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Hermite-Hadamard Type Inequalities for Generalized Convex Functions on Fractal Sets Style*

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ABSTRACT: We aim to establish certain generalized Hermite-Hadamard's inequalities for generalized convex functions via local fractional integral. As special cases of some of the results presented here, certain interesting inequalities involving generalized arithmetic and logarithmic means are obtained.

Key Words: Fractal sets, Generalized convex functions, Generalized Hermite–Hadamard inequality, Generalized Hölder inequality

Contents

1. Introduction and Preliminaries

In order to describe the definition of the local fractional derivative and local fractional integral, recently, one has introduced to define the following sets (see, *e.g.*, [15,19]; see also [2]): For $0 < \alpha \leq 1$,

(i) the α-type set of integers $\mathbb{Z}^\alpha$ is defined by

$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha \,:\, m \in \mathbb{N}\};$$

(ii) the α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by

$$\mathbb{Q}^\alpha := \{q^\alpha \,:\, q \in \mathbb{Q}\} = \left\{\left(\frac{m}{n}\right)^\alpha \,:\, m \in \mathbb{Z},\, n \in \mathbb{N}\right\};$$

(iii) the α-type set of irrational numbers $\mathbb{J}^\alpha$ is defined by

$$\mathbb{J}^\alpha := \{r^\alpha \,:\, r \in \mathbb{J}\} = \left\{r^\alpha \neq \left(\frac{m}{n}\right)^\alpha \,:\, m \in \mathbb{Z},\, n \in \mathbb{N}\right\};$$

(iv) the α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by $\mathbb{R}^\alpha := \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha$.

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Here and in the following, let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of real and positive real numbers, rational numbers, integers and positive integers, respectively, and

$$\mathbb{J} := \mathbb{R} \setminus \mathbb{Q} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

and, whenever the α-type set $\mathbb{R}^\alpha$ of real line numbers is involved, the α is assumed to be tacitly $0 < \alpha \leq 1$.

One has also defined two binary operations the addition $+$ and the multiplication $\cdot$ (which is conventionally omitted) on the α-type set $\mathbb{R}^\alpha$ of real line numbers as follows (see, *e.g.*, [15,19]; see also [2]): For a^α, $b^\alpha \in \mathbb{R}^\alpha$,

$$a^\alpha + b^\alpha := (a+b)^\alpha \quad \text{and} \quad a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha. \tag{1.1}$$

Then one finds that

- $(\mathbb{R}^\alpha, +)$ is a commutative group: For a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$,

  (A$_1$) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$;

  (A$_2$) $a^\alpha + b^\alpha = b^\alpha + a^\alpha$;

  (A$_3$) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$;

  (A$_4$) 0^α is the identity for $(\mathbb{R}^\alpha, +)$: For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$;

  (A$_5$) For each $a^\alpha \in \mathbb{R}^\alpha$, $(-a)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, +)$: $a^\alpha + (-a)^\alpha = (a + (-a))^\alpha = 0^\alpha$;

- $(\mathbb{R}^\alpha \setminus \{0^\alpha\}, \cdot)$ is a commutative group: For a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$,

  (M$_1$) $a^\alpha b^\alpha \in \mathbb{R}^\alpha$;

  (M$_2$) $a^\alpha\, b^\alpha = b^\alpha\, a^\alpha$;

  (M$_3$) $a^\alpha\,(b^\alpha c^\alpha) = (a^\alpha\, b^\alpha)\, c^\alpha$;

  (M$_4$) 1^α is the identity for $(\mathbb{R}^\alpha, \cdot)$: For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$;

  (M$_5$) For each $a^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, $(1/a)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, \cdot)$: $a^\alpha (1/a)^\alpha = (a(1/a))^\alpha = 1^\alpha$;

- Distributive law holds: $a^\alpha\,(b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.

Furthermore we observe some additional properties for $(\mathbb{R}^\alpha, +, \cdot)$ which are stated in the following proposition (see [2]).

