(d(Tx_n, Tx_{n+1})). \quad (2.5)$$

On the other hand, if $m(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1})$, then from (2.4) we get

$$\begin{aligned} \psi(d(Tx_{n+1}, Tx_{n+2})) \leq& \alpha(d(Tx_n, Tx_{n+1}))\psi(d(Tx_n, Tx_{n+1})) \\ &+ \beta(d(Tx_n, Tx_{n+1}))\psi(d(Tx_n, Tx_{n+1})). \end{aligned}$$

In view of above inequality we get

$$\begin{aligned} \psi(d(Tx_{n+1}, Tx_{n+2})) \leq& (\alpha(d(Tx_n, Tx_{n+1})) + \beta(d(Tx_n, Tx_{n+1}))) \\ &\times \psi(d(Tx_n, Tx_{n+1})). \end{aligned} \quad (2.6)$$

From (2.5) and (2.6), and by using the properties of the functions α and β we obtain

$$\psi(d(Tx_{n+1}, Tx_{n+2})) < \psi(d(Tx_n, Tx_{n+1})).$$

Thus, $(z_n) = (\psi(d(Tx_n, Tx_{n+1})))$ is a decreasing sequence of positive numbers bounded below by zero, and so converges to $a \geq 0$. Now, if $a > 0$ then by taking limsup on both sides of the above inequality we have a contradiction. Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \psi(d(Tx_n, Tx_{n+1})) = 0.$$

Consequently, from the stability condition at zero (2.2) we conclude that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = \lim_{n\to\infty} d(Tx_{n+1}, Tx_{n+2}) = 0. \tag{2.7}$$

To prove (2), we are going to suppose that $(y_n) \subset T(M)$ is not a Cauchy sequence. Then, there exists $\varepsilon_0 > 0$ and sequences $(m(k))$ and $(n(k))$ with $m(k) \geq n(k) > k$ such that

$$d(y_{m(k)}, y_{n(k)}) \geq \varepsilon_0 \quad \text{and} \quad d(y_{m(k)-1}, y_{n(k)}) < \varepsilon_0.$$

From Lemma 1.2 and the continuity of ψ we have

$$\begin{aligned}\psi(\varepsilon_0) &= \limsup_{k\to\infty} \psi(d(Tx_{m(k)+1}, Tx_{n(k)+1})) = \limsup_{k\to\infty} \psi(d(Sx_{m(k)}, Sx_{n(k)}))\\ &\leq \limsup_{k\to\infty} \alpha(d(Tx_{m(k)}, Tx_{n(k)}))\psi(d(Tx_{m(k)}, Tx_{n(k)}))\\ &\quad + \limsup_{k\to\infty} \beta(d(Tx_{m(k)}, Tx_{n(k)}))\psi(m(x_{m(k)}, x_{n(k)})),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&m(x_{m(k)}, x_{n(k)}) =\\ &\max\left\{d(Sx_{n(k)}, Tx_{n(k)})\frac{1 + d(Sx_{m(k)}, Tx_{m(k)})}{1 + d(Tx_{m(k)}, Tx_{n(k)})}, d(Tx_{m(k)}, Tx_{n(k)})\right\}\\ &=\\ &\max\left\{d(Tx_{n(k)}, Tx_{n(k)+1})\frac{1 + d(Tx_{m(k)}, Tx_{m(k)+1})}{1 + d(Tx_{m(k)}, Tx_{n(k)})}, d(Tx_{m(k)}, Tx_{n(k)})\right\}.\end{aligned} \tag{2.9}$$

Letting $k \to \infty$ in (2.9), and by (2.7) we obtain that

$$\lim_{k\to\infty} m(x_{m(k)}, x_{n(k)}) = \max\{0, \varepsilon_0\} = \varepsilon_0.$$

Therefore, (2.8) is now

$$\begin{aligned}\psi(\varepsilon_0) &\leq \limsup_{k\to\infty} \alpha(d(Tx_{m(k)-1}, Tx_{n(k)-1}))\psi(\varepsilon_0)\\ &\quad + \limsup_{k\to\infty} \beta(d(Tx_{m(k)-1}, Tx_{n(k)-1}))\psi(\varepsilon_0) < \psi(\varepsilon_0).\end{aligned}$$

Which is a contradiction, hence $(Tx_n) \subset T(M)$ is a Cauchy sequence. □

Lemma 2.4. *Let S and T be two self-mappings on a metric space (M, d). Let us assume that the pair (S, T) is a $\psi - (\alpha, \beta, m)$-contraction pair. If S and T have a POC in M then it is unique.*

