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Existence of Solution for Nonlinear Fourth-order Three-point Boundary Value Problem

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ABSTRACT: In this paper, we study the existence of solution for the fourth-order three-point boundary value problem having the following form

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = \alpha u(\eta), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = \beta u(\eta),$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, and $f(t,0) \neq 0$. We give sufficient conditions that allow us to obtain the existence of solution. And by using the Leray-Schauder nonlinear alternative we prove the existence of at least one solution of the posed problem. As an application, we also given some examples to illustrate the results obtained.

Key Words: Green's function, Existence of solution, Leary-Schauder nonlinear alternative, Fixed point theorem, Boundary value problem.

Contents

1. Introduction

The study of fourth-order three-point boundary value problems (BVP) for ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

Many authors studied the existence of positive solutions for nth-order m-point boundary value problems using different methods such that fixed point theorems in cones, nonlinear alternative of Leray-Schauder, and Krasnoselskii's fixed point theorem, see ([2,3,4,5]) and the references therein.

In 2003, by using the Leray-Schauder degree theory, Yuji Liu and Weigao Ge ([6]) proved the existence of positive solutions for $(n-1,1)$ three-point boundary

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value problems with coefficient that changes sign given as follows

$$u^{(n)}(t) + \lambda a(t) f(u(t)) = 0, \quad \text{t} \in (0,1),$$
$$u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta), \quad u^{(i)}(0) = 0 \ for \ i = 1, 2, ..., n-2, \ and$$
$$u^{(n-2)}(0) = \alpha u^{(n-2)}(\eta), \ u^{(n-2)}(1) = \beta u^{(n-2)}(\eta), \ u^{(i)}(0) = 0 \ for \ i = 1, 2, .., n-3,$$

where $\eta \in (0,1)$, $\alpha \geq 0$, $\beta \geq 0$, and $a : (0,1) \to \mathbb{R}$ may change sign and $\mathbb{R} = (-\infty, \infty)$, $f(0) > 0$, $\lambda > 0$ is a parameter.

In 2005, Paul W. Eloea and Bashir Ahmad ([7]) studied the existence of positive solutions of the following nonlinear nth-order boundary value problem with nonlocal conditions

$$u^{(n)}(t) + a(t) f(u(t)) = 0, \quad \text{t} \in (0,1),$$
$$u(0) = 0, \quad u^{'}(0) = 0, ..., u^{(n-2)}(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$, $0 < \alpha \eta^{n-1} < 1$, $a : [0,1] \to [0, \infty)$ is continuous, and f is either superlinear or sublinear. The methods used is based on the fixed point theorem in cones due to Krasnoselkiî and Guo.

Then in the year 2009, Xie, Liu and Bai ([8]) used fixed-point theory to study the existence of positive solutions for a singular nth-order three-point boundary value problem on time scales represented in the following figure

$$u^{(n)}(t) + h(t) f(u(t)) = 0, \quad \text{t} \in (0,1),$$
$$u(a) = \alpha u(\eta), \quad u^{'}(a) = 0, ..., u^{(n-2)}(a) = 0, \quad u(b) = \beta u(\eta),$$

where $a < \eta < b$, $0 \leq \alpha < 1$, $0 < \beta(\eta - a)^{n-1} < (1-\alpha)(b-a)^{n-1} + \alpha(\eta - a)^{n-1}$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and $h \in C([a,b], [0,\infty))$ may be singular at $t = a$ and $t = b$.

In 2013, Yan Sun and Cun Zhu ([9]), considered the singular fourth-order three point boundary value problem

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 \leq t \leq 1,$$
$$u(0) = u^{'}(0) = u^{''}(0) = 0, \quad u^{''}(1) - \alpha u^{''}(\eta) = \lambda,$$

where $\eta \in (0,1)$ and $\alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in [0, \infty)$ is a parameter, The authors presented the existence of positive solutions by using the Krasnosel'skii fixed point theorem.

For Some other results on fourth-order boundary value problem, we refer the reader to the papers ([10,11,12,13,14]).