Proposition 1.1. *Each of the following statements holds true:*

(i) *Like the usual real number system* $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}^\alpha, +, \cdot)$ *is a field;*

(ii) *The additive identity* 0^α *and the multiplicative identity* 1^α *are unique, respectively;*

(iii) *The additive inverse element and the multiplicative inverse element are unique, respectively;*

(iv) *For each* $a^\alpha \in \mathbb{R}^\alpha$*, its inverse element* $(-a)^\alpha$ *may be written as* $-a^\alpha$*; for each* $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$*, its inverse element* $(1/b)^\alpha$ *may be written as* $1^\alpha/b^\alpha$ *but not as* $1/b^\alpha$*;*

(v) *If the order* $<$ *is defined on* $(\mathbb{R}^\alpha, +, \cdot)$ *as follows:* $a^\alpha < b^\alpha$ *in* $\mathbb{R}^\alpha$ *if and only if* $a < b$ *in* $\mathbb{R}$*, then* $(\mathbb{R}^\alpha, +, \cdot, <)$ *is an ordered field like* $(\mathbb{R}, +, \cdot, <)$*.*

In order to introduce the local fractional calculus on $\mathbb{R}^\alpha$, we begin with the concept of the local fractional continuity as in Definition 1.1.

Definition 1.1. *A non-differentiable function* $f : \mathbb{R} \to \mathbb{R}^\alpha$*,* $x \mapsto f(x)$*, is called to be local fractional continuous at* x_0 *if for any* $\varepsilon \in \mathbb{R}^+$*, there exists* $\delta \in \mathbb{R}^+$ *such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_\alpha(a, b)$.

Among several attempts to have defined local fractional derivative and local fractional integral (see [14, Section 2.1]), we choose to recall the following definitions of local fractional calculus (see, *e.g.*, [3,14,15]):

Definition 1.2. *The local fractional derivative of* $f(x)$ *of order* α *at* $x = x_0$ *is defined by*

$$f^{(\alpha)}(x_0) = {}_{x_0}D_x^\alpha f(x) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha\left(f(x) - f(x_0)\right)}{(x - x_0)^\alpha},$$

where $\Delta^\alpha\left(f(x) - f(x_0)\right) \cong \Gamma(\alpha + 1)\left(f(x) - f(x_0)\right)$ *and* Γ *is the familiar Gamma function (see, e.g.,* [13, Section 1.1]*).*

Let $f^{(\alpha)}(x) = D_x^\alpha f(x)$*. If there exists* $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \ldots D_x^\alpha}^{k+1\text{ times}} f(x)$ *for any* $x \in I \subseteq \mathbb{R}$*, then we denote* $f \in D_{(k+1)\alpha}(I)$ $(k \in \mathbb{N}_0)$*.*

Definition 1.3. *Let* $f \in C_\alpha[a, b]$*. Also let* $P = \{t_0, \ldots, t_N\}$ $(N \in \mathbb{N})$ *be a partition of the interval* $[a, b]$ *which satisfies* $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$*. Further, for this partition* P*, let* $\Delta t := \max_{0\le j\le N-1} \Delta t_j$ *where* $\Delta t_j := t_{j+1} - t_j$ $\{j = 0, \ldots, N-1\}$*. Then the local fractional integral of* f *on the interval* $[a, b]$ *of order* α *(denoted by* ${}_aI_b^{(\alpha)} f$*) is defined by*

$$ {}_aI_b^{(\alpha)} f(t) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha := \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \tag{1.2}$$

provided the limit exists (in fact, this limit exists if $f \in C_\alpha\,[a,b]$).

Here, it follows that ${}_aI_b^{(\alpha)} f = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f = -{}_bI_a^{(\alpha)} f$ if $a < b$.

If ${}_aI_x^{(\alpha)} g$ exists for any $x \in [a, b]$ and a function $g : [a, b] \to \mathbb{R}^\alpha$, then we denote $g \in I_x^{(\alpha)}\,[a, b]$.

We give some of the features related to the local fractional calculus that will be required for our main results (see [15]).