Proof: Let $w \in M$ be a POC of the pair (S,T). Then there exits $x \in M$ such that $Sx = Tx = w$. Suppose that for some $y \in M$, $Sy = Ty = v$ with $v \neq w$. Then,

$$\begin{aligned}\psi(d(w,v)) &= \psi(d(Sx,Sy)) \\ &\leq \alpha(d(Tx,Ty))\psi(d(Tx,Ty)) + \beta(d(Tx,Ty))\psi(m(x,y)).\end{aligned}$$

It follows that,

$$\psi(d(w,v)) \leq \alpha(d(w,v))\psi(d(w,v)) + \beta(d(w,v))\psi(m(x,y)). \tag{2.10}$$

Using (2.3) we have

$$\begin{aligned}m(x,y) =& \max\left\{d(Sy,Ty)\frac{1+d(Sx,Tx)}{1+d(Tx,Ty)}, d(Tx,Ty)\right\} \\ =& \max\left\{d(v,v)\frac{1+d(w,w)}{1+d(w,v)}, d(w,v)\right\} = d(w,v).\end{aligned}$$

Substituting it into (2.10) we get

$$\begin{aligned}\psi(d(w,v)) \leq& \alpha(d(w,v))\psi(d(w,v)) + \beta(d(w,v))\psi(d(w,v)) \\ &\leq (\alpha(d(w,v) + \beta(d(w,v))))\,\psi(d(w,v)) < \psi(d(w,v)),\end{aligned}$$

which is a contradiction, therefore $w = v$. □

Theorem 2.5. *Let S and T be self-mappings on a metric space (M,d) such that*

(i) $S(M) \subset T(M)$.

(ii) $T(M) \subset M$ *is a complete subspace of* M.

(iii) The pair (S,T) *is a* $\psi - (\alpha,\beta,m)$*-contraction pair.*

Then, the pair (S,T) *has a unique POC.*

Proof: Let $y_n = Sx_n = Tx_{n+1}$, $n = 0,1,\ldots$, be a Cauchy sequence defined in Proposition 2.3 which, as was proved, satisfies that $(y_n) = (Tx_{n+1}) \subset T(M)$. Since $T(M) \subset M$ is a complete subspace of M, then there exists $z \in T(M)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = z,$$

thus we can find $u \in M$ such that $Tu = z$. Now, we are going to show that $Tu = Su$. Suppose that $Tu \neq Su$. Then,

$$\begin{aligned}\psi(d(Sx_{n+1},Su)) \leq& \alpha(d(Tx_{n+1},Tu))\psi(d(Tx_{n+1},Tu)) \\ &+ \beta(d(Tx_{n+1},Tu))\psi(m(x_{n+1},u)),\end{aligned} \tag{2.11}$$

where

$$m(x_{n+1},u) = \max\left\{d(Su,Tu)\frac{1+d(Sx_{n+1},Tu)}{1+d(Tx_{n+1},Tu)}, d(Tx_{n+1},Tu)\right\}. \tag{2.12}$$

Taking the limits $n \to \infty$ in (2.11) and (2.12) we obtain

$$\psi(d(z, Su)) \leq \limsup_{n\to\infty} \alpha(d(Tx_{n+1}, Tu))\psi(d(z, Tu)) + \limsup_{n\to\infty} \beta(d(Tx_{n+1}, Tu))\psi(d(Su, Tu)).$$

From above inequality we get

$$\psi(d(z, Su)) \leq \limsup_{n\to\infty} \beta(d(Tx_{n+1}, Tu))\psi(d(Su, Tu)) < \psi(d(Su, Tu)),$$

which is a contradiction. Hence, $Su = Tu = z$. Therefore, z is a POC of S and T. From Lemma 2.4 we conclude that z is a unique POC. □

3. Common fixed points for $\psi - (\alpha, \beta, m)$-contraction pairs

In this section we prove general common fixed point results for a pair of mappings belonging to the $\psi - (\alpha, \beta, m)$-contraction class, under a minimal commutativity condition.

Theorem 3.1. *Let (M, d) be a metric space and $S, T : M \longrightarrow M$ mappings satisfying the hypotheses of Theorem 2.5. Moreover, let us suppose that the pair (S, T) is non-trivially weakly compatible pair, then S and T have a unique common fixed point.*

Proof: Since the pair (S, T) is non-trivially weakly compatible, then they commute at their unique coincidence point. Hence, $SSu = STu = TSu = TTu$, using uniqueness of the POC, we obtain that Su is a common fixed point of (S, T). Uniqueness of the common fixed point can be proved using the same reasoning as above. □

Now, we drop the condition $S(M) \subset T(M)$ from the above theorem and obtain the following result.