Motivated by the above works, the aim of this paper is to establish some sufficient conditions for the existence of solution for the fourth-order three-point boundary value problem (BVP)

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1. \tag{1.1}$$
$$u(0) = \alpha u(\eta), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = \beta u(\eta), \tag{1.2}$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $f(t,0) \neq 0$, and $\mathbb{R} = (-\infty, +\infty)$.

This paper is organized as follows. In section 2, we present two lemmas that will be used to prove the results. Then, in section 3, we present and prove our main results which consists of existence theorems and corollary for nontrivial solution of the BVP (1.1) − (1.2), and we establish some existence criteria of at least one solution by using the Leray-Schauder nonlinear alternative. Finally, in section 4, as an application, we give some examples to illustrate the results we obtained.

2. Preliminaries

Let $E = C([0,1])$ with the norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. A solution $u(t)$ of the BVP (1.1) − (1.2) is called nontrivial solution if $u(t) \neq 0$. To get our results, we need to provide the following lemma.

Lemma 2.1. *Let $y \in C([0,1])$, $\alpha \neq 1$, $\beta\eta^3 \neq 1$, and $\zeta = (1-\alpha) + \eta^3(\alpha - \beta) \neq 0$, then three-point BVP*

$$u^{(4)}(t) + y(t) = 0, \quad 0 < t < 1,$$
$$u(0) = \alpha u(\eta), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = \beta u(\eta),$$

has a unique solution

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)ds + \frac{t^3(1-\alpha) + \alpha\eta^3}{6\zeta}\int_0^1 (1-s)^3 y(s)ds +$$
$$\frac{t^3(\alpha-\beta) - \alpha}{6\zeta}\int_0^\eta (\eta - s)^3 y(s)ds.$$

Proof. Rewriting the differential equation as $u^{(4)}(t) = -y(t)$, and integrating four times from 0 to t, we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)ds + \frac{t^3}{6}c_0 + \frac{t^2}{2}c_1 + tc_2 + c_3. \tag{2.1}$$

By the boundary conditions (1.2), we have $u'(0) = u''(0) = 0$, i.e. $c_1 = c_2 = 0$,
and $u(0) = \alpha u(\eta)$, implies

$$c_3 = -\frac{\alpha}{6(1-\alpha)}\int_0^\eta (\eta - s)^3 y(s)ds + \frac{\alpha\eta^3}{6(1-\alpha)}c_0, \tag{2.2}$$

also $u(1) = \beta u(\eta)$, we find

$$c_0 = \frac{1}{(1-\beta\eta^3)}\int_0^1 (1-s)^3 y(s)ds - \frac{\beta}{(1-\beta\eta^3)}\int_0^\eta (\eta-s)^3 y(s)ds + \frac{6(\beta-1)}{(1-\beta\eta^3)}c_3. \tag{2.3}$$

Injecting equation (2.2) in (2.3), we obtain

$$c_0 = \frac{(1-\alpha)}{(1-\alpha)+\eta^3(\alpha-\beta)}\int_0^1 (1-s)^3 y(s)ds + \frac{(\alpha-\beta)}{(1-\alpha)+\eta^3(\alpha-\beta)}\int_0^\eta (\eta-s)^3 y(s)ds,$$

and $c_3 =$

$$\frac{\alpha\eta^3}{6((1-\alpha)+\eta^3(\alpha-\beta))}\int_0^1(1-s)^3y(s)ds-\frac{\alpha}{6((1-\alpha)+\eta^3(\alpha-\beta))}\int_0^\eta(\eta-s)^3y(s)ds.$$

Substituting c_0 and c_3 by their values in (2.1), we obtain the solution in the statement of the lemma. this completes the proof. □

Define the integral operator $T : E \longrightarrow E$, by

$$Tu(t) = -\frac{1}{6}\int_0^t(t-s)^3f(s,u(s))ds + \frac{t^3(1-\alpha)+\alpha\eta^3}{6\zeta}\int_0^1(1-s)^3f(s,u(s))ds+$$
$$\frac{t^3(\alpha-\beta)-\alpha}{6\zeta}\int_0^\eta(\eta-s)^3f(s,u(s))ds. \tag{2.4}$$

By Lemma 2.1, the BVP $(1.1)-(1.2)$ has a solution if and only if the operator T has a fixed point in E. So we only need to seek a fixed point of T in E. By Ascoli-Arzela theorem, we can prove that T is a completely continuous operator. Now we cite the Leray-Schauder nonlinear alternative.