Lemma 1.2. *The following identities hold true:*

(a) (*1α-local fractional derivative of $x^{k\alpha}$*)

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\,\alpha)} x^{(k-1)\alpha}.$$

(b) (*Local fractional integration is anti-differentiation*)
Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha\,[a, b]$. Then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(c) (*Local fractional integration by parts*)
Suppose that $f(x), g(x) \in D_\alpha\,[a, b]$ and $f^{(\alpha)}(x),\ g^{(\alpha)}(x) \in C_\alpha\,[a, b]$. Then we have

$${}_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b -{}_a I_b^\alpha f^{(\alpha)}(x) g(x).$$

(d) (*Local fractional definite integrals of $x^{k\alpha}$*)

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\,\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right) \quad (k \in \mathbb{R}).$$

For further details on local fractional calculus, one may refer to [14]- [18].

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}^\alpha$ is said to be convex on I if

$$f\,(tx + (1-t)\,y) \leq tf\,(x) + (1-t)\,f\,(y) \tag{1.3}$$

holds for every $x,\ y \in I$ and $t \in [0, 1]$.

If a function $f : I \subset \mathbb{R} \to \mathbb{R}$ (I an interval) is convex on I, then, for $a,\, b \in I$ with $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f\,(x)\,dx \leq \frac{f\,(a) + f\,(b)}{2}, \tag{1.4}$$

which is known as the Hermite-Hadamard inequality.

Mo *et al.* [8] introduced the following generalized convex function.

Definition 1.4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ *(I an interval) be a function. If, for any* $x_1, x_2 \in I$ *and* $\lambda \in [0,1]$, *the following inequality*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda^{\alpha} f(x_1) + (1-\lambda)^{\alpha} f(x_2)$$

holds, then f *is called a generalized convex function on* I.

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$ $(p > 1)$;

(2) $g(x) = E_{\alpha}(x^{\alpha})$ $(x \in \mathbb{R})$, where $E_{\alpha}(x^{\alpha}) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function.

Recently the fractal theory has received a significant attention (see, *e.g.*, [1, 3,6,7,9,10,11,12]). Mo *et al.* [8] proved the following analogue of the Hermite-Hadamard inequality (1.4) for generalized convex functions: Let $f(x) \in I_x^{\alpha}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}. \tag{1.5}$$

Remark 1.3. The double inequality (1.5) is known in the literature as generalized Hermite-Hadamard integral inequality for generalized convex functions. Some of the classical inequalities for means can be derived from (1.5) with appropriate selections of the mapping f. Both inequalities in (1.4) and (1.5) hold in the reverse direction if f is concave and generalized concave, respectively. For some more results which generalize, improve and extend the inequalities (1.5), one may refer to the recent papers [1,6,7], [9]-[11] and references therein.

An analogue in the fractal set $\mathbb{R}^{\alpha}$ of the classical Hölder's inequality has been established by Yang [15], which is asserted by the following lemma.

Lemma 1.4. *Let* $f, g \in C_{\alpha}[a,b]$, *with* $\frac{1}{p} + \frac{1}{q} = 1$ $(p, q > 1)$. *Then we have*

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| \, (dx)^{\alpha} \\ &\leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p \, (dx)^{\alpha}\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q \, (dx)^{\alpha}\right)^{\frac{1}{q}}. \end{aligned} \tag{1.6}$$

Here, in this paper, we establish certain generalized Hermite-Hadamard's inequalities for generalized convex functions via local fractional integral. As special cases of some of the results presented here, certain interesting inequalities involving generalized arithmetic and logarithmic means are obtained.

