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Approximation Properties of Modified Srivastava-Gupta Operators Based on Certain Parameter

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ABSTRACT: In the present article, we give a modified form of generalized Srivastava-Gupta operators based on certain parameter which preserve the constant as well as linear functions. First, we estimate moments of the operators and then prove Voronovskaja type theorem. Next, direct approximation theorem, rate of convergence and weighted approximation by these operators in terms of modulus of continuity are studied. Then, we obtain point-wise estimate using the Lipschitz type maximal function. Finaly, we study the A-statistical convergence of these operators.

Key Words: Srivastava-Gupta operators, Modulus of continuity, Weighted approximation, Rate of convergence, A-statistical convergence.

Contents

1. Introduction

The approximation of functions by linear positive operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solutions of differential equations.

In order to approximate Lebesgue integrable functions on $[0,\infty)$, Srivastava and Gupta [31] introduced a general family of summation-integral type operators which includes some well-known operators as special cases. They obtained the rate of convergence for functions of bounded variation. After that several researchers studied different approximation properties of these operators (see [1], [3], [12], [14], [22], [34], [35]).

For $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma), \gamma > 0\}$, Verma [33] define the following generalization of Srivastava-Gupta operators based on certain parameter $\rho > 0$ as:

$$L_{n,\rho}(f;x) = \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^\infty \Theta_{n,k}^{\rho}(t,c) f(t)dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

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where

$$p_{n,k}(x,c)=\frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), \tag{1.2}$$

$$\Theta_{n,k}^{\rho}(t,c)=\begin{cases}\frac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1}, & c=0,\\ \frac{\Gamma(\frac{n\rho}{c}+k\rho)}{\Gamma(k\rho)\Gamma(\frac{n\rho}{c})}\frac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}}, & c\in N.\end{cases}$$

and

$$\phi_{n,c}(x)=\begin{cases}e^{-nx}, & c=0,\\ (1+cx)^{-n/c}, & c\in N.\end{cases}$$

For the properties of $\phi_{n,c}(x)$, we refer the readers to [31]. For $\rho=1$ the operators (1.1) reduced to the Srivastava-Gupta operators [31]. In [33], Verma studied some results in simultaneous approximation by the operators $L_{n,\rho}$.
It is observed that the operators (1.1) reproduce only constant functions. So here we modify the operators (1.1) so that they may be capable to reproduce constant as well as linear function. King [20] gave an approach for modification of the classical Bernstein polynomials and he achieved better approximation. Here we give some alternate approach and we propose the modification of the operators (1.1) as follows:

$$L_{n,\rho}^{*}(f;x)=\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f\left(\frac{(n\rho-c)t}{n\rho}\right)dt+p_{n,0}(x,c)f(0). \tag{1.3}$$

In the present paper, we study the basic convergence theorem, Voronovskaja type asymptotic formula, local approximation, rate of convergence, weighted approximation, pointwise estimation and A-statistical convergence of the operators (1.3).

2. Preliminaries

In this section we collect some results about the operators $L_{n,\rho}^{*}$ useful in the sequel.

Lemma 2.1. *[33] For $L_{n,\rho}(t^m;x)$, $m=0,1,2$, we have*

1. $L_{n,\rho}(1;x)=1$;
2. $L_{n,\rho}(t;x)=\frac{n\rho x}{(n\rho-c)}$;
3. $L_{n,\rho}(t^2;x)=\frac{n(n+c)\rho^2x^2+n\rho(1+\rho)x}{(n\rho-c)(n\rho-2c)}$.

Lemma 2.2. *For the operators $L_{n,\rho}^{*}(f;x)$ as defined in (1.3), the following equalities holds for $n\rho>2c$*

1. $L_{n,\rho}^{*}(1;x)=1$;
2. $L_{n,\rho}^{*}(t;x)=x$;
3. $L_{n,\rho}^{*}(t^2;x)=\left\{\frac{(n\rho-c)(n+c)}{n(n\rho-2c)}\right\}x^2+\left\{\frac{(n\rho-c)(1+\rho)}{n\rho(n\rho-2c)}\right\}x$.