Theorem 3.2. *Let (M, d) be a metric space and $S, T : M \longrightarrow M$ mappings satisfying the property (E.A.). Let us suppose that the pair (S, T) is non-trivially weakly compatible $\psi - (\alpha, \beta, m)$-contraction pair. If $T(M) \subset M$ is closed, then S and T have a unique common fixed point.*

Proof: Since the pair (S, T) satisfies the property (E. A.), there exists a sequence $(x_n) \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in M$. Since $T(M)$ is closed, then $z \in T(M)$ and $z = Tu$ for some $u \in M$. As in the proof of the Theorem 2.5, we can prove that $z = Tu = Su$ and that z is a unique POC of S and T. The existence of the unique common fixed point follows as in the proof of Theorem 3.1. □

Remark 3.3. *Since noncompatible mappings on a metric space (M, d) satisfy the property (E. A.) . Therefore, conclusion of Theorem 3.2 remains valid if we consider S and T, noncompatible mappings.*

We can replace conditions (i) and (ii) of Theorem 2.5 by a single condition and obtain the following result. Here $\overline{S(M)}$ denotes the closure of the range of the mapping S.

Theorem 3.4. *Let S and T be self-mappings on a metric space (M, d) such that*

(i) $\overline{S(M)} \subset M$ *is a complete subspace of* M.

(ii) The pair (S, T) *is a* $\psi - (\alpha, \beta, m)$*-contraction pair.*

Then the pair (S, T) has a unique POC. Furthermore, if the pair (S, T) is non-trivially weakly compatible, then S and T have a unique common fixed point.

In the next result, we drop the closedness of the range of mapping and replace the property (E. A.) by CLR_T property.

Theorem 3.5. *Let (M, d) be a metric space and $S, T : M \longrightarrow M$ satisfying the CLR_T property. Let us suppose that the pair (S, T) is a $\psi - (\alpha, \beta, m)$-contraction pair. If the pair (S, T) is non-trivially weakly compatible, then S and T have a unique common fixed point.*

Proof: Since the pair (S, T) satisfies the CLR_T property, then there exists a sequence $(x_n) \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tz, \quad \text{for some } z \in M.$$

The rest of the proof runs with similarities to the proof of the previous results. □

Remark 3.6. *Notice that by considering particular functions, as constants, for the functions α, β as well as by considering $\psi = id$ (the identity mapping), or by choosing a particular form for $m(x, y)$ in the class of $\psi - (\alpha, \beta, m)$-contraction pairs, we can obtain several subclasses of mappings, including various important classes of contraction-type of mappings, as the given by B.K. Das and S. Gupta [4], G. Jungck [7], M.S. Khan et al [10], J.R. Morales and E.M. Rojas [12,13] among other authors.*

4. On the existence of solutions for a class of nonlinear integral equations

In this section we will study the existence of solutions for a class of nonlinear integral equations by using the existence of coincidence and common fixed points for mappings belonging to the $\psi - (\alpha, \beta, m)$-contraction class.

Let $M = C([0,T],\mathbb{R})$ denote the space of all continuous functions on $[0,T]$, which, as it is well-known, is a complete metric space when it is equipped with the uniform metric d

$$d(u,v) = \sup_{t\in[0,T]} \{|u(t) - v(t)|\}, \quad u, v \in M. \tag{4.1}$$

Now, following the idea in ([15], see also [5]) , we discuss an application of fixed point techniques to the solution of the nonlinear integral equation:

$$x(t) = g_1(t) - g_2(t) + \mu \int_0^t V_1(t,s)h_1(s,x(s))\, ds + \Lambda \int_0^T V_2(t,s)h_2(s,x(s))ds, \tag{4.2}$$

where $t \in [0,T]$, μ, Λ are real numbers, $g_1, g_2 \in C([0,T],\mathbb{R})$ and $V_1(t,s)$, $V_2(t,s)$, $h_1(t,s)$, $h_2(t,s)$ are continuous real-valued functions in $[0,T]\times\mathbb{R}$.

To attain our aim, we will use some functional associated with h-concave and quasilinear functions [14]. Let C be a convex cone in the linear space X over $\mathbb{R}$ and let L be a real number $L \neq 0$. A functional $\psi : C \longrightarrow \mathbb{R}$ is called *L-superadditive* on C if

$$f(x+y) \geq L\left(f(x) + f(y)\right), \quad \text{for any } x, y \in C.$$

Let K be a real non-negative function, a functional ψ satisfying

$$\psi(tx) = K(t)\psi(x)$$

for any $t \geq 0$ and $x \in C$, is called *K-positive homogeneous*. Notice that necessarily $K(1) = 1$.

The existence of solutions for the nonlinear integral equation (4.2) will be analyzed by using some auxiliary operators S and T (see, (4.4)-(4.5) below) belonging to the $\psi - (\alpha,\beta,m)$-contraction class. The conclusion is obtained from the existence of coincidence points or common fixed points for (S,T). We would like to point out that our results remain valid if in equation (4.2) we replace the kernels $h_i(s,x(s))$ for ones of the form $h_i(t,x(s))$.