2. Main Results

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \to \mathbb{R}^\alpha$ (I° is interior of I) such that $f \in D_\alpha(I^\circ)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^\circ$ with $a < b$. Then, for all $x \in [a,b]$, the following identity holds true:*

$$\begin{aligned} & f(x) + \frac{(b-x)^\alpha f(b) + (x-a)^\alpha f(a)}{(b-a)^\alpha} - \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^\alpha f(t) \\ = \; & \frac{(x-a)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)(dt)^\alpha \\ - \; & \frac{(x-a)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)(dt)^\alpha \\ - \; & \frac{(b-x)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)(dt)^\alpha \\ + \; & \frac{(b-x)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)(dt)^\alpha. \end{aligned} \tag{2.1}$$

Proof: Using the local fractional integration by parts, we have

$$\begin{aligned} & \frac{(x-a)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)(dt)^\alpha \\ = \; & \frac{(x-a)^{2\alpha}}{(b-a)^\alpha} \left[\frac{t^\alpha}{(x-a)^\alpha} f\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\Bigg|_0^1 \right. \\ & \left. - \frac{\Gamma(1+\alpha)}{(x-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 f\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)(dt)^\alpha \right] \\ = \; & \frac{(x-a)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(1+\alpha)}{(x-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 f\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)(dt)^\alpha \\ = \; & \left(\frac{x-a}{b-a}\right)^\alpha f(x) - \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_{\frac{a+x}{2}}I_x^\alpha f(t). \end{aligned} \tag{2.2}$$

Similarly, we also get the following identities:

$$\begin{aligned} - \; & \frac{(x-a)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)(dt)^\alpha \\ = \; & \left(\frac{x-a}{b-a}\right)^\alpha f(a) - \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_{\frac{a+x}{2}}^\alpha f(t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} - \; & \frac{(b-x)^{2\alpha}}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)(dt)^\alpha \\ = \; & \left(\frac{b-x}{b-a}\right)^\alpha f(x) - \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_xI_{\frac{x+b}{2}}^\alpha f(t), \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
& - \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^{\alpha} f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)(dt)^{\alpha} \\
& = \left(\frac{b-x}{b-a}\right)^{\alpha} f(b) - \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}} \; {}_{\frac{x+b}{2}}I_b^{\alpha} f(t). \qquad (2.5)
\end{aligned}
$$

By adding identities (2.2)–(2.5), we get the desired identity (2.1). □

Theorem 2.2. *Suppose that assumptions of* Lemma 2.1 *are satisfied. If* $\left|f^{(\alpha)}\right|$ *is generalized convex on* $[a,b]$*, then the following inequality holds:*

$$
\begin{aligned}
& \left| f(x) + \frac{(b-x)^{\alpha} f(b) + (x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}} \; {}_aI_b^{\alpha} f(t) \right| \\
\leq & \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \left(\frac{\left|f^{(\alpha)}(x)\right| + \left|f^{(\alpha)}(a)\right|}{2^{\alpha}} \right) \right. \\
& \left. + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \left(\frac{\left|f^{(\alpha)}(x)\right| + \left|f^{(\alpha)}(b)\right|}{2^{\alpha}} \right) \right] \qquad (2.6)
\end{aligned}
$$

for $x \in [a,b]$.

Proof: Using Lemma 2.1 and taking the modulus, we have

$$
\begin{aligned}
& \left| f(x) + \frac{(b-x)^{\alpha} f(b) + (x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}} \; {}_aI_b^{\alpha} f(t) \right| \\
& \leq \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^{\alpha} \left| f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right) \right| (dt)^{\alpha} \\
& + \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^{\alpha} \left| f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right) \right| (dt)^{\alpha} \qquad (2.7) \\
& + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^{\alpha} \left| f^{(\alpha)}\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right) \right| (dt)^{\alpha} \\
& + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{t}{2}\right)^{\alpha} \left| f^{(\alpha)}\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right) \right| (dt)^{\alpha}.
\end{aligned}
$$

If we use generalized convexity of $\left|f^{(\alpha)}\right|$ on $[a,b]$, we get from the inequality

(2.7) that

$$\begin{aligned}
&\left|f(x)+\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right| \\
\leq\ &\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left[\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|+\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha} \\
&+\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left[\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|+\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha} \\
&+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left[\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|+\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(b)\right|\right](dt)^{\alpha} \\
&+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left[\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|+\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(b)\right|\right](dt)^{\alpha}
\end{aligned} \tag{2.8}$$

holds for all $x \in [a, b]$.