Proof: For $x \in [0, \infty)$, in view of Lemma 2.1, we have

$$L^*_{n,\rho}(1; x) = 1.$$

Next, for $f(t) = t$, we get

$$L^*_{n,\rho}(t; x) = \sum_{k=1}^{\infty} p_{n,k}(x, c) \int_0^{\infty} \Theta^{\rho}_{n,k}(t, c) \frac{(n\rho - c)t}{n\rho} dt = \frac{(n\rho - c)}{n\rho} L_{n,\rho}(t, x) = x.$$

Proceeding similarly, we have

$$\begin{aligned} L^*_{n,\rho}(t^2; x) &= \sum_{k=1}^{\infty} p_{n,k}(x, c) \int_0^{\infty} \Theta^{\rho}_{n,k}(t, c) \left(\frac{(n\rho - c)t}{n\rho}\right)^2 dt \\ &= \left(\frac{n\rho - c}{n\rho}\right)^2 L_{n,\rho}(t^2, x) \\ &= \left\{\frac{(n\rho - c)(n + c)}{n(n\rho - 2c)}\right\} x^2 + \left\{\frac{(n\rho - c)(1 + \rho)}{n\rho(n\rho - 2c)}\right\} x. \end{aligned}$$

□

Remark 2.3. *For every $x \in [0, \infty)$ and $n\rho > 2c$, we have*

$$L^*_{n,\rho}\left((t - x); x\right) = 0$$

and

$$L^*_{n,\rho}\left((t - x)^2; x\right) = \left\{\frac{n\rho c + nc - c^2}{n(n\rho - 2c)}\right\} x^2 + \left\{\frac{(n\rho - c)(1 + \rho)}{n\rho(n\rho - 2c)}\right\} x = \xi_{n,\rho}(x), \text{ (say)}.$$

Lemma 2.4. *For $f \in C_B[0, \infty)$ (space of all real valued bounded and uniformly continuous functions on $[0, \infty)$ endowed with norm $\| f \|_{C_B[0,\infty)} = \sup_{x \in [0,\infty)} |f(x)|$),*

$$\| L^*_{n,\rho}(f; x) \| \leq \| f \| .$$

Proof: In view of (1.3) and Lemma 2.2, the proof of this lemma easily follows. □

For $C_B[0, \infty)$, let us define the following Peetre's K-functional:

$$K_2(f, \delta) = \inf_{g \in W^2} \{\| f - g \|_{C_B[0,\infty)} + \delta \| g'' \|_{C_B[0,\infty)}\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By, p. 177, Theorem 2.4 in [4], there exists an absolute constant $M > 0$ such that

$$K_2(f, \delta) \leq M\omega_2(f, \sqrt{\delta}), \tag{2.1}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<|h|\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |\ f(x+2h) - 2f(x+h) + f(x)\ |$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<|h|\leq\delta} \sup_{x\in[0,\infty)} |\ f(x+h) - f(x)\ |,$$

we denote the usual modulus of continuity of $f \in C_B[0,\infty)$.

3. Main results

In this section we establish some approximation properties in several settings.

Theorem 3.1. *(Voronovskaja type theorem) Let f be bounded and integrable on $[0,\infty)$, second derivative of f exists at a fixed point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n\left(L^*_{n,\rho}(f;x) - f(x)\right) = \frac{x(1+cx)}{2}\left(1+\frac{1}{\rho}\right) f''(x).$$

Proof: Using Taylor's theorem, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{3.1}$$

where $r(t,x)$ is the remainder term and $\lim_{t\to x} r(t,x) = 0$.
Applying $L^*_{n,\rho}(f,x)$ to (3.1), we get

$$n\left(L^*_{n,\rho}(f;x) - f(x)\right) = nf'(x)L^*_{n,\rho}\left((t-x);x\right) + \frac{1}{2}nf''(x)L^*_{n,\rho}\left((t-x)^2;x\right) + nL^*_{n,\rho}\left(r(t,x)(t-x)^2;x\right).$$

In view of Remark 2.3, we have

$$\lim_{n\to\infty} nL^*_{n,\rho}\left((t-x);x\right) = 0 \tag{3.2}$$

and

$$\lim_{n\to\infty} nL^*_{n,\rho}\left((t-x)^2;x\right) = x(1+cx)\left(1+\frac{1}{\rho}\right). \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} nL^*_{n,\rho}\Big(r(t,x)(t-x)^2;x\Big) = 0.$$