To prove our result we will make use of the following lemma.

Lemma 4.1 ([14]). *Let $u, v \in C$ and $\psi : C \longrightarrow \mathbb{R}$ be a non-negative, L-superadditive and K-positive homogeneous functional on C. If $M \geq m > 0$ are such that $u - mv$ and $Mv - u \in C$, then*

$$LK(m)\psi(v) \leq \psi(u) \leq \frac{1}{L}K(M)\psi(v).$$

The existence result can be formulated as follows.

Theorem 4.2 (Existence). *Suppose the following assumptions are satisfied:*

(i) $\int_0^T \sup_{t\in[0,T]} |V_i(t,s)|ds = L_i < \infty$, $i \in \{1,2\}$,

(ii) for each $s \in [0,T]$ and for all $x, y \in M$, there is $M_i \geq 0$ such that

$$|h_i(s,x(s)) - h_i(s,y(s))| \leq M_i|x(s) - y(s)|, \quad i \in \{1,2\}.$$

Then the integral equation (4.2) has at least one solution in M, provided that

$$|\mu|L_1M_1 + |\Lambda|M_2L_2 = 1. \tag{4.3}$$

Proof: We define the following operators, for each $x \in M$,

$$Sx(t) = -g_2(t) + \mu\int_0^t V_1(t,s)h_1(s,x(s))ds \tag{4.4}$$

and

$$Tx(t) = x(t) - g_1(t) - \Lambda\int_0^T V_2(t,s)h_2(s,x(s))\,ds, \tag{4.5}$$

where $t \in [0,T]$, μ, Λ are real numbers, $g_1, g_2 \in C([0,T],\mathbb{R})$ and $V_1(t,s)$, $V_2(t,s)$, $h_1(t,s)$, $h_2(t,s)$ are continuous real-valued functions in $[0,T]\times\mathbb{R}$ satisfying assumptions (i)–(ii) above.

Clearly, S and T are self-operators on M. Now, for all $x, y \in M$ by using (i)–(ii), we get

$$\begin{aligned}
|Sx(t) - Sy(t)| &\leq |\mu|\int_0^t |V_1(t,s)||h_1(s,x(s)) - h_1(s,y(s))|ds \\
&\leq |\mu|\int_0^t \sup_{t\in[0,T]} |V_1(t,s)||h_1(s,x(s)) - h_1(s,y(s))|ds \\
&\leq |\mu|\int_0^t \sup_{t\in[0,T]} |h_1(t,s)|M_1|x(s) - y(s)|ds \\
&\leq |\mu|L_1\|x-y\|\int_0^t \sup_{t\in[0,T]} |h_1(t,s)|ds \\
&\leq |\mu|M_1L_1\|x-y\|.
\end{aligned}$$

This implies that

$$\|Sx - Sy\| = \sup_{t\in[0,T]} |Sx(t) - Sy(t)| \leq |\mu|M_1L_1\|x-y\|. \tag{4.6}$$

By a similar reasoning we get

$$\begin{aligned}
&\left|\Lambda\int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda\int_0^T V_2(t,s)h_2(s,y(s))\,ds\right| \\
&\le |\Lambda|\int_0^T |V_2(t,s)||h_2(s,x(s)) - h_2(s,y(s))|ds \\
&\le |\Lambda|\int_0^T \sup_{t\in[0,T]} |V_2(t,s)||h_2(s,x(s)) - h_2(s,y(s))|ds \\
&\le |\Lambda|\int_0^T \sup_{t\in[0,T]} |V_2(t,s)|M_2|x(s) - y(s)|ds \\
&\le |\Lambda|M_2L_2\|x-y\|,
\end{aligned}$$

which implies

$$\sup_{t\in[0,T]}\left|\Lambda\int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda\int_0^T V_2(t,s)h_2(s,y(s))\,ds\right| \le |\Lambda|L_2M_2\|x-y\|.$$

Consequently, we note that

$$\begin{aligned}
&\|Tx-Ty\| \\
&\ge \|x-y\| - \sup_{t\in[0,T]}\left|\Lambda\int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda\int_0^T V_2(t,s)h_2(s,y(s))\,ds\right| \\
&\ge (1-|\Lambda|L_2M_2)\|x-y\|, \qquad (4.7)
\end{aligned}$$

since condition (4.3) implies that $|\Lambda|L_2M_2 < 1$, the above inequality gives

