



Lacunary Statistical and Lacunary Strongly Convergence of Generalized Difference Sequences in Intuitionistic Fuzzy Normed Linear Spaces

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ABSTRACT: In this article we introduce the concepts of lacunary statistical convergence and lacunary strongly convergence of generalized difference sequences in intuitionistic fuzzy normed linear spaces and give their characterization. We obtain some inclusion relation relating to these concepts. Further some necessary and sufficient conditions for equality of the sets of statistical convergence and lacunary statistical convergence of generalized difference sequences have been established. The notion of strong Cesàro summability in intuitionistic fuzzy normed linear spaces has been introduced and studied. Also the concept of lacunary generalized difference statistically Cauchy sequence has been introduced and some results are established.

Key Words: Statistical convergence, Lacunary sequence, Intuitionistic fuzzy normed linear space, Difference sequence, Cesàro convergence.

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1. Introduction

Ever since the theory of fuzzy sets was introduced by Zadeh [38] in 1965, the potential of the introduced notion was realised by researchers and it has been applied in various branches of science like Statistics, Artificial Intelligence, Computer Programming, Operation Research, Quantum Physics, Pattern Recognition, Decision Making etc. Some of its application can be found in ([8], [12], [14], [17], [19], [24]). An important development of the classical fuzzy sets theory is the theory of intuitionistic fuzzy sets (IFS) proposed by Atanassov [7]. IFS give us a very natural tool for modeling imprecision in real life situations and found applications in various areas of science and engineering. Generalizing the idea of ordinary normed linear

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space, Saadati and Park [28] introduced the notion of intuitionistic fuzzy normed linear space. There after the theory has emerged as an active area of research in many branches of Mathematics like approximation theory, stability of functional equations, summability theory etc. The idea of statistical convergence was first introduced by Steinhaus [34] and Fast [13] which was later on studied by many authors. Schoenberg [30] studied statistical convergence as a summability method and studied some properties of statistical convergence. Altin et al. [5] have studied statistical summability method $(C, 1)$ for sequences of fuzzy real numbers. Karakus et al. [20] generalized the concept of statistical convergence on intuitionistic fuzzy normed spaces. Some works in this field can be found in ([1], [22], [23], [25], [32]). Generalizing the idea of statistical convergence, Fridy and Orhan [15] introduced the idea of lacunary statistical convergence. Some works in lacunary statistical convergence can be found in ([2], [15], [16], [18], [26], [27], [29], [33], [36]). The idea of difference sequence was introduced by Kizmaz [21] and later on it was further investigated by different researchers in classical as well as fuzzy sequence spaces ([3], [4], [6], [9], [10], [11], [35], [37]).

The aim of the present paper is to introduce the concepts of lacunary statistical convergence and lacunary strongly convergence of generalized difference sequences in intuitionistic fuzzy normed linear spaces (IFNLS) and obtain some important results on this concept. Also we have introduced the concept of lacunary generalized difference statistically Cauchy sequences and given some new characterizations of it.

2. Preliminaries

Throughout this paper R and N will denote the set of real numbers and the set of natural numbers respectively.

Using the definitions of continuous t -norm and continuous t -conorm found in [31], an IFNLS is defined as follows:

Definition 2.1. *An intuitionistic fuzzy normed linear space (in short, IFNLS) is a five-tuple $(X, \mu, \nu, *, \circ)$, where X is a linear space, $*$ is a continuous t -norm, \circ is a continuous t -conorm and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:*

- (i) $\mu(x, t) + \nu(x, t) \leq 1$,
- (ii) $\mu(x, t) > 0$,
- (iii) $\mu(x, t) = 1$ if and only if $x = 0$,
- (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
- (v) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
- (vi) $\mu(x, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t ,
- (vii) $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
- (viii) $\nu(x, t) < 1$,
- (ix) $\nu(x, t) = 0$ if and only if $x = 0$,

- (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
- (xi) $\nu(x, t) \circ \nu(y, s) \geq \nu(x + y, t + s)$,
- (xii) $\nu(x, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t ,
- (xiii) $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$.

Et and Colak [11] introduced the notion of generalized difference sequences as follows :

Definition 2.2. Let m be a non-negative integer, then the generalized difference operator $\Delta^m x_k$ is defined as $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$, where $\Delta^0 x_k = x_k$, for all $k \in N$.