From Lemma 1.2, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left(\frac{1+t}{2}\right)^{\alpha}(dt)^{\alpha}=\frac{1}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \tag{2.9}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left(\frac{1-t}{2}\right)^{\alpha}(dt)^{\alpha}=\frac{1}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]. \tag{2.10}$$

If we substitute equalities (2.9) and (2.10) in the right-side of the inequality (2.8), we get the desired inequality (2.6). This completes the proof.

□

In (2.6), if we set $x = \frac{a+b}{2}$ and use the convexity of $\left|f^{(\alpha)}\right|$, we obtain the following inequality in (2.11).

Corollary 2.3. *The following inequality holds true:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right| \\
&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{b-a}{4}\right)^{\alpha}\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right).
\end{aligned} \tag{2.11}$$

Theorem 2.4. *Suppose that assumptions of* Lemma 2.1 *are satisfied. If* $\left|f^{(\alpha)}\right|^q$ *is generalized convex on* $[a, b]$ *for some fixed* $q > 1$*, then the following inequality holds* *true:*

$$\left|f(x)+\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|$$
$$\le\left(\frac{1}{2}\right)^{(1+\frac{1}{q})\alpha}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\right.$$
$$\times\left[\left(A\left|f^{(\alpha)}(x)\right|^{q}+B\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}+\left(B\left|f^{(\alpha)}(x)\right|^{q}+A\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$\left.+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left[\left(A\left|f^{(\alpha)}(x)\right|^{q}+B\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}+\left(B\left|f^{(\alpha)}(x)\right|^{q}+A\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}, \tag{2.12}$$

where

$$A:=\frac{1}{\Gamma(1+\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\quad\textit{and}\quad B:=\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{2.13}$$

for all $x\in[a,b]$.

Proof: From Lemma 2.1 and using generalized Hölder integral inequality, we have

$$\begin{aligned}&\left|f(x)+\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|\\&\quad\le A(\alpha;a,b)+B(\alpha;a,b)+C(\alpha;a,b)+D(\alpha;a,b),\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}A(\alpha;a,b):=&\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}B(\alpha;a,b):=&\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}C(\alpha;a,b):=&\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}},\end{aligned}$$

and

$$D(\alpha;a,b):=\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}$$

for all $x\in[a,b]$.

Since $\left|f^{(\alpha)}\right|^q$ is generalized convex on $[a,b]$, in view of Lemma 1.2, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q(dt)^{\alpha}\\ &\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|^q+\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha}\\ &\quad=\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(x)\right|^q\\ &\qquad+\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(a)\right|^q.\end{aligned}\tag{2.15}$$

Similarly,

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q(dt)^{\alpha}\\ &\quad=\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(x)\right|^q\\ &\qquad+\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(a)\right|^q,\end{aligned}\tag{2.16}$$

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q(dt)^{\alpha}\\ &\quad=\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(x)\right|^q\\ &\qquad+\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(b)\right|^q,\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q(dt)^{\alpha}\\ =\;&\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(x)\right|^q\\ &+\frac{1}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|f^{(\alpha)}(b)\right|^q\end{aligned}\tag{2.18}$$

If we substitute the inequalities from (2.15) to (2.18) in the inequality (2.14), we obtain the desired inequality (2.12). Note that we have also used the following identity:

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(\frac{t}{2}\right)^{\alpha p}(dt)^\alpha = \frac{\Gamma(1+p\alpha)}{2^{\alpha p}\Gamma(1+(p+1)\alpha)}.$$

This completes the proof. □

Taking $x=\frac{a+b}{2}$ in the result in Theorem 2.4 and using the convexity of $\left|f^{(\alpha)}\right|^q$, we get an inequality asserted in Corollary 2.5.