$$\|x-y\| \le \frac{1}{1-|\Lambda|L_2M_2}\|Tx-Ty\|. \qquad (4.8)$$

Finally, by (4.6), (4.8) and condition (4.3), we get

$$\|Sx-Sy\| \le \|Tx-Ty\|.$$

Moreover, there exists $0 \le m < 1$ depending of x and y such that

$$m\|Tx-Ty\| \le \|Sx-Sy\| \le \|Tx-Ty\|. \qquad (4.9)$$

Now, let ψ be a non-negative, continuous, 2-superadditive and K-positive homogeneous functional on the cone $\mathbb{R}_+$ satisfying (2.2). For $u = \|Sx-Sy\|$, $v = \|Tx-Ty\|$ and the inequality (4.9), the Lemma 4.1 allows us to conclude that,

$$\psi(\|Sx-Sy\|) \le \frac{1}{2}\psi(\|Tx-Ty\|).$$

Let $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0,1)$ satisfying (2.1) with $\frac{1}{2} \leq \alpha(t)$ for any $t \in \mathbb{R}_+$. Hence, we obtain

$$\begin{aligned}\psi(\|Sx - Sy\|) \leq & \frac{1}{2}\psi(\|Tx - Ty\|)\\ \leq & \alpha(\|Tx - Ty\|)\psi(\|Tx - Ty\|) + \beta(\|Tx - Ty\|)\psi(m(x, y)).\end{aligned}$$

Therefore, (S, T) is a $\psi - (\alpha, \beta, m)$-contraction pair.

Since S is a continuous map and M is complete, $\overline{S(M)}$ is a complete subspace of M, therefore from Theorem 3.4, the pair (S, T) has a unique POC (say y_0); i.e., $y_0 = Sx_*(t) = Tx_*(t)$. Thus,

$$-g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, x_*(s))ds = x_*(t) - g_1(t) - \Lambda \int_0^T V_2(t, s)h_2(s, x_*(s))ds$$

or equivalently,

$$x_*(t) = g_1(t) - g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, x_*(s))ds + \Lambda \int_0^T V_2(t, s)h_2(s, x_*(s))ds.$$

Therefore, $x_* \in M$ is a solution of the nonlinear integral equation (4.2). □

Remark 4.3. *In the light of above theorem, we note that if the auxiliary pair (S, T) associated to equation (4.2) and defined by formulae (4.4)-(4.5) has a unique POC, then all CP related with the POC is a solution of the equation.*

Under the notion of non-trivial weak compatibility of the pair (S, T) given in (4.4)-(4.5), the next result shows that there exists a (unique) solution of the equation (4.2) satisfying a certain integral equation.

Proposition 4.4. *Under the hypotheses of Theorem 4.2, if the pair of mappings (S, T) defined in (4.4)-(4.5) is non-trivially weakly compatible, then there is a unique solution ϕ of the equation (4.2) satisfying the integral equation*

$$g_1(t) = -\Lambda \int_0^T V_2(t, s)h_2(s, \phi(s))ds.$$

Proof: Since the pair (S, T) is non-trivially weakly compatible, from Theorem 3.1 there is a unique solution ϕ satisfying that $S\phi(t) = T\phi(t) = \phi(t)$, moreover $ST\phi(t) = TS\phi(t)$, where

$$\begin{aligned}ST\phi(t) =& - g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, \phi(s))ds\\ TS\phi(t) =& - g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, \phi(s))ds - g_1(t) - \Lambda \int_0^T V_2(t, s)h_2(s, \phi(s))ds.\end{aligned}$$

From here we obtain,

$$\begin{aligned}-g_2(t)+\mu\int_0^t V_1(t,s)h_1(s,\phi(s))ds =& -g_2(t)+\mu\int_0^t V_1(t,s)h_1(s,\phi(s))ds - g_1(t)\\ &-\Lambda\int_0^T V_2(t,s)h_2(s,\phi(s))ds\\ g_1(t) =& -\Lambda\int_0^T V_2(t,s)h_2(s,\phi(s))ds.\end{aligned}$$

This completes the proof. □

Remark 4.5. *In view of the proof of Proposition 4.3, one can observe that the only solution which satisfies the equation* $g_1(t)=-\Lambda\int_0^T V_2(t,s)h_2(s,\phi(s))ds$, *is a unique common fixed point of the pair* (S,T) *defined in* (4.4)-(4.5).

4.1. The equation (4.2) on compact subspaces of (M,d)

In that follows by $(\mathcal{K},d)$ we denote a compact subspace of M endowed with the induced uniform metric d defined in (4.1).