Using this concept, we can define Δ^m -convergent and Δ^m -Cauchy sequences in IFNLS as follows:

Definition 2.3. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be Δ^m -convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in N$ such that $\mu(\Delta^m x_k - L, t) > 1 - \varepsilon$ and $\nu(\Delta^m x_k - L, t) < \varepsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu) - \lim \Delta^m x_k = L$.

Definition 2.4. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. We say that a sequence $x = \{x_k\}$ in X is Δ^m -Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in N$ such that $\mu(\Delta^m x_k - \Delta^m x_n, t) > 1 - \varepsilon$ and $\nu(\Delta^m x_k - \Delta^m x_n, t) < \varepsilon$ for all $k, n \geq k_0$.

Definition 2.5. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be Δ^m -bounded with respect to the intuitionistic fuzzy norm (μ, ν) if, there exists $\varepsilon \in (0, 1)$ and $t > 0$, such that $\mu(\Delta^m x_k, t) > 1 - \varepsilon$ and $\nu(\Delta^m x_k, t) < \varepsilon$. Let $\ell_\infty^{(\mu, \nu)}(\Delta^m)$ denotes the set of all Δ^m -bounded sequences in IFNLS $(X, \mu, \nu, *, \circ)$.

Definition 2.6. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r . Let $K \subseteq N$. The number

$$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{k \in I_r : k \in K\}|$$

is said to be the θ -density of K , provided the limit exists.

Definition 2.7. Let θ be a lacunary sequence. A sequence $x = \{x_k\}$ of numbers is said to be lacunary statistically convergent (briefly S_θ -convergent) to the number L if for every $\varepsilon > 0$, the set $K(\varepsilon)$ has θ -density zero, where

$$K(\varepsilon) = \{k \in I_r : |x_k - L| \geq \varepsilon\}.$$

In this case we write $S_\theta\text{-lim } x = L$.

3. Lacunary Δ^m -statistical convergence in IFNLS

In this section we define lacunary generalized difference statistical convergence in IFNLS and obtain our main results.

Definition 3.1. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. A sequence $x = \{x_k\}$ in X is said to be lacunary Δ^m -statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - L, t) \geq \varepsilon\}) = 0$.

In this case we write $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

The following lemma can be easily obtained using Definition 3.1 and properties of the θ -density.

Lemma 3.2. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. Then for every $\varepsilon \in (0, 1)$ and $t > 0$, the following statements are equivalent:

- (i) $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$,
- (ii) $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) \leq 1 - \varepsilon\}) = \delta_\theta(\{k \in N : \nu(\Delta^m x_k - L, t) \geq \varepsilon\}) = 0$,
- (iii) $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) > 1 - \varepsilon \text{ and } \nu(\Delta^m x_k - L, t) < \varepsilon\}) = 1$,
- (iv) $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) > 1 - \varepsilon\}) = \delta_\theta(\{k \in N : \nu(\Delta^m x_k - L, t) < \varepsilon\}) = 1$,
- (v) $S_\theta - \lim \mu(\Delta^m x_k - L, t) = 1$ and $S_\theta - \lim \nu(\Delta^m x_k - L, t) = 0$.

We formulate the following two preliminary results without proof.

Theorem 3.3. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. If $x = \{x_k\}$ is a sequence in X such that $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k$ exists, then the limit is unique.

Theorem 3.4. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. If $(\mu, \nu) - \lim \Delta^m x_k = L$, then $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

The converse of Theorem 3.4 is not true in general which follows from the following example.

Example 3.5. Consider $(R, |\cdot|)$, the space of real numbers with usual norm. Let $a * b = ab$, $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $t > 0$ and $x \in R$, let us define $\mu(x, t) = \frac{t}{t+|x|}$ and $\nu(x, t) = \frac{|x|}{t+|x|}$. Then $(R, \mu, \nu, *, \circ)$ is an IFNLS. Let $\theta = \{k_r\}$ be a lacunary sequence.