Corollary 2.5. *The following inequality holds true:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^\alpha}-\frac{2^\alpha\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right| \qquad (2.19)\\
\leq\ & \left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{q}}\left(\frac{b-a}{4}\right)^\alpha\left(\frac{1}{2}\right)^{(1+\frac{1}{q})^\alpha}\\
&\times\left\{\left(A\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+B\left|f^{(\alpha)}(a)\right|^q\right)^{\frac{1}{q}}+\left(B\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+A\left|f^{(\alpha)}(a)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(A\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+B\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}+\left(B\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+A\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right\}\\
\leq\ & \left(\frac{1}{2}\right)^{\left(1+\frac{2}{q}\right)^\alpha}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{b-a}{4}\right)^\alpha\\
&\times\left[\left(A^{\frac{1}{q}}+B^{\frac{1}{q}}+(A+2B)^{\frac{1}{q}}+(2A+B)^{\frac{1}{q}}\right)\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right)\right],
\end{aligned}$$

where A and B are given as in (2.13).

Remark 2.6. For the last part of the inequality (2.19), we used the following known inequality (see, *e.g.* [5, p.54]):

$$\sum_{k=1}^n (u_k+v_k)^s \leq \sum_{k=1}^n (u_k)^s+\sum_{k=1}^n (v_k)^s \qquad (2.20)$$

$$(u_k, v_k \geq 0,\ \ 1\leq k\leq n;\ \ 0\leq s\leq 1).$$

If we take $\alpha=1$ in Corollary 2.5, the inequality (2.19) reduces to the inequality (2.12) in [4].

Theorem 2.7. *Suppose that assumptions of* Lemma 2.1 *are satisfied. If $\left|f^{(\alpha)}\right|^q$ is generalized convex on $[a,b]$ for some fixed $q\geq 1$, then the following inequality holds* *true:*

$$\left|f(x)+\frac{(b-a)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \tag{2.21}$$
$$\leq \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{(1+\frac{1}{q})\alpha}$$
$$\times\left\{\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left[\left(K\left|f^{(\alpha)}(x)\right|^{q}+L\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}+\left(L\left|f^{(\alpha)}(x)\right|^{q}+K\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left[\left(K\left|f^{(\alpha)}(x)\right|^{q}+L\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}+\left(L\left|f^{(\alpha)}(x)\right|^{q}+K\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}$$

where

$$K:=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\quad \textit{and}\quad L:=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}. \tag{2.22}$$

Proof: Suppose that $q \geq 1$. From Lemma 1.1 and using the well-known power-mean inequality, we have

$$\left|f(x)+\frac{(b-a)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \tag{2.23}$$
$$\leq \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1+t}{2}b\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$

for all $x \in [a,b]$. Since $\left|f^{(\alpha)}\right|^{q}$ is convex on $[a,b]$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|$$
$$\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left[\left(\frac{1+t}{2}\right)^{\alpha}\left|f^{(\alpha)}(x)\right|+\left(\frac{1-t}{2}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha}$$
$$=\frac{1}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left|f^{(\alpha)}(x)\right|^{q}$$
$$+\frac{1}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left|f^{(\alpha)}(a)\right|^{q}$$
$$=\frac{K\left|f^{(\alpha)}(x)\right|^{q}+L\left|f^{(\alpha)}(a)\right|^{q}}{4^{\alpha}}. \tag{2.24}$$

Similarly,

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^{q}(dt)^{\alpha}$$
$$\leq\frac{L\left|f^{(\alpha)}(x)\right|^{q}+K\left|f^{(\alpha)}(a)\right|^{q}}{4^{\alpha}}, \tag{2.25}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^{q}(dt)^{\alpha}$$
$$\leq\frac{K\left|f^{(\alpha)}(x)\right|^{q}+L\left|f^{(\alpha)}(b)\right|^{q}}{4^{\alpha}} \tag{2.26}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}\right)^{\alpha}\left|f^{(\alpha)}\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^{q}(dt)^{\alpha}$$
$$\leq\frac{L\left|f^{(\alpha)}(x)\right|^{q}+K\left|f^{(\alpha)}(a)\right|^{q}}{4^{\alpha}}. \tag{2.27}$$

Substituting the inequalities (2.24)-(2.27) in (2.23), we get the inequality (2.23). Hence the proof is complete.

□

In Theorem 2.7, if we take $x=\frac{a+b}{2}$ and use a similar argument as in Corollary 2.5, we get the following inequality in Corollary 2.8.