Lemma 2.2. ([1, 15]). *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*
(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or
(ii) there exists a fixed point $u^* \in \overline{\Omega}$ of T.

3. Existence of solution

In this section, we prove the existence of a nontrivial solution for the BVP $(1.1)-(1.2)$. Suppose that $f \in C([0,1]\times\mathbb{R},\mathbb{R})$.

Theorem 3.1. *Suppose that $f(t,0) \neq 0$, $\zeta \neq 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1]\times\mathbb{R},$$

$$(\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1(1-s)^3k(s)ds + \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta(\eta-s)^3k(s)ds < 1.$$

Then the BVP $(1.1)-(1.2)$ has at least one nontrivial solution $u^* \in C([0,1])$.

Proof. Let

$$M = (\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1(1-s)^3k(s)ds + \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta(\eta-s)^3k(s)ds,$$

$$N = (\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1(1-s)^3h(s)ds + \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta(\eta-s)^3h(s)ds.$$

Then $M < 1$. Since $f(t,0) \neq 0$, there exists an interval $[a,b] \subset [0,1]$ such that $\min_{a \le t \le b} |f(t,0)| > 0$. And as $h(t) \ge |f(t,0)|$, a.e. $t \in [0,1]$, we know that $N > 0$.

Let $A = N(1-M)^{-1}$ and $\Omega = \{u \in E : \|u\| < A\}$. Assume that $u \in \partial\Omega$ and $\lambda > 1$ such that $Tu = \lambda u$, then

$$\begin{aligned}
\lambda A = \lambda\|u\| =& \|Tu\| = \max_{0\le t\le 1} |(Tu)(t)| \\
\le& \frac{1}{6}\max_{0\le t\le 1}\int_0^t (t-s)^3|f(s,u(s))|ds \\
&+ \max_{0\le t\le 1}|\frac{t^3(1-\alpha)+\alpha\eta^3}{6\zeta}|\int_0^1 (1-s)^3|f(s,u(s))|ds \\
&+ \max_{0\le t\le 1}|\frac{t^3(\alpha-\beta)-\alpha}{6\zeta}|\int_0^\eta (\eta-s)^3|f(s,u(s))|ds \\
\le& (\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1 (1-s)^3|f(s,u(s))|ds \\
&+ \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta (\eta-s)^3|f(s,u(s))|ds \\
\le& [(\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1 (1-s)^3 k(s)|u(s)|ds \\
&+ \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta (\eta-s)^3 k(s)|u(s)|ds] \\
&+ \left[(\frac{1}{6}+\frac{1+|\alpha|(1+\eta^3)}{6|\zeta|})\int_0^1 (1-s)^3 h(s)ds\right. \\
&\left.+ \frac{|\beta|+2|\alpha|}{6|\zeta|}\int_0^\eta (\eta-s)^3 h(s)ds\right] \\
=& M\|u\| + N.
\end{aligned}$$

Therefore,

$$\lambda \le M + \frac{N}{A} = M + \frac{N}{N(1-M)^{-1}} = M + (1-M) = 1.$$

This contradicts $\lambda > 1$. By Lemma 2.3, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0) \neq 0$, the BVP $(1.1)-(1.2)$ has a nontrivial solution $u^* \in E$.