Define a sequence $x = \{x_k\}$ whose terms are given by

$$\Delta^m x_k = \begin{cases} k & \text{if } n - [\sqrt{h_r}] + 1 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

For every $\varepsilon \in (0, 1)$ and $t > 0$, let $K_r(\varepsilon, t) = \{k \in I_r : \mu(\Delta^m x_k, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k, t) \geq \varepsilon\}$.

Now $K_r(\varepsilon, t) = \{k \in I_r : |\Delta^m x_k| \geq \frac{\varepsilon t}{1-\varepsilon} > 0\} \subseteq \{k \in I_r : \Delta^m x_k = k\}$. Thus

$\frac{1}{h_r}|\{k \in I_r : k \in K_r(\varepsilon, t)\}| \leq \frac{\sqrt{h_r}}{h_r} \rightarrow 0$ as $r \rightarrow \infty$. Hence $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = 0$. However the sequence $\{\Delta^m x_k\}$ is not convergent in $(R, |\cdot|)$, it is not convergent with respect to the intuitionistic fuzzy norm (μ, ν) .

We state the following result without proof.

Lemma 3.6. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then*

(i) *If $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L_1$ and $S_\theta^{(\mu, \nu)} - \lim \Delta^m y_k = L_2$, then $S_\theta^{(\mu, \nu)} - \lim \Delta^m (x_k + y_k) = L_1 + L_2$.*

(ii) *If $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$ and $\alpha \in R$, then $S_\theta^{(\mu, \nu)} - \lim \Delta^m (\alpha x_k) = \alpha L$.*

The following result can easily be proved using standard techniques.

Theorem 3.7. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$ if and only if there exists an increasing index sequence $K = \{k_i\}$ of natural numbers such that $\delta_\theta(K) = 1$ and $(\mu, \nu) - \lim \Delta^m x_{k_i} = L$.*

Theorem 3.8. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$ if and only if there exists a sequence $y = \{y_k\}$ such that $(\mu, \nu) - \lim \Delta^m y_k = L$ and $\delta_\theta(\{k \in N : \Delta^m x_k = \Delta^m y_k\}) = 1$.*

Proof. Let $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$. By Theorem 3.7, we get an increasing index sequence $K = \{k_i\}$ of the natural numbers such that $\delta_\theta(K) = 1$ and $(\mu, \nu) - \lim \Delta^m x_{k_i} = L$. Consider the sequence y defined by

$$\Delta^m y_k = \begin{cases} \Delta^m x_k & \text{if } k \in K \\ L & \text{otherwise} \end{cases}$$

Then y serves our purpose.

Conversely suppose that x and y are be sequences such that $(\mu, \nu) - \lim \Delta^m y_k = L$ and $\delta_\theta(\{k \in N : \Delta^m x_k = \Delta^m y_k\}) = 1$. Then for every $\varepsilon \in (0, 1)$ and $t > 0$, we have

$$\begin{aligned} & \{k \in N : \mu(\Delta^m x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - L, t) \geq \varepsilon\} \\ & \subseteq \{k \in N : \mu(\Delta^m y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m y_k - L, t) \geq \varepsilon\} \cup \{k \in N : \Delta^m x_k \neq \Delta^m y_k\}. \end{aligned}$$

Since $(\mu, \nu) - \lim \Delta^m y_k = L$, so the set $\{k \in N : \mu(\Delta^m y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m y_k - L, t) \geq \varepsilon\}$ contains at most finitely many terms. Also by assumption, $\delta_\theta(\{k \in N : \Delta^m x_k \neq \Delta^m y_k\}) = 0$.

Hence $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - L, t) \geq \varepsilon\}) = 0$.

and so $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$. □

Theorem 3.9. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$ if and only if there exist sequences $\{y_k\}$ and $\{z_k\}$ in X such that $\Delta^m x_k = \Delta^m y_k + \Delta^m z_k$, for all $k \in N$ where $(\mu, \nu) - \lim \Delta^m y_k = L$ and $S_\theta^{(\mu, \nu)} - \lim \Delta^m z_k = 0$*

Proof. Let $S_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L$. By Theorem 3.7, there exists an increasing sequence $K = \{k_i\}$ of natural numbers such that $\delta_{\theta}(K) = 1$ and $(\mu, \nu) - \lim \Delta^m x_{k_i} = L$.