In order to establish the existence result in this case, we will use the operator S given in (4.4) and the next auxiliary mapping:

$$Rx(t)=\kappa x(t)-g_1(t)-\Lambda\int_0^T V_2(t,s)h_2(s,x(s))ds,\quad 0\le\kappa<1. \tag{4.10}$$

Theorem 4.6. *Under assumptions (i)-(ii) of Theorem 4.2, if* S,R *defined in* (4.4) *and* (4.10) *are non-trivially weakly compatible self-mappings of* $(\mathcal{K},d)$, *then the equation*

$$x(t)=g_1(t)-g_2(t)+\mu\int_0^t V_1(t,s)h_1(s,x(s))\ ds+\Lambda\int_0^T V_2(t,s)h_2(s,x(s))ds,\quad x\in\mathcal{K}.$$

has a unique solution $\phi\in\mathcal{K}$ *satisfying*

$$g_1(t)=-\Lambda\int_0^T V_2(t,s)h_2(s,\phi(s))ds,$$

provided that

$$|\mu|L_1M_1+|\Lambda|M_2L_2=\kappa,\quad (0\le\kappa<1),\quad \textit{holds.} \tag{4.11}$$

Proof: We claim that (S,R) has the property (E. A.) if it is non-trivially weakly compatible. In fact, let $\phi_n\to\phi$ a sequence of functions on $\mathcal{K}$ converging to ϕ, where the function ϕ is a unique point of coincidence of the weakly compatible pair (S,R). From the continuity of the function $h_i(t,s)$ we have

$$\begin{aligned}\lim_{n\to\infty}S\phi_n(t) =& -g_2(t)+\mu\int_0^t V_1(t,s)h_1(s,\lim_{n\to\infty}\phi_n(s))ds=S\phi(t),\\ \lim_{n\to\infty}R\phi_n(t) =& \kappa\phi_n(t)-g_1(t)-\Lambda\int_0^T V_2(t,s)h_2(s,\lim_{n\to\infty}\phi_n(t))ds=R\phi(t).\end{aligned}$$

Then, we conclude that (S, T) has the property (E. A.).

On the other hand, it is easy to check that the operator R is continuous on $(\mathcal{K}, d)$. Since $\mathcal{K}$ is a compact and Hausdorff space, the Closed Map Lemma implies that $R(\mathcal{K})$ is closed. Thus, from Theorem 3.2, R and S have a unique common fixed point $\phi \in \mathcal{K}$. The existence of a unique solution satisfying the above relation is obtained from the proof of Theorem 4.2, replacing the mapping T by R. The representation for the solution follows from the proof of Proposition 4.4, upon replacing T by R. □

4.2. The equation (4.2) on non-complete metric space

The existence Theorem 4.2 was proved by applying Theorem 3.4, since $\overline{S(M)}$ is a complete subspace. However, if equation (4.2) is posed in a non-complete metric subspace $(\mathcal{X}, d)$ of (M, d), we are not able to apply such theorem. By imposing an extra condition we obtain the following existence result for this case.

Theorem 4.7. *(Existence: non-complete metric space). Suppose the following assumptions are satisfied:*

(i) $\int_0^T \sup_{t\in[0,T]} |V_i(t,s)|ds = L_i < \infty$, $i \in \{1, 2\}$,

(ii) for each $s \in [0, T]$ *and for all* $x, y \in M$, *there is* $M_i \geq 0$ *such that*

$$|h_i(s, x(s)) - h_i(s, y(s))| \leq M_i|x(s) - y(s)|, \quad i \in \{1, 2\},$$

(iii) $\Lambda \int_0^T V_2(t,s)h_2\left(s, \mu \int_0^s V_1(s,\kappa)k_1(\kappa, x(\kappa))d\kappa + g_1(s) - g_2(s)\right) ds = 0.$

Then, the integral equation (4.2) has a unique solution, $\phi \in \mathcal{X}$, *satisfying*

$$g_1(t) = -\Lambda \int_0^T V_2(t,s)h_2(s, \phi(s))ds$$

provided that

$$|\mu|L_1M_1 + |\Lambda|M_2L_2 = 1.$$

Proof: From the proof of Theorem 4.2, it is sufficient to show that the pair (S, T) defined in (4.4)-(4.5) has a POC in $\mathcal{X}$. To do so we will apply Theorem 2.5, thus we prove that $S(M) \subseteq T(M)$.

In fact, adopting the same reasoning as in [15], by assumption (iii), for $x(t) \in \mathcal{X}$ we have

$$\begin{aligned}
&T(Sx(t) + g_1(t)) \\
&= Sx(t) + g_1(t) - g_1(t) - \Lambda \int_0^T V_2(t,s)h_2(s, Sx(s) + g_1(s))ds \\
&= Sx(t) - \Lambda \int_0^T V_2(t,s)h_2\left(s, \mu \int_0^s V_1(s,\kappa)h_1(\kappa, x(\kappa))d\kappa + g_1(s) - g_2(s)\right) ds \\
&= Sx(t).
\end{aligned}$$

Thus, from Theorem 2.5 S and T have a unique POC, so all coincidence point related with the POC is a solution of the integral equation (4.2) in $\mathfrak{X}$. As the proof of Proposition 4.4, the formula for the solution is a consequence of the non-trivially weakly compatibility and the existence of a unique common fixed point. $\square$

5. Examples

In this section we are going to consider some nonlinear integral equations on $C([0,1],\mathbb{R})$ defined in (4.2). The existence of solutions will be established as an application of the previous results.