Applying the Cauchy-Schwarz inequality, we have

$$L^*_{n,\rho}\Big(r(t,x)(t-x)^2;x\Big) \leq \sqrt{L^*_{n,\rho}(r^2(t,x);x)}\sqrt{L^*_{n,\rho}((t-x)^4;x)}. \tag{3.4}$$

We observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} L^*_{n,\rho}(r^2(t,x);x) = r^2(x,x) = 0, \tag{3.5}$$

in view of fact that $L^*_{n,\rho}((t-x)^4;x) = O\left(\frac{1}{n^2}\right)$. Now, from (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} nL^*_{n,\rho}\left(r(t,x)(t-x)^2;x\right) = 0. \tag{3.6}$$

From (3.2), (3.3) and (3.6), we get the required result. □

Theorem 3.2. *For every $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, there exist an absolute constant M such that*

$$| L^*_{n,\rho}(f;x) - f(x) | \leq M\omega_2\left(f, \sqrt{\xi_{n,\rho}(x)}\right),$$

Proof: Let $g \in W^2$ and $x,t \in [0,\infty)$. Using Taylor's series, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v)g''(v)dv.$$

Applying $L^*_{n,\rho}$ on both sides and using Lemma 2.2, we get

$$L^*_{n,\rho}(g;x) - g(x) = L^*_{n,\rho}\left(\int_x^t (t-v)g''(v)dv;x\right).$$

Obviously, we have $\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2\|g''\|$.
Therefore

$$| L^*_{n,\rho}(g;x) - g(x) | \leq L^*_{n,\rho}((t-x)^2;x) \parallel g'' \parallel = \xi_{n,\rho}(x) \parallel g'' \parallel .$$

Since $| L^*_{n,\rho}(f;x) | \leq \|f\|$, we have

$$\begin{aligned} | L^*_{n,\rho}(f;x) - f(x) | &\leq | L^*_{n,\rho}(f-g;x) | + | (f-g)(x) | + | L^*_{n,\rho}(g;x) - g(x) | \\ &\leq 2\|f-g\| + \xi_{n,\rho}(x)\|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (2.1) we obtain

$$| L^*_{n,\rho}(f;x) - f(x) | \leq M\omega_2\left(f, \sqrt{\xi_{n,\rho}(x)}\right),$$

which proves the theorem. □

Definition 3.3. *The modulus of continuity of f on the closed interval $[0,b], b > 0$ is denoted by $\omega_b(f,\delta)$ and defined as*

$$\omega_b(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,b]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero.

Now, we give a rate of convergence theorem for the operators $L^*_{n,\rho}$.

Theorem 3.4. *Let $f \in C_B[0,\infty)$ and $\omega_{b+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$, where $b>0$. Then, we have*

$$|L^*_{n,\rho}(f;x)-f(x)| \leq 6M_f(1+b^2)\xi_{n,\rho}(b)+2\omega_{b+1}\left(f,\sqrt{\xi_{n,\rho}(b)}\right),$$

where $\xi_{n,\rho}(b)$ is defined in Remark 2.3 and M_f is a constant depending only on f.

Proof: For $x \in [0,b]$ and $t>b+1$. Since $t-x>1$, we have

$|f(t)-f(x)| \leq M_f(2+x^2+t^2) \leq M_f(t-x)^2(2+3x^2+2(t-x)^2) \leq 6M_f(1+b^2)(t-x)^2.$

For $x\in[0,b]$ and $t \leq b+1$, we have

$$|f(t)-f(x)| \leq \omega_{b+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta)$$

with $\delta>0$.
From the above, we have

$$|f(t)-f(x)| \leq 6M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),$$

for $x\in[0,b]$ and $t\geq 0$.
Applying Cauchy-Schwarz inequality, we have
$|L^*_{n,\rho}(f;x)-f(x)|$

$$\begin{aligned} &\leq 6M_f(1+b^2)(L^*_{n,\rho}(t-x)^2;x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}(L^*_{n,\rho}(t-x)^2;x)^{\frac{1}{2}}\right)\\ &\leq 6M_f(1+b^2)\xi_{n,\rho}(b)+2\omega_{b+1}\left(f,\sqrt{\xi_{n,\rho}(b)}\right),\end{aligned}$$

on choosing $\delta=\sqrt{\xi_{n,\rho}(b)}$. This completes the proof of the theorem. □

Next, we obtain the Korovkin type weighted approximation by the operators defined in (1.3). The weighted Korovkin-type theorems were proved by Gadzhiev [5].