Define the sequences $\{y_k\}$ and $\{z_k\}$ as follows:

$$\Delta^m y_k = \begin{cases} \Delta^m x_k, & \text{if } k \in K \\ L, & \text{otherwise.} \end{cases}$$

and

$$\Delta^m z_k = \begin{cases} 0, & \text{if } k \in K \\ \Delta^m x_k - L, & \text{otherwise.} \end{cases}$$

Then $\{y_k\}$ and $\{z_k\}$ serves our purpose.

Conversely if such two sequences $\{y_k\}$ and $\{z_k\}$ exist with the required properties, then the result follows immediately from Theorem 3.4 and Lemma 3.6. \square

Definition 3.10. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be Δ^m -statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$,

$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \mu(\Delta^m x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - L, t) \geq \varepsilon\}| = 0$ and we write $S^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

Let $S(\Delta^m)$ and $S_{\theta}(\Delta^m)$ denote the sets of all Δ^m -statistically and lacunary Δ^m -statistically convergent sequences respectively in an IFNLS $(X, \mu, \nu, *, \circ)$.

For $x \in X, t > 0$ and $\alpha \in (0, 1)$, the ball centered at x with radius α is defined by

$$B^Y(x, \alpha, t) = \{y \in X : \mu(x - y, t) > 1 - \alpha \text{ and } \nu(x - y, t) < \alpha\}.$$

Theorem 3.11. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For any lacunary sequence θ , $S_{\theta}(\Delta^m) \subseteq S(\Delta^m)$ if and only if $\limsup_r q_r < \infty$.

Proof. If $\limsup_r q_r < \infty$, then there exists $H > 0$ such that $q_r < H$ for all r . Suppose that $x \in S_{\theta}(\Delta^m)$ and $S_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L$. For $t > 0$ and $\lambda \in (0, 1)$, let $N_r = |\{k \in I_r : \mu(\Delta^m x_k - L, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - L, t) \geq \lambda\}|$.

Then for $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \varepsilon \text{ for all } r > r_0. \quad (3.1)$$

Now let $K = \max\{N_r : 1 \leq r \leq r_0\}$ and choose n such that $k_{r-1} < n \leq k_r$. Then we have

$$\begin{aligned} & \frac{1}{n} |\{k \leq n : \mu(\Delta^m x_k - L, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - L, t) \geq \lambda\}| \\ & \leq \frac{1}{k_{r-1}} |\{k \leq k_r : \mu(\Delta^m x_k - L, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - L, t) \geq \lambda\}| \\ & = \frac{1}{k_{r-1}} \{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{K}{k_{r-1}}r_0 + \frac{1}{k_{r-1}}\{h_{r_0+1}\frac{N_{r_0+1}}{h_{r_0+1}} + \dots + h_r\frac{N_r}{h_r}\} \\ &\leq \frac{r_0K}{k_{r-1}} + \varepsilon q_r \\ &\leq \frac{r_0K}{k_{r-1}} + \varepsilon H. \end{aligned}$$

To prove the converse suppose that $\limsup_r q_r = \infty$. Since θ is a lacunary sequence, we can select a subsequence $\{k_{r(j)}\}$ of θ such that $q_{r(j)} > j$. Let $\xi (\neq 0) \in X$. We define a sequence $x = \{x_k\}$ as follows:

$$\Delta^m x_k = \begin{cases} \xi & \text{if } k_{r(j)-1} < k \leq 2k_{r(j)-1} \text{ for some } j = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Since $\xi (\neq 0)$, we can choose $t > 0$, $\lambda \in (0, 1)$ such that $\xi \notin B^Y(0, \lambda, t)$. Now for $j > 1$,

$$\frac{1}{h_{r(j)}}|\{k \leq k_{r(j)} : \mu(\Delta^m x_k, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k, t) \geq \lambda\}| < \frac{1}{j-1}.$$

Thus $x \in S_\theta(\Delta^m)$. But $x \notin S(\Delta^m)$. For

$$\frac{1}{2k_{r(j)-1}}|\{k \leq 2k_{r(j)-1} : \mu(\Delta^m x_k, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k, t) \geq \lambda\}| > \frac{1}{2}$$

and

$$\frac{1}{k_{r(j)}}|\{k \leq k_{r(j)} : \mu(\Delta^m x_k - \xi, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - \xi, t) \geq \lambda\}| \geq 1 - \frac{2}{j},$$

which is a contradiction. □

Theorem 3.12. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For any lacunary sequence θ , $S(\Delta^m) \subseteq S_\theta(\Delta^m)$ if and only if $\liminf_r q_r > 1$*