Example 5.1. *Let us consider the following the nonlinear integral equation:*

$$\begin{aligned} x(t) =& g_1(t) - g_2(t) + \frac{1}{2}\int_0^t 2t\frac{s}{2}x(s)ds + \frac{1}{2}\int_0^1 4t^3 s\frac{-x(s)}{4}ds. \\ =& g_1(t) - g_2(t) + \frac{t}{2}\int_0^t sx(s)ds - \frac{1}{2}\int_0^1 t^3 sx(s)ds, \quad t \in [0,1]. \end{aligned} \tag{5.1}$$

Taking $\mu = \Lambda = \frac{1}{2}$ and the kernel $C([0,1]\times\mathbb{R},\mathbb{R})$-functions $h_i(s,x(s))$ and $V_i(s,x(s))$, $i \in \{1,2\}$ given by

$$h_1(s,x(s)) = \frac{s}{2}x(s), \qquad h_2(s,x(s)) = -\frac{x(s)}{4}$$

and

$$V_1(t,s) = 2t, \qquad V_2(t,s) = 4t^3 s.$$

Notice that the functions $h_i(s,x(s))$, $i \in \{1,2\}$ satisfy

$$|h_i(s,x(s)) - h_i(s,y(s))| \le \frac{1}{2}|x(s)-y(s)|, \quad \text{for all } x,y \in C([0,1],\mathbb{R}), \quad i \in \{1,2\}$$

and the functions $V_i(s,x(s))$, $i \in \{1,2\}$ satisfy

$$\int_0^1 \sup_{t\in[0,1]} |V_i(t,s)|ds = 2.$$

Thus, Theorem 4.2 guarantees that this equation has at least one solution, and from the proof of the mentioned theorem, the solution is the CP of the mappings S and T defined by

$$\begin{aligned} Sx(t) =& -g_2(t) + \frac{1}{2}\int_0^t tsx(s)ds, \\ Tx(t) =& x(t) - g_1(t) + \frac{1}{2}\int_0^1 t^3 sx(s)ds. \end{aligned}$$

Now, let $x^\star$ be a coincidence point of (S,T), and we assume that the following system is satisfied

$$\begin{cases} x^\star(t) = g_1(t) - g_2(t) \\ \frac{1}{2}\int_0^1 t^3 s x^\star(s)ds = \frac{1}{2}\int_0^t tsx^\star(s)ds \end{cases} \quad \text{for all } t \in [0,1]. \tag{5.2}$$

Since $t = 0$ obviously holds, we assume $t \neq 0$. Notice that the second equality of the system is equivalent to

$$\begin{aligned} t^3 \int_0^1 sx^\star(s)ds &= t\int_0^t sx^\star(s)ds \\ t^2 \int_0^1 sx^\star(s)ds &= \int_0^t sx^\star(s)ds. \end{aligned}$$

Differentiating with respect to t, equality above is equivalent to

$$\begin{aligned} 2t \int_0^1 sx^\star(s)ds &= tx^\star(t) \\ 2 \int_0^1 sx^\star(s)ds &= x^\star(t). \end{aligned}$$

That means, the constant functions are the only coincidence point of (T,S) satisfying (5.2), provided $g_1(t) - g_2(t)$ is also constant. Let $x^\star(t) \equiv \varsigma \in \mathbb{R}$, we obtain

$$\begin{aligned} S\varsigma =& -g_2(t) + \frac{1}{2}\int_0^t ts\varsigma ds = -g_2(t) + \frac{1}{4}t^3\varsigma, \\ T\varsigma =& \varsigma - g_1(t) + \frac{1}{2}\int_0^1 t^3 s\varsigma ds = \varsigma - g_1(t) + \frac{1}{4}t^3\varsigma. \end{aligned}$$

Therefore, equation (5.1) for $g_1(t) - g_2(t) \equiv \varsigma \in \mathbb{R}$ has a solution which is nothing but the constant function $x^\star(t) \equiv \varsigma$.