Definition 3.5. *A real function $\nu(x)=1+x^2$ is called a weight function if it is continuous on R and $\lim_{|x|\to\infty}\nu(x)=\infty$, $\nu(x)\geq 1$ for all $x\in R$.*

Let $B_\nu(R)$ denote the weighted space of real-valued functions f defined on R with the property $|f(x)| \leq M_f\nu(x)$ for all $x\in R$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu(R)$ of $B_\nu(R)$ given by $C_\nu(R)=\{f\in B_\nu(R) : f \text{ is continuous on } R\}$ and $C^*_\nu[0,\infty)$ denotes the subspace of all functions $f\in C_\nu[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{f(x)}{\nu(x)}$ exists finitely.

Theorem 3.6. *For each $f \in C_\nu^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \| L_{n,\rho}^*(f) - f \|_\nu = 0.$$

Proof: From [5], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| L_{n,\rho}^*(t^k;x) - x^k \|_\nu = 0, \ \ k = 0,1,2. \tag{3.7}$$

Since $L_{n,\rho}^*(1;x) = 1$, the condition in (3.7) holds for $k = 0$.
By Lemma 2.2, we have

$$\| L_{n,\rho}^*(t;x) - x \|_\nu = \sup_{x\in[0,\infty)} \frac{|L_{n,\rho}^*(t;x) - x|}{1+x^2} = 0$$

which implies that the condition in (3.7) holds for $k = 1$.
Similarly, we can write for $n\rho > 2c$

$$\begin{aligned} \| L_{n,\rho}^*(t^2;x) - x^2 \|_\nu &= \sup_{x\in[0,\infty)} \frac{|L_{n,\rho}^*(t^2;x) - x^2|}{1+x^2} \\ &\leq \left|\frac{n\rho c + nc - c^2}{n(n\rho - 2c)}\right| + \left|\frac{(n\rho - c)(1+\rho)}{n\rho(n\rho - 2c)}\right|, \end{aligned}$$

which implies that $\lim_{n\to\infty} \| L_{n,\rho}^*(t^2;x) - x^2 \|_\nu = 0$, the equation (3.7) holds for $k = 2$.
This completes the proof of theorem. □

Now we give the following theorem to approximate all functions in C_ν^*. Such type of results are given in [6] for locally integrable functions.

Theorem 3.7. *For each $f \in C_\nu^*$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|L_{n,\rho}^*(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0.$$

Proof: For any fixed $x_0 > 0$,

$$\sup_{x\in[0,\infty)} \frac{|L_{n,\rho}^*(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} = \sup_{x\leq x_0} \frac{|L_{n,\rho}^*(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x> x_0} \frac{|L_{n,\rho}^*(f;x) - f(x)|}{(1+x^2)^{1+\alpha}}$$

$$\sup_{x\in[0,\infty)} \frac{|L_{n,\rho}^*(f;x) - f(x)|}{(1+x^2)^{1+\alpha}}$$

$$\leq \| L_{n,\rho}^*(f) - f \|_{C[0,x_0]} + \| f \|_\nu \sup_{x>x_0} \frac{|L_{n,\rho}^*(1+t^2;x)|}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}.$$

The first term of the above inequality tends to zero from Theorem 3.4. By Lemma 2.2, for any fixed $x_0 > 0$, it is easily prove that

$$\sup_{x>x_0} \frac{|L^*_{n,\rho}(1+t^2;x)|}{(1+x^2)^{1+\alpha}} \to 0$$

as $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be small.
Hence the proof is completed. □

Definition 3.8. *A function $f \in C_B[0,\infty)$ is in $Lip_M(\eta)$ on E, $\eta \in (0,1]$, $E\subset [0,\infty)$ if it satisfies the condition*

$$|f(t) - f(x)| \leq M|t-x|^{\eta},\ t \in [0,\infty) \text{ and } x \in E,$$

where M is a constant depending only on η and f.

Now, we obtain some pointwise estimates of the operators $L^*_{n,\rho}$.