This completes the proof. □

Theorem 3.2. *Suppose that $f(t,0) \neq 0$, $\zeta > 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is fulfilled

(1) *There exists a constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < [\frac{6\zeta(1+3q)^{1/q}}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{(1+3q)/q}}]^p, \quad (\frac{1}{p}+\frac{1}{q}=1).$$

(2) *There exists a constant* $\mu > -1$ *such that*

$$k(s) \leq \frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}} s^{\mu}, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}} s^{\mu}\} > 0.$$

(3) *There exists a constant* $\mu > -4$ *such that*

$$k(s) \leq \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}(1-s)^{\mu}, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}(1-s)^{\mu}\} > 0.$$

(4) $k(s)$ *satisfies*

$$k(s) \leq \frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}\} > 0.$$

(5) $f(t,x)$ *satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}.$$

Then the BVP $(1.1)-(1.2)$ *has at least one nontrivial solution* $u^* \in E$.

Proof. Let M be defined as in the proof of Theorem 3.1. To prove Theorem 3.2, we only need to prove that $M < 1$. Since $\zeta > 0$, we have

$$M = \frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}\int_0^1 (1-s)^3 k(s)ds + \frac{|\beta|+2|\alpha|}{6\zeta}\int_0^{\eta}(\eta-s)^3 k(s)ds.$$

(1) Using the Hölder inequality, we have

$$M \leq [\int_0^1 k^p(s)ds]^{1/p}\{\frac{\zeta+1+|\alpha|((1+\eta^3)}{6\zeta}[\int_0^1 (1-s)^{3q}ds]^{1/q} + \frac{|\beta|+2|\alpha|}{6\zeta} \times$$

$$[\int_0^{\eta}(\eta-s)^{3q}ds]^{1/q}\}$$

$$M \leq [\int_0^1 k^p(s)ds]^{1/p}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}(\frac{1}{1+3q})^{1/q}+\frac{|\beta|+2|\alpha|}{6\zeta}(\frac{\eta^{1+3q}}{1+3q})^{1/q}]$$
$$< \frac{6\zeta(1+3q)^{1/q}}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{(1+3q)/q}}\times$$
$$\frac{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{(1+3q)/q}}{6\zeta(1+3q)^{1/q}} = 1$$

(2) In this case, we have

$$M < \frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}\int_0^1(1-s)^3 s^\mu ds+$$
$$\frac{|\beta|+2|\alpha|}{6\zeta}\int_0^\eta(\eta-s)^3 s^\mu ds]$$
$$\leq \frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{\zeta}\times$$
$$\frac{1}{(1+\mu)(2+\mu)(3+\mu)(4+\mu)}+\frac{|\beta|+2|\alpha|}{\zeta}\frac{\eta^{4+\mu}}{(1+\mu)(2+\mu)(3+\mu)(4+\mu)}] =$$
$$\frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}}.\frac{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}}{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)} = 1.$$

(3) In this case, we have

$$M < \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}\int_0^1(1-s)^{3+\mu}ds+\frac{|\beta|+2|\alpha|}{6\zeta}\times$$
$$\int_0^\eta(\eta-s)^3(1-s)^\mu ds]$$
$$\leq \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}\int_0^1(1-s)^{3+\mu}ds+\frac{|\beta|+2|\alpha|}{6\zeta}\times$$
$$\int_0^1(1-s)^{3+\mu}ds]$$
$$= \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}.\frac{1}{4+\mu}+\frac{|\beta|+2|\alpha|}{6\zeta}.\frac{1}{4+\mu}]$$
$$= \frac{6\zeta(4+\mu)}{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}.\frac{\zeta+1+|\alpha|(3+\eta^3)+|\beta|}{6\zeta(4+\mu)} = 1.$$

(4) In this case, we have

$$M < \frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{6\zeta}\int_0^1(1-s)^3 ds+$$

$$\frac{|\beta|+2|\alpha|}{6\zeta}\int_0^\eta(\eta-s)^3ds]$$

$$=\frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}[\frac{\zeta+1+|\alpha|(1+\eta^3)}{24\zeta}+\frac{(|\beta|+2|\alpha|)\eta^4}{24\zeta}]=1.$$

(5) Let $\epsilon=\frac{1}{2}[\frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}-Q]$, then there exists $c>0$ such that

$$|f(t,x)|\leq[\frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}-\epsilon]|x|,\ (t,x)\in[0,1]\times\mathbb{R}\setminus(-c,c).$$

Set $A=\max\{|f(t,x)|:(t,x)\in[0,1]\times[-c,c]\}$, then

$$|f(t,x)|\leq[\frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}-\epsilon]|x|+A,\ (t,x)\in[0,1]\times\mathbb{R}.$$

Set $k(s)=\frac{24\zeta}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^4}-\epsilon$, $h(s)=A$, then (4) holds.