On the other hand, notice that the pair (S,T) is not weakly compatible. In fact,

$$\begin{aligned} ST\varsigma =& -g_2(t) + \frac{1}{2}\int_0^t ts(-g_2(s) + \frac{1}{4}\varsigma s^3)ds \\ =& -g_2(t) + \frac{\varsigma}{32}t^4 - \frac{t}{2}\int_0^t sg_2(s)ds \end{aligned}$$

and

$$\begin{aligned} TS\varsigma =& -g_2(t) - g_1(t) + \frac{1}{4}t^3\varsigma - \frac{t^3}{2}\int_0^1 g_2(s)sds + \frac{t^3}{8}\varsigma\int_0^1 s^4 ds \\ =& \varsigma + \frac{11}{40}t^3\varsigma - \frac{t^3}{2}\int_0^1 g_2(s)sds. \end{aligned}$$

Therefore, the solution $x(t)^\star \equiv \varsigma$ does not satisfy the integral equation given in Proposition 4.4.

Example 5.2. *Now, we will consider the following integral equation*

$$\begin{aligned} x(t) =&\frac{(\cos(1)-\cos(e))e^{-t}}{e-1} - \frac{e^{4t}-e^{t}}{6(e^2-e)} + e^t + \frac{1}{e-1}\int_0^1 e^{s-t}\sin(x(s))ds \\ &+ \frac{1}{e^2-e}\int_0^t e^{t+s}e^s\frac{x(s)}{2}ds, \quad t\in[0,1]. \end{aligned} \tag{5.3}$$

Equation (5.3) *is of the form* (4.2), *for*

$$h_1(t,x(t)) = e^t\frac{x(t)}{2},\ V_1(t,s)=e^{t+s},\ h_2(t,x(t))=\sin(x(t)),\ V_2(t,s)=e^{s-t},$$

$$g_1(t) = \frac{(\cos(1)-\cos(e))e^{-t}}{e-1}, \quad g_2(t) = \frac{e^{4t}-e^t}{6(e^2-e)} - e^t$$

and $\Lambda = -\frac{1}{e-1}$, $\mu = \frac{1}{e^2-e}$. *Notice that* $L_1 = e^2-e$, $L_2 = e-1$, $M_1 = \frac{1}{2}$ *and* $M_2 = 1$. *Also note that* $|\Lambda|M_1L_1 + |\mu|L_2M_2 = 1$. *Let the mappings* (S,T) *given in this case by*

$$\begin{aligned} Sx(t) =&e^t - \frac{e^{4t}-e^t}{6(e^2-e)} + \frac{1}{e^2-e}\int_0^t e^{s+t}e^s\frac{x(s)}{2}ds \\ Tx(t) =&x(t) - \frac{(\cos(1)-\cos(e))e^{-t}}{e-1} + \frac{1}{e-1}\int_0^1 e^{s-t}\sin(x(s))ds. \end{aligned}$$

We are going to find the coincidence points of (S,T). *A point* $x^\star\in M$ *is a CP of* (S,T) *if*

$$\begin{aligned} &e^t\left[1-\frac{e^{3t}-1}{6(e^2-e)} + \frac{1}{e^2-e}\int_0^t e^{2s}\frac{x^\star(s)}{2}ds\right] \\ &= e^{-t}\left[x(t)e^t - \frac{\cos(1)-\cos(e)}{e-1} + \frac{1}{e-1}\int_0^1 e^s\sin(x^\star(s))ds\right], \end{aligned}$$

equivalently,

$$\begin{aligned} &e^{2t}\left[1-\frac{e^{3t}-1}{6(e^2-e)} + \frac{1}{e^2-e}\int_0^t e^{2s}\frac{x^\star(s)}{2}ds\right] \\ &= x(t)e^t - \frac{\cos(1)-\cos(e)}{e-1} + \frac{1}{e-1}\int_0^1 e^s\sin(x^\star(s))ds. \end{aligned} \tag{5.4}$$

Since the term

$$-\frac{cos(1)-\cos(e)}{e-1} + \frac{1}{e-1}\int_0^1 e^s\sin(x^\star(s))ds$$

is constant and the left side of equality (5.4) *depends of* t, *necessarily we have that*

$$\frac{1}{e-1}\int_0^1 e^s\sin(x^\star(s))ds = \frac{cos(1)-\cos(e)}{e-1}$$

whose solution is $x^\star(t) = e^t$*. Notice that equality* (5.4) *is satisfied for this function. Therefore,* $x^\star(t) = e^t$ *is a unique coincidence point of* (T, S)*, moreover,*

$$Se^t = Te^t = e^t, \quad STe^t = TSe^t = e^t.$$

Thus, the pair (S, T) *is non-trivially weakly compatible, so from Proposition 4.4, equation* (5.3) *has a solution satisfying the integral equation*

$$\begin{aligned} g_1(t) =& -\Lambda \int_0^1 V_2(t,s)h_2(s,x(s))ds \\ \frac{(cos(1) - \cos(e))e^{-t}}{e-1} =& \frac{1}{e-1}\int_0^1 e^{s-t}\sin(x(s))ds, \end{aligned}$$

whose solution is $x(t) = e^t$.

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