Proof. Let $\liminf_r q_r > 1$. Then there exists $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r , which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

Let $S^{(\mu, \nu)} - \lim \Delta^m x_k = L$. Then for each $t > 0$, $\lambda \in (0, 1)$ and sufficiently large r , we have

$$\begin{aligned} &\frac{1}{k_r}|\{k \leq k_r : \mu(\Delta^m x_k - L, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - L, t) \geq \lambda\}| \\ &\geq \frac{\delta}{1+\delta} \frac{1}{h_r}|\{k \in I_r : \mu(\Delta^m x_k - L, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - L, t) \geq \lambda\}|. \end{aligned}$$

Thus $S_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

To prove the converse suppose that $\liminf_r q_r = 1$. Since θ is a lacunary sequence, we can select a subsequence $\{k_{r(j)}\}$ of $\theta = \{k_r\}$ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r(j)} - 1}{k_{r(j-1)}} > j \text{ where } r(j) \geq r(j-1) + 2.$$

Let $\xi (\neq 0) \in X$. Consider the sequence $x = \{x_k\}$ defined by:

$$\Delta^m x_k = \begin{cases} \xi & \text{if } k \in I_{r(j)} \text{ for some } j = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

We show that $x \in S(\Delta^m)$. Let $t > 0$ and $\lambda \in (0, 1)$. Choose $t_1 > 0$ and $\lambda_1 \in (0, 1)$ such that $B(0, \lambda_1, t_1) \subset B(0, \lambda, t)$ and $\xi \notin B(0, \lambda_1, t_1)$. Also for each $n \in N$ we can find a positive integer j_n such that $k_{r(j_n)-1} < n \leq k_{r(j_n)}$. Then for each $n \in N$, we have

$$\frac{1}{n} |\{k \leq n : \mu(\Delta^m x_k, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k, t) \geq \lambda\}| \leq \frac{k_{r(j_n-1)} + h_{r(j_n)}}{k_{r(j_n)-1}} \leq \frac{2}{j_n}.$$

Thus $S^{(\mu, \nu)} - \lim \Delta^m x_k = L$. Next we show that $x \notin S_\theta(\Delta^m)$. Since $\xi (\neq 0)$, so we can choose $t > 0$ and $\lambda \in (0, 1)$ such that $\xi \notin B(0, \lambda, t)$. Thus

$$\lim_{j \rightarrow \infty} \frac{1}{h_{r(j)}} |\{k \in I_{r(j)} : \mu(\Delta^m x_k, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k, t) \geq \lambda\}| = 1$$

and for $r \neq r_j$,

$$\lim_{j \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \mu(\Delta^m x_k - \xi, t) \leq 1 - \lambda \text{ or } \nu(\Delta^m x_k - \xi, t) \geq \lambda\}| = 1.$$

Hence $x \notin S_\theta(\Delta^m)$, a contradiction. \square

Theorem 3.11 and Theorem 3.12 together give the following corollary.

Corollary 3.13. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For any lacunary sequence θ , $S(\Delta^m) = S_\theta(\Delta^m)$ if and only if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.*

4. Lacunary strongly Δ^m convergence in IFNLS

In this section we define lacunary strongly Δ^m convergence in IFNLS.

Definition 4.1. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. A sequence $x = \{x_k\}$ in X is said to be lacunary strongly Δ^m -convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exist $r_0 \in N$ such that*

$$\frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L, t) > 1 - \varepsilon \text{ and } \frac{1}{h_r} \sum_{k \in I_r} \nu(\Delta^m x_k - L, t) < \varepsilon \text{ for all } r \geq r_0.$$

In this case we write $N_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

Definition 4.2. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and θ be a lacunary sequence. A sequence $x = \{x_k\}$ in X is strongly Δ^m -Cesàro summable to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that*

$$\frac{1}{n} \sum_{k=1}^n \mu(\Delta^m x_k - L, t) > 1 - \varepsilon \text{ and } \frac{1}{n} \sum_{k=1}^n \nu(\Delta^m x_k - L, t) < \varepsilon \text{ for all } n \geq n_0.$$

In this case we write $|\sigma|^{(\mu, \nu)} - \lim \Delta^m x_k = L$.