This completes the proof. □

Corollary 3.3. *Suppose $f(t,0)\neq 0$, $\zeta>0$, and there exist nonnegative functions $k,h\in L^1[0,1]$ such that*

$$|f(t,x)|\leq k(t)|x|+h(t),\quad a.e.\ (t,x)\in[0,1]\times\mathbb{R}.$$

If one of following conditions is holds

(1) *There exists a constant $p>1$ such that*

$$\int_0^1 k^p(s)ds<[\frac{6\zeta(1+3q)^{1/q}}{\zeta+1+4|\alpha|+|\beta|}]^p,\quad(\frac{1}{p}+\frac{1}{q}=1).$$

(2) *There exists a constant $\mu>-1$ such that*

$$k(s)\leq\frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+4|\alpha|+|\beta|}s^\mu,\quad a.e.\ s\in[0,1],$$

$$meas\{s\in[0,1]:k(s)<\frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+4|\alpha|+|\beta|}s^\mu\}>0.$$

(3) *$k(s)$ satisfies*

$$k(s)\leq\frac{24\zeta}{\zeta+1+4|\alpha|+|\beta|},\quad a.e.\ s\in[0,1],$$

$$meas\{s\in[0,1]:k(s)<\frac{24\zeta}{\zeta+1+4|\alpha|+|\beta|}\}>0.$$

(4) *$f(t,x)$ satisfies*

$$Q=\limsup_{|x|\to\infty}\max_{t\in[0,1]}|\frac{f(t,x)}{x}|<\frac{24\zeta}{\zeta+1+4|\alpha|+|\beta|}.$$

Then the BVP (1.1) − (1.2) has at least one nontrivial solution $u^*\in E$.

Proof. In this case, we have

$$M = \frac{\zeta + 1 + |\alpha|(1+\eta^3)}{6\zeta}\int_0^1 (1-s)^3 k(s)ds + \frac{|\beta| + 2|\alpha|}{6\zeta}\int_0^\eta (\eta - s)^3 k(s)ds$$

$$\le \frac{\zeta + 1 + 2|\alpha|}{6\zeta}\int_0^1 (1-s)^3 k(s)ds + \frac{|\beta| + 2|\alpha|}{6\zeta}\int_0^1 (1-s)^3 k(s)ds$$

$$= \frac{\zeta + 1 + 4|\alpha| + |\beta|}{6\zeta}\int_0^1 (1-s)^3 k(s)ds.$$

Proof of this Corollary 3.3 is the same method in the proof Theorem 3.2. □

Theorem 3.4. *Suppose $f(t,0) \neq 0$, $\alpha > 0$, $\beta > 0$, $\zeta < 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is fulfilled

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < [\frac{-6\zeta(1+3q)^{1/q}}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^{(1+3q)/q}}]^p, \quad (\frac{1}{p} + \frac{1}{q} = 1).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \le \frac{-\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^{4+\mu}} s^\mu, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^{4+\mu}} s^\mu\} > 0.$$

(3) *There exists a constant $\mu > -4$ such that*

$$k(s) \le \frac{-6\zeta(4+\mu)}{4\alpha + \beta(1+\eta^3)}(1-s)^\mu, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-6\zeta(4+\mu)}{4\alpha + \beta(1+\eta^3)}(1-s)^\mu\} > 0.$$

(4) *$k(s)$ satisfies*

$$k(s) \le \frac{-24\zeta}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^4}, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-24\zeta}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^4}\} > 0.$$

(5) *$f(t,x)$ satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t \in [0,1]} |\frac{f(t,x)}{x}| < \frac{-24\zeta}{2\alpha + \beta\eta^3 + (\beta + 2\alpha)\eta^4}.$$

Then the BVP (1.1) – (1.2) has at least one nontrivial solution $u^* \in E$.