Theorem 3.9. *Let $f \in C_B[0,\infty) \cap Lip_M(\eta)$, $E \subset [0,\infty)$ and $0 < \eta \leq 1$. Then, we have*

$$|L^*_{n,\rho}(f;x) - f(x)| \leq M\bigg(\left(\xi_{n,\rho}(x)\right)^{\eta/2} + 2(d(x,E))^{\eta}\bigg),\ x \in [0,\infty),$$

where M is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E) = \inf\{|t-x|; t \in E\}.$$

Proof: Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $t_0 \in \overline{E}$ such that

$$d(x,E) = |x - t_0|.$$

By our hypothesis and the monotonicity of $L^*_{n,\rho}$, we get

$$\begin{aligned} |L^*_{n,\rho}(f;x) - f(x)| &\leq L^*_{n,\rho}(|f(t) - f(t_0)|;x) + L^*_{n,\rho}(|f(x) - f(t_0)|;x) \\ &\leq M\left(L^*_{n,\rho}(|t-t_0|^{\eta};x) + |x-t_0|^{\eta}\right) \\ &\leq M\left(L^*_{n,\rho}(|t-x|^{\eta};x) + 2|x-t_0|^{\eta}\right). \end{aligned}$$

Now, applying Hölder's inequality with $p = \dfrac{2}{\eta}$ and $q = \dfrac{2}{2-\eta}$, we obtain

$$|L^*_{n,\rho}((f;x) - f(x)| \leq M\left((L^*_{n,\rho}(|t-x|^2;x))^{\eta/2} + 2(d(x,E))^{\eta}\right),$$

from which the desired result immediate. □

Next, we obtain a local direct estimate of the operators defined in (1.3), using the Lipschitz-type maximal function of order η introduced by B. Lenze [21] as

$$\widetilde{\omega}_\eta(f,x)=\sup_{t\neq x,\ t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^\eta},\quad x\in[0,\infty)\ \text{ and }\ \eta\in(0,1]. \tag{3.8}$$

Theorem 3.10. *Let $f\in C_B[0,\infty)$ and $0<\eta\leq 1$, then for all $x\in[0,\infty)$ we have*

$$|L^*_{n,\rho}(f;x)-f(x)|\leq\widetilde{\omega}_\eta(f,x)\left(\xi_{n,\rho}(x)\right)^{\eta/2}.$$

Proof: From the equation (3.8), we have

$$|L^*_{n,\rho}(f;x)-f(x)|\leq L^*_{n,\rho}(|f(t)-f(x)|;x)\leq\widetilde{\omega}_\eta(f,x)L^*_{n,\rho}(|t-x|^\eta;x).$$

Now, using the Hölder's inequality with $p=\dfrac{2}{\eta}$ and $\dfrac{1}{q}=1-\dfrac{1}{p}$, we obtain

$$|L^*_{n,\rho}(f;x)-f(x)|\leq\widetilde{\omega}_\eta(f,x)L^*_{n,\rho}((t-x)^2;x)^{\frac{\eta}{2}}\leq\widetilde{\omega}_\eta(f,x)\left(\xi_{n,\rho}(x)\right)^{\eta/2}.$$

Thus, the proof is completed. □

For $a,b>0$, Özarslan and Aktuğlu [30] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a,b)}(\eta)=\left(f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^\eta}{(t+ax^2+bx)^{\eta/2}};\ x,t\in[0,\infty)\right),$$

where M is any positive constant and $0<\eta\leq 1$.

Theorem 3.11. *For $f\in Lip_M^{(a,b)}(\eta)$. Then, for all $x>0$, we have*

$$|L^*_{n,\rho}(f;x)-f(x)|\leq M\left(\frac{\xi_{n,\rho}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Proof: First we prove the theorem for $\eta=1$. Then, for $f\in Lip_M^{(a,b)}(1)$ and $x\in[0,\infty)$, we have

$$\begin{aligned}|L^*_{n,\rho}(f;x)-f(x)| &\leq L^*_{n,\rho}(|f(t)-f(x)|;x)\\ &\leq ML^*_{n,\rho}\left(\frac{|t-x|}{(t+ax^2+bx)^{1/2}};x\right)\\ &\leq \frac{M}{(ax^2+bx)^{1/2}}L^*_{n,\rho}(|t-x|;x).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$|L^*_{n,\rho}(f;x)-f(x)|\leq\frac{M}{(ax^2+bx)^{1/2}}\left(L^*_{n,\rho}((t-x)^2;x)\right)^{1/2}\leq M\left(\frac{\xi_{n,\rho}(x)}{ax^2+bx}\right)^{1/2}.$$