Let $N_\theta(\Delta^m)$ and $|\sigma(\Delta^m)|$ denote the sets of all lacunary strongly Δ^m -convergent sequences and strongly Δ^m -Cesàro summable sequences in the IFNLS $(X, \mu, \nu, *, \circ)$.

Theorem 4.3. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and $\theta = (k_r)$ be a lacunary sequence. Then*

$$|\sigma(\Delta^m)| \subseteq N_\theta(\Delta^m) \text{ if } \liminf_r q_r > 1.$$

Proof. Let $\liminf_r q_r > 1$ and $x = (x_k) \in |\sigma(\Delta^m)|$. Then there exists $\delta > 0$ such that $q_r \geq 1 + \delta$ for all $r \geq 1$. Then

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L, t) - 1 \\ &= \frac{1}{h_r} \sum_{k=1}^{k_r} \mu(\Delta^m x_k - L, t) - \frac{1}{h_r} \sum_{k=1}^{k_{r-1}} \mu(\Delta^m x_k - L, t) - 1 \\ &= \frac{k_r}{h_r} \left[\frac{1}{k_r} \sum_{k=1}^{k_r} \mu(\Delta^m x_k - L, t) - 1 \right] - \frac{k_{r-1}}{h_r} \left[\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \mu(\Delta^m x_k - L, t) - 1 \right] \end{aligned}$$

Since $h_r = k_r - k_{r-1}$, $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$

Also the terms $\frac{1}{k_r} \sum_{k=1}^{k_r} \mu(\Delta^m x_k - L, t) - 1$ and $\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \mu(\Delta^m x_k - L, t) - 1$ both converges to 0.

So, $\frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L, t) \rightarrow 1$.

Similarly $\frac{1}{h_r} \sum_{k \in I_r} \nu(\Delta^m x_k - L, t) \rightarrow 0$.

So $(x_k) \in N_\theta(\Delta^m)$. □

Theorem 4.4. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and $\theta = (k_r)$ be a lacunary sequence. Then*

$N_\theta(\Delta^m) \subseteq |\sigma(\Delta^m)|$ if $\liminf_r q_r = 1$.

Proof. Let $\liminf_r q_r = 1$ and $x = (x_k) \in N_\theta(\Delta^m)$. Then for $t > 0$, we have

$$H_r = \frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L, t) \rightarrow 1$$

and $H'_r = \frac{1}{h_r} \sum_{k \in I_r} \nu(\Delta^m x_k - L, t) \rightarrow 0$ as $r \rightarrow \infty$.

Then for $\varepsilon > 0$, there exists $r_0 \in N$ such that $H_r < 1 + \varepsilon$ for all $r \geq r_0$. Also we can find $T > 0$ such that $H_r < T$ and $H'_r < T$, $r = 1, 2, \dots$. Let n be an integer with $k_{r-1} < n \leq k_r$. Then

$$\begin{aligned} & \frac{1}{n} \sum_{k=1}^n \mu(\Delta^m x_k - L, t) \\ & \leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_r} \mu(\Delta^m x_k - L, t) \\ & = \frac{1}{k_{r-1}} \left[\sum_{i \in I_1} \mu(\Delta^m x_k - L, t) + \dots + \sum_{i \in I_2} \mu(\Delta^m x_k - L, t) \right] \\ & = \sup_{1 \leq r \leq r_0} H_r \frac{k_{r_0}}{k_{r-1}} + \frac{h_{r_0+1}}{k_{r-1}} H_{r_0+1} + \dots + \frac{h_r}{k_{r-1}} H_r \\ & < T \frac{k_{r_0}}{k_{r-1}} + (1 + \varepsilon) \frac{k_r - k_{r_0}}{k_{r-1}}. \end{aligned}$$