Proof. Let M be given as in the proof of Theorem 3.1. To prove Theorem 3.4, we only need to prove that $M < 1$. Since $\alpha > 0$, $\beta > 0$, and $\zeta < 0$, we have

$$M = \frac{2\alpha + \beta\eta^3}{-6\zeta} \int_0^1 (1-s)^3 k(s)ds + \frac{\beta + 2\alpha}{-6\zeta} \int_0^\eta (\eta - s)^3 k(s)ds.$$

Proof of this Theorem 3.4 is the same method in the proof Theorem 3.2. □

Corollary 3.5. *Suppose $f(t,0) \neq 0$, $\alpha > 0$, $\beta > 0$, $\zeta < 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is holds

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < [\frac{-3\zeta(1+3q)^{1/q}}{2\alpha + \beta}]^p, \quad (\frac{1}{p} + \frac{1}{q} = 1).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \leq \frac{-\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2(2\alpha+\beta)} s^\mu, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2(2\alpha+\beta)} s^\mu\} > 0.$$

(3) *$k(s)$ satisfies*

$$k(s) \leq \frac{-12\zeta}{2\alpha+\beta}, \quad a.e.\ s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-12\zeta}{2\alpha+\beta}\} > 0.$$

(4) *$f(t,x)$ satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{-12\zeta}{2\alpha+\beta}.$$

Then the BVP $(1.1)-(1.2)$ has at least one nontrivial solution $u^* \in E$.

Proof. In this case, we have

$$\begin{aligned} M &= \frac{2\alpha + \beta\eta^3}{-6\zeta} \int_0^1 (1-s)^3 k(s)ds + \frac{\beta + 2\alpha}{-6\zeta} \int_0^\eta (\eta - s)^3 k(s)ds \\ &\leq \frac{2\alpha+\beta}{-6\zeta} \int_0^1 (1-s)^3 k(s)ds + \frac{\beta+2\alpha}{-6\zeta} \int_0^1 (1-s)^3 k(s)ds \\ &= \frac{2\alpha+\beta}{-3\zeta} \int_0^1 (1-s)^3 k(s)ds. \end{aligned}$$

The rest procedure is the same as for Theorem 3.4. □

4. Examples

In order to illustrate the above results, we consider some examples.

Example 4.1. *Consider the following problem*

$$\begin{aligned} &u^{(4)} + \tfrac{t}{5}|u| \cos \sqrt[3]{u} + 2t + 1 = 0, \quad 0 < t < 1, \\ &u(0) = 2u(1/2), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = -14u(1/2). \end{aligned} \tag{4.1}$$

Set $\eta = \frac{1}{2}$, $\alpha = 2$, $\beta = -14$, *and*

$$f(t, x) = \frac{t}{5}|x| \cos \sqrt[3]{x} + 2t + 1,$$

$$k(t) = \frac{t}{2}, \quad h(t) = 2t + 1,$$

It is easy to prove that $k, h \in L^1[0, 1]$ *are nonnegative functions, and*

$$|f(t, x)| \le k(t)|x| + h(t), \quad a.e.\ (t, x) \in [0, 1] \times \mathbb{R},$$

and

$$\zeta = (1 - \alpha) + \eta^3(\alpha - \beta) = 1 \neq 0.$$

Moreover, we have

$$M = (\frac{1}{6} + \frac{1 + |\alpha|(1 + \eta^3)}{6|\zeta|}) \int_0^1 (1 - s)^3 k(s) ds + \frac{|\beta| + 2|\alpha|}{6|\zeta|} \int_0^{\eta} (\eta - s)^3 k(s) ds$$

$$M = \frac{17}{24} \int_0^1 (1 - s)^3 . \frac{s}{2} ds + 3 \int_0^{1/2} (\frac{1}{2} - s)^3 . \frac{s}{2} ds = \frac{17}{960} + \frac{3}{1280} = 0.019 < 1.$$

Hence, by Theorem 3.1, *the BVP* (4.1) *has at least one nontrivial solution* u^* *in* E.