Thus, the result holds for $\eta = 1$.
Now, we prove that the result is true for $0 < \eta < 1$. Then, for $f \in Lip_M^{(a,b)}(\eta)$ and $x \in [0, \infty)$, we get

$$|L^*_{n,\rho}(f;x) - f(x)| \le \frac{M}{(ax^2+bx)^{\eta/2}} L^*_{n,\rho}(|t-x|^\eta; x).$$

Taking $p = \frac{1}{\eta}$ and $q = \frac{p}{p-1}$, applying the Hölders inequality, we have

$$|L^*_{n,\rho}(f;x) - f(x)| \le \frac{M}{(ax^2+bx)^{\eta/2}} \left(L^*_{n,\rho}(|t-x|; x)\right)^\eta.$$

Finally by Cauchy-Schwarz inequality, we get

$$|L^*_{n,\rho}(f;x) - f(x)| \le M \left(\frac{\xi_{n,\rho}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Thus, the proof is completed. □

Definition 3.12. *Let $A = (a_{nk})$ be a non-negative infinite summability matrix. For a given sequence $x := (x)_n$, the A-transform of x denoted by $Ax : (Ax)_n$ is defined as*

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$$

provided the series converges to each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_n (x)_n = L$. Then $x = (x)_n$ is said to be A- statistically convergent to L i.e. $st_A - \lim_n (x)_n = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \ge \epsilon} a_{nk} = 0$. If we replace A by C_1 then A is a Cesáro matrix of order one and A- statistical convergence is reduced to the statistical convergence. Similarly, if $A = I$, the identity matrix, then A- statistical convergence coincides with the ordinary convergence.

Many researchers have investigated the statistical convergence properties for several sequences and classes of linear positive operators (see [2], [7], [13], [19], [26]). In the following result we prove a weighted Korovkin theorem via A-statistical convergence.
Throughout this section, let us assume that $e_i(t) = t^i, i = 0, 1, 2$.

Theorem 3.13. *Let (a_{nk}) be a non-negative regular infinite summability matrix and $x \in [0, \infty)$. Let $\nu_\varsigma \ge 1$ be a continuous function such that*

$$\lim_{x \to \infty} \frac{\nu(x)}{\nu_\varsigma(x)} = 0.$$

Then, for all $f \in C^*_\nu[0, \infty)$, we have

$$st_A - \lim_n \| L^*_{n,\rho}(f) - f \|_{\nu_\varsigma} = 0.$$

Proof: From ([7] p. 195, Th. 6), it is enough to show that

$$st_A - \lim_n \| L^*_{n,\rho}(e_i) - e_i \|_\nu = 0.$$

Using Lemma 2.2, obviously for $i = 0, 1$, we have

$$st_A - \lim_n \| L^*_{n,\rho}(e_i) - e_i \|_\nu = 0.$$

Now

$$\begin{aligned}
\| L^*_{n,\rho}(e_2) - e_2 \|_\nu &\leq \frac{(n\rho c + nc - c^2)}{n(n\rho - 2c)} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
&\quad + \frac{(n\rho - c)(1+\rho)}{n\rho(n\rho - 2c)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
&\leq \frac{(n\rho^2 + n\rho - c\rho)(c+1)}{n\rho(n\rho - 2c)} + \frac{c}{n\rho(n\rho - 2c)}.
\end{aligned}$$

Now, we define the following sets:

$$S := \left\{ n : \| L^*_{n,\rho}(e_2) - e_2 \|_\nu \geq \epsilon \right\},$$

$$S_1 := \left\{ n : \frac{(n\rho^2 + n\rho - c\rho)(c+1)}{n\rho(n\rho - 2c)} \geq \frac{\epsilon}{2} \right\}$$

and

$$S_2 := \left\{ n : \frac{c}{n\rho(n\rho - 2c)} \geq \frac{\epsilon}{2} \right\}.$$

Then, we get $S \subseteq S_1 \cup S_2$ which implies that

$$\sum_{k\in S} a_{nk} \leq \sum_{k\in S_1} a_{nk} + \sum_{k\in S_2} a_{nk}$$

and hence

$$st_A - \lim_n \| L^*_{n,\rho}(e_2) - e_2 \|_\nu = 0.$$

This completes the proof of the theorem. □

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