Since $k_{r-1} \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $\frac{1}{n} \sum_{k=1}^n \mu(\Delta^m x_k - L, t) \rightarrow 1$. Similarly we can show that $\frac{1}{n} \sum_{k=1}^n \nu(\Delta^m x_k - L, t) \rightarrow 0$. Hence $x \in |\sigma(\Delta^m)|$. □

Theorem 4.5. *If $x = \{x_k\} \in N_\theta(\Delta^m) \cap |\sigma(\Delta^m)|$, then $N_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = |\sigma|^{(\mu, \nu)} - \lim \Delta^m x_k$.*

Proof. Let $N_\theta^{(\mu, \nu)} - \lim \Delta^m x_k = L_1$ and $|\sigma|^{(\mu, \nu)} - \lim \Delta^m x_k = L_2$.

Given $\varepsilon > 0$, choose $\lambda \in (0, 1)$ such that $(1 - \lambda) * (1 - \lambda) > 1 - \varepsilon$ and $\lambda \circ \lambda < \varepsilon$.

Now for $t > 0$, there exists $r_0 \in N$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L_1, \frac{t}{2}) > 1 - \lambda \text{ and } \frac{1}{h_r} \sum_{k \in I_r} \nu(\Delta^m x_k - L_1, \frac{t}{2}) < \lambda \text{ for all } r \geq r_0.$$

Also, there exists $n_0 \in N$ such that

$$\frac{1}{n} \sum_{k=1}^n \mu(\Delta^m x_k - L_2, \frac{t}{2}) > 1 - \lambda \text{ and } \frac{1}{n} \sum_{k=1}^n \nu(\Delta^m x_k - L_2, \frac{t}{2}) < \lambda \text{ for all } n \geq n_0.$$

Consider $r_1 = \max(r_0, n_0)$. Then we will get a $l \in N$ such that

$$\mu(\Delta^m x_l - L_1, \frac{t}{2}) \geq \frac{1}{h_r} \sum_{k \in I_r} \mu(\Delta^m x_k - L_1, \frac{t}{2}) > 1 - \lambda$$

$$\text{and } \mu(\Delta^m x_l - L_2, \frac{t}{2}) \geq \frac{1}{n} \sum_{k=1}^n \mu(\Delta^m x_k - L_2, \frac{t}{2}) > 1 - \lambda.$$

Therefore $\mu(L_1 - L_2, t) \geq (1 - \lambda) * (1 - \lambda) > 1 - \varepsilon$.

Since ε is arbitrary, $\mu(L_1 - L_2, t) = 1$ for all $t > 0$ and so $L_1 = L_2$. \square

The following theorem can be proved using the standard techniques, so we state without proof.

Theorem 4.6. *Let $\theta = (k_r)$ be a lacunary sequence and $x = \{x_k\}$ be a sequence in $(X, \mu, \nu, *, \circ)$. Then*

$$(i) N_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L \text{ implies } S_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L$$

$$(ii) x \in \ell_{\infty}^{(\mu, \nu)}(\Delta^m) \text{ and } S_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L \text{ implies } N_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L$$

$$(iii) \ell_{\infty}^{(\mu, \nu)}(\Delta^m) \cap S_{\theta}(\Delta^m) = \ell_{\infty}^{(\mu, \nu)}(\Delta^m) \cap N_{\theta}(\Delta^m).$$

5. Lacunary Δ^m -statistically Cauchy sequences in IFNLS

Definition 5.1. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be lacunary Δ^m -statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if there is a subsequence $(x_{k'(r)}) \in I_r$ for each r , $(\mu, \nu) - \lim \Delta^m x_{k'(r)} = L$ and for each $\varepsilon \in (0, 1)$, $t > 0$,*

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \mu(\Delta^m x_k - \Delta^m x_{k'(r)}, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - \Delta^m x_{k'(r)}, t) \geq \varepsilon\}| = 0.$$

Theorem 5.2. *The sequence $x = \{x_k\}$ in an IFNLS $(X, \mu, \nu, *, \circ)$ is Δ^m -statistically convergent if and only if it is lacunary Δ^m -statistically Cauchy in X .*

Proof. Let $S_{\theta}^{(\mu, \nu)} - \lim \Delta^m x_k = L$ and for each n we write

$$K_n = \{k \in N : \mu(\Delta^m x_k - L, t) > 1 - \frac{1}{n} \text{ and } \nu(\Delta^m x_k - L, t) < \frac{1}{n}\}.$$