Example 4.2. *Consider the following problem*

$$\begin{aligned} &u^{(4)} + \tfrac{2/3 \sqrt[3]{7 + tu}}{1 + u^5} \sin u^2 - e^t - 3 = 0, \quad 0 < t < 1, \\ &u(0) = 1/2u(1/2), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = 1/4u(1/2). \end{aligned} \tag{4.2}$$

Set $\eta = 1/2$, $\alpha = 1/2$, $\beta = 1/4$, *and*

$$f(t, x) = \frac{2/3 \sqrt[3]{7 + tx}}{1 + x^5} \sin x^2 - e^t - 3,$$

$$k(t) = \frac{2}{3} \sqrt[3]{7 + t}, \quad h(t) = e^t + 3.$$

It is easy to prove that $k, h \in L^1[0, 1]$ *are nonnegative functions, and*

$$|f(t, x)| \le k(t)|x| + h(t), \quad a.e.\ (t, x) \in [0, 1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta^3(\alpha-\beta) = \frac{17}{32} > 0.$$

Let $p = 3, q = \frac{3}{2}$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_0^1 k^p(s)ds = \int_0^1 \frac{8}{27}(7+s)ds = \frac{60}{27}.$$

Moreover, we have

$$[\frac{6\zeta(1+3q)^{1/q}}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{(1+3q)/q}}]^p = 23.788.$$

Therefore,

$$\int_0^1 k^p(s)ds < [\frac{6\zeta(1+3q)^{1/q}}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{(1+3q)/q}}]^p.$$

Hence, by Theorem 3.2 (1)*, the BVP* (4.2) *has at least one nontrivial solution* u^* *in* E.

Example 4.3. *Consider the following problem*

$$\begin{aligned} &u^{(4)} + \tfrac{u^3}{(9+2u^2)\sqrt[3]{t}}e^{-\sin u^2} - \sqrt{t} - 1 = 0, \quad 0 < t < 1, \\ &u(0) = 2u(1/3), \quad u^{'}(0) = u^{''}(0) = 0, \quad u(1) = -79u(1/3). \end{aligned} \tag{4.3}$$

Set $\eta = 1/3$, $\alpha = 2$, $\beta = -79$, *and*

$$f(t,x) = \frac{x^3}{(9+2x^2)\sqrt[3]{t}}e^{-\sin x^2} - \sqrt{t} - 1,$$

$$k(t) = \frac{1}{9\sqrt[3]{t}}, \quad h(t) = \sqrt{t} + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta^3(\alpha-\beta) = 2 > 0.$$

Let $\mu = -\frac{1}{3} > -1$, *then*

$$\frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta+1+|\alpha|(1+\eta^3)+(|\beta|+2|\alpha|)\eta^{4+\mu}} = 3.35.$$

Therefore,

$$k(s) = \frac{1}{9\sqrt[3]{s}} = \frac{1}{9}s^{-\frac{1}{3}} < 3.35.s^{-\frac{1}{3}},$$

$$meas\{s \in [0,1] : k(s) < \frac{\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{\zeta + 1 + |\alpha|(1+\eta^3) + (|\beta| + 2|\alpha|)\eta^{4+\mu}} s^{\mu}\} > 0.$$

Hence, by Theorem 3.2 (2)*, the BVP* (4.3) *has at least one nontrivial solution* u^* *in* E*.*

Example 4.4. *Consider the following problem*

$$\begin{aligned} &u^{(4)} + \tfrac{u}{7(3+u^2)\sqrt[5]{(1-t)^2}} e^{-2t}\cos u + t^3 - 2 = 0, \quad 0 < t < 1, \\ &u(0) = 1/3u(1/2), \quad u'(0) = u''(0) = 0, \quad u(1) = 1/4u(1/2). \end{aligned} \tag{4.4}$$