Corollary 2.8. *The following inequality holds true:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^{\alpha}} - \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t) \right| \\
&\leq \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\left(1-\frac{1}{q}\right)} \left(\frac{1}{2}\right)^{\left(1+\frac{2}{q}\right)^{\alpha}} \left(\frac{b-a}{4}\right)^{\alpha} \\
&\quad \times \left(K^{\frac{1}{q}} + L^{\frac{1}{q}} + (K+2L)^{\frac{1}{q}} + (L+2K)^{\frac{1}{q}}\right) \left(\left|f^{(\alpha)}(a)\right| + \left|f^{(\alpha)}(b)\right|\right),
\end{aligned} \tag{2.28}$$

where k and L are given as in (2.22).

Remark 2.9. The special case of (2.28) when $\alpha = 1$ reduces to a known inequality (2.15) in [4].

3. Applications

Here we apply some of the results in the previous section to the following generalized means (see [3,12]):

- (The generalized arithmetic mean)

$$A(a,b) := \frac{a^{\alpha} + b^{\alpha}}{2^{\alpha}}.$$

- (The generalized logarithmic mean)

$$L_n(a,b) := \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left(\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right)\right]^{\frac{1}{n}}$$

$$(n \in \mathbb{Z} \setminus \{-1, 0\};\ a, b \in \mathbb{R} \text{ with } a \neq b).$$

Choosing the function

$$f(x) = x^{n\alpha}\ (x \in \mathbb{R};\ n \in \mathbb{Z},\ |n| \geq 2) \tag{3.1}$$

in Corollaries 2.3, 2.5 and 2.8, we obtain the following inequalities asserted by Propositions 3.1, 3.2 and 3.3, respectively. Here the constants A, B, K, and L are given as in (2.13) and (2.22).

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a,b]$, and $n \in \mathbb{N} \setminus \{1\}$. Then the following inequality holds true:*

$$\begin{aligned}
&|A^n(a,b) + A(a^n, b^n) - 2^{\alpha}\Gamma(1+\alpha) L_n^n(a,b)| \\
&\leq \frac{2^{\alpha}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(\frac{b-a}{4}\right)^{\alpha} \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} A\left(|a|^{n-1}, |b|^{n-1}\right).
\end{aligned} \tag{3.2}$$

Proposition 3.2. *Let a, $b \in \mathbb{R}$ with $a < b$, $0 \notin [a,b]$, and $n \in \mathbb{N} \setminus \{1\}$. Then, for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds true:*

$$\begin{aligned}
&|A^n(a,b) + A(a^n, b^n) - 2^{\alpha}\Gamma(1+\alpha)L_n^n(a,b)| \\
&\quad \le \left(\frac{1}{2}\right)^{\frac{2\alpha}{q}} \left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}} \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \\
&\times \left(\frac{b-a}{4}\right)^{\alpha} \left(A^{\frac{1}{q}} + B^{\frac{1}{q}} + (A+2B)^{\frac{1}{q}} + (2A+B)^{\frac{1}{q}}\right) A\left(|a|^{n-1}, |b|^{n-1}\right).
\end{aligned}$$

Proposition 3.3. *Let a, $b \in \mathbb{R}$ with $a < b$, $0 \notin [a,b]$, and $n \in \mathbb{N} \setminus \{1\}$. Then, for $q \ge 1$, the following inequality holds true:*

$$\begin{aligned}
&|A^n(a,b) + A(a^n, b^n) - 2^{\alpha}\Gamma(1+\alpha)L_n^n(a,b)| \\
\le\ & \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\left(1-\frac{1}{q}\right)} \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \left(\frac{1}{2}\right)^{\frac{2\alpha}{q}} \left(\frac{b-a}{4}\right)^{\alpha} \\
&\times \left(K^{\frac{1}{q}} + L^{\frac{1}{q}} + (K+2L)^{\frac{1}{q}} + (L+2K)^{\frac{1}{q}}\right) A\left(|a|^{n-1}, |b|^{n-1}\right).
\end{aligned}$$

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