Then $K_{n+1} \subseteq K_n$ for each n and $\lim_{r \rightarrow \infty} \frac{|K_n \cap I_r|}{h_r} = 1$. So there exists p_1 such that $r > p_1$ and $\frac{|K_1 \cap I_r|}{h_r} > 0$ i.e. $K_1 \cap I_r \neq \emptyset$. We next choose $p_2 > p_1$ such that $r \geq p_2$ implies $K_2 \cap I_r \neq \emptyset$. Then for each r satisfying $p_1 \leq r \leq p_2$ we choose $k'(r) \in I_r$ such that $k'(r) \in K_1 \cap I_r$. In general we choose $p_{n+1} > p_n$ such that $r \geq p_{n+1}$ implies $k'(r) \in K_n \cap I_r$. Thus $k'(r) \in I_r$ for each r and

$$\mu(\Delta^m x_{k'(r)} - L, t) > 1 - \frac{1}{n} \text{ and } \nu(\Delta^m x_{k'(r)} - L, t) < \frac{1}{n}.$$

Hence $(\mu, \nu) - \lim \Delta^m x_{k'(r)} = L$.

Using Theorem and 3.4 Lemma 3.6, it can be easily seen that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \mu(\Delta^m x_k - \Delta^m x_{k'(r)}, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta^m x_k - \Delta^m x_{k'(r)}, t) \geq \varepsilon\}| = 0.$$

Conversely suppose that $\{x_k\}$ is a lacunary Δ^m -statistically Cauchy sequence in X . For $\varepsilon > 0$ choose $\lambda \in (0, 1)$ such that $(1 - \lambda) * (1 - \lambda) > 1 - \varepsilon$ and $\lambda \circ \lambda < \varepsilon$.

Then for any $t > 0$ define

$$K_{\mu, 1} = \{k \in N : \mu(\Delta^m x_k - \Delta^m x_{k'(r)}, \frac{t}{2}) > 1 - \lambda\}$$

$$\text{and } K_{\mu, 2} = \{k \in N : \mu(\Delta^m x_{k'(r)} - L, \frac{t}{2}) > 1 - \lambda\}$$

Let $K_{\mu} = K_{\mu, 1} \cap K_{\mu, 2}$. Then $\delta_{\theta}(K_{\mu}) = 1$ and for $k \in K_{\mu}$,

$\mu(\Delta^m x_k - L, t) \geq \mu(\Delta^m x_k - \Delta^m x_{k'(r)}, \frac{t}{2}) * \mu(\Delta^m x_{k'(r)} - L, \frac{t}{2}) > 1 - \varepsilon$
 Similarly if we define
 $K_{\nu,1} = \{k \in N : \nu(\Delta^m x_k - \Delta^m x_{k'(r)}, \frac{t}{2}) < \lambda\}$
 and $K_{\nu,2} = \{k \in N : \nu(\Delta^m x_{k'(r)} - L, \frac{t}{2}) < \lambda\}$
 Then $\delta_\theta(K_\nu) = \delta_\theta(K_{\nu,1} \cap K_{\nu,2}) = 1$ and for $k \in K_\nu$, $\nu(\Delta^m x_k - L, t) < \varepsilon$.
 Therefore
 $\delta_\theta(\{k \in N : \mu(\Delta^m x_k - L, t) > 1 - \varepsilon\}) = 1$
 $\delta_\theta(\{k \in N : \nu(\Delta^m x_k - L, t) < \varepsilon\}) = 1$.
 Hence $x = \{x_k\}$ is Δ^m - statistically convergent. □

Corollary 5.3. *Any lacunary Δ^m - statistically convergent sequence has a Δ^m -convergent subsequence.*

6. Conclusion

In this paper, we have introduced the notion of lacunary Δ^m -statistically convergent and lacunary strongly Δ^m -convergent and strongly Δ^m -Cesàro summable sequences in IFNLS and proved several useful results for these notions. Also we have introduced and studied the concept of lacunary Δ^m -statistically Cauchy sequences in IFNLS. As every crisp norm can induce an intuitionistic fuzzy norm, the results obtained here are more general than the corresponding results for normed spaces.

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