Set $\eta = 1/2$*,* $\alpha = 1/3$*,* $\beta = 1/4$*, and*

$$f(t,x) = \frac{x}{7(3+x^2)\sqrt[5]{(1-t)^2}} e^{-2t}\cos x + t^3 - 2,$$

$$k(t) = \frac{1}{6\sqrt[5]{(1-t)^2}}, \quad h(t) = t^3 + 2.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta^3(\alpha - \beta) = \frac{65}{96} > 0.$$

Let $\mu = -\frac{2}{5} > -4$*, then*

$$\frac{6\zeta(4+\mu)}{\zeta + 1 + |\alpha|(3+\eta^3) + |\beta|} = \frac{1404}{285}.$$

Therefore,

$$k(s) = \frac{1}{6\sqrt[5]{(1-s)^2}} = \frac{1}{6}(1-s)^{-\frac{2}{5}} < \frac{1404}{285}(1-s)^{-\frac{2}{5}},$$

$$meas\{s \in [0,1] : k(s) < \frac{6\zeta(4+\mu)}{\zeta + 1 + |\alpha|(3+\eta^3) + |\beta|}(1-s)^{\mu}\} > 0.$$

Hence, by Theorem 3.2 (3)*, the BVP* (4.4) *has at least one nontrivial solution* u^* *in* E*.*

Example 4.5. *Consider the following problem*

$$\begin{aligned} &u^{(4)} + \tfrac{tu^5}{8(1+u^2)} - e^{4t} - 1 = 0, \quad 0 < t < 1, \\ &u(0) = -3u(1/5), \quad u'(0) = u''(0) = 0, \quad u(1) = -2u(1/5). \end{aligned} \tag{4.5}$$

Set $\eta = 1/5$, $\alpha = -3$, $\beta = -2$, *and*

$$f(t,x) = \frac{tx^5}{8(1+x^2)} - e^{4t} - 1,$$

$$k(t) = \frac{t}{5}, \quad h(t) = e^{4t} + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta^3(\alpha - \beta) = \frac{499}{125} > 0.$$

Moreover, we have

$$\frac{24\zeta}{\zeta + 1 + |\alpha|(1+\eta^3) + (|\beta| + 2|\alpha|)\eta^4} = \frac{29940}{2509}.$$

Therefore,

$$k(s) = \frac{s}{5} < \frac{29940}{2509}, \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24\zeta}{\zeta + 1 + |\alpha|(1+\eta^3) + (|\beta| + 2|\alpha|)\eta^4}\} > 0.$$

Hence, by Theorem 3.2 (4), *the BVP* (4.5) *has at least one nontrivial solution* u^* *in* E.

Example 4.6. *Consider the following problem*

$$\begin{aligned} & u^{(4)} + \tfrac{7t^2 u}{3(1+e^t)^2} - t + 1 = 0, \quad 0 < t < 1, \\ & u(0) = -u(1/2), \quad u'(0) = u''(0) = 0, \quad u(1) = -3u(1/2). \end{aligned} \tag{4.6}$$

Set $\eta = 1/2$, $\alpha = -1$, $\beta = -3$, *and*

$$f(t,x) = \frac{7t^2 x}{3(1+e^t)^2} - t + 1,$$

$$k(t) = 7t^2, \quad h(t) = t + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta^3(\alpha - \beta) = \frac{9}{4} > 0.$$

Moreover, we have

$$\frac{24\zeta}{\zeta + 1 + |\alpha|(1+\eta^3) + (|\beta| + 2|\alpha|)\eta^4} = \frac{864}{75},$$

and

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| = \limsup_{|x|\to\infty} (\frac{7}{3(1+e)^2} + \frac{2}{|x|}) = 0.17.$$

Therefore,

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{24\zeta}{\zeta + 1 + |\alpha|(1+\eta^3) + (|\beta| + 2|\alpha|)\eta^4}.$$

Hence, by Theorem 3.2 (5)*, the BVP* (4.6) *has at least one nontrivial solution* u^* *in* E*.*

Remark: We can give examples similar in relation to the Corollary 3.3, Theorem 3.4, and Corollary 3.5.

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