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# On Primitive $\mathbb{Z}_p$-algebra

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abstract: Our main purpose is to provide for primitive associative $\mathbb{Z}_p$-algebras a structure theory analogous to that for algebras and superalgebras [1,7,8,9,10] and to classify primitive $\mathbb{Z}_p$-rings having a minimal one sided $\mathbb{Z}_p$-ideal.

Key Words: Primitive $\mathbb{Z}_p$-algebras, Division $\mathbb{Z}_p$-algebras, $\mathbb{Z}_p$-rings.

## Contents



## 1. Introduction

The group grading on associative algebras was described by many authors such as Y. A. Bahturin, A. Giambruno, S. Sehgal, M. Zaicev and I. Shestakov, see [12,13,14,15]. They described all group gradings by an arbitrary finite group $G$ on non-simple finite-dimensional superinvolution simple associative superalgebras over an algebraically closed field $F$ of characteristic 0.
The existence of superinvolution on associative superalgebras was studied by many authors, for example Michel Racine in [9] and A.Elduque and O. Villa in [1] studied superalgebras, which are $\mathbb{Z}_2$-graded algebras, and existence of superinvolutions on finite dimensional central simple superalgebras, and they found that non-trivial central division superalgebras are never endowed with superinvolution of the first kind, but they prove the graded version of the classical Albert and Albert-Riehm Theorem of existence of superinvolution of the second kind.
Continuing on the studying of superalgebras, in [2,3,4] we developed the theory of existence of pseudo-superinvolutions of the first kind and superinvolutions of the first and second kinds on finite dimensional central simple associative superalgebras over a field $K$ of characteristic not 2. We proved that a central division superalgebra $\mathcal{D}$, over a field $K$ of characteristic not 2, of even type has a pseudo-superinvolution of the first kind if and only if $\mathcal{D}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$.
Moreover we proved that if $\mathcal{D}$ is of type, then $\mathcal{D}$ has a pseudo-superinvolution of the first kind if and only if $\sqrt{-1} \in K$ and $\mathcal{D}$ is of order 2 in the Brauer-Wall group $\mathrm{BW}(K)$.
In this paper we generalize the above work about superalgebras to introduce the

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following article about primitive associative $\mathbb{Z}_p$-algebras with $\mathbb{Z}_p$-involution, and the purpose of this article is to provide a structure theory analogous to that for algebras and superalgebras [1,7,8,9,10] and to classify primitive $\mathbb{Z}_p$-rings having a minimal one sided $\mathbb{Z}_p$-ideal.
In section 2 we introduced some examples of associative $\mathbb{Z}_p$-algebras, and in section 3 we proved that artinian simple associative $\mathbb{Z}_p$-ring is isomorphic to $M_n(\mathcal{D})$, where $\mathcal{D}$ is an associative division $\mathbb{Z}_p$-algebra.

## 2. Examples of $\mathbb{Z}_p$-algebras

Let $p$ be any prime number. An *associative* $\mathbb{Z}_p$-ring $R = \bigoplus_{i=0}^{p-1} R_i$ is nothing but a $(\mathbb{Z}/p\mathbb{Z})$-graded *associative* ring. A $(\mathbb{Z}/p\mathbb{Z})$-graded ideal $I = \bigoplus_{i=0}^{p-1} I_i$ of an associative $\mathbb{Z}_p$-ring $R$ is called a $\mathbb{Z}_p$*-ideal* of $R$. An associative $\mathbb{Z}_p$-ring $R$ is *simple* if it has no non-trivial $\mathbb{Z}_p$-ideals. An *associative* $\mathbb{Z}_p$-ring $R$ is a commutative $\mathbb{Z}_p$-ring if

$$a_\alpha b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha \quad \forall a_\alpha \in R_\alpha,\ b_\beta \in R_\beta,$$

where the product $\alpha\beta$ is taken modulo $p$. We will say that such elements $\mathbb{Z}_p$-commute.

Let $R$ be an associative $\mathbb{Z}_p$-ring with $1 \in R_0$, then $R$ is said to be a *division* $\mathbb{Z}_p$-ring if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_\alpha \in R_\alpha$ has an inverse $r_\alpha^{-1}$, necessarily in $R_{p-\alpha}$, where the subscript $p-\alpha$ is taken modulo $p$.

Let $K$ be a field of characteristic 0. An *associative* $(\mathbb{Z}/p\mathbb{Z})$-graded $K$-algebra $\mathcal{A} = \bigoplus_{i=0}^{p-1} \mathcal{A}_i$ is a *finite dimensional central simple* $\mathbb{Z}_p$-algebra over a field $K$, if $Z(\mathcal{A}) \cap \mathcal{A}_0 = K$, where $Z(\mathcal{A}) = \{a \in \mathcal{A} \mid ab = ba\ \forall\ b \in \mathcal{A}\}$ is the *center* of $\mathcal{A}$, and the only $\mathbb{Z}_p$-ideals of $\mathcal{A}$ are $(0)$ and $\mathcal{A}$ itself.

**Example 2.1.** Let $\mathcal{A} = K(\sqrt[3]{a})$ be an algebraic field extension of the field $K$ of degree 3, that is $[\mathcal{A} : K] = 3$. We can make $\mathcal{A}$ into a $\mathbb{Z}_3$-algebra by setting

$$\mathcal{A}_0 = K,\ \mathcal{A}_1 = K.\sqrt[3]{a},\ \mathcal{A}_2 = K.\sqrt[3]{a^2}.$$

Note that $\mathcal{A}$ is a central simple $\mathbb{Z}_3$-algebra, since $\mathcal{A}$ is a field and $\mathcal{A} \cap \mathcal{A}_0 = K$.

**Example 2.2.** Let $p$ be any prime number. A $\mathbb{Z}_p$-space over a field $K$ is a left $K$-vector space $V$ which is $\mathbb{Z}_p$-graded $V = \bigoplus_{i=0}^{p-1} V_i$. The associative algebra $\mathrm{End}_K V = \bigoplus_{i=0}^{p-1} \mathrm{End}_i V$, where

$$\mathrm{End}_i V := \{a \in \mathrm{End}_K V \ :\ v_j a \in V_{i+j}\},$$

is an associative $\mathbb{Z}_p$-algebra.

**Example 2.3.** Let $p$ be any prime number. Let $\mathcal{D} = \bigoplus_{i=0}^{p-1} \mathcal{D}_i$ be a division $\mathbb{Z}_p$-algebra, then $\mathcal{A} = M_k(\mathcal{D})$ can be made into a $\mathbb{Z}_p$-algebra by setting

$$\mathcal{A}_0 = M_k(\mathcal{D}_0),\ \mathcal{A}_1 = M_k(\mathcal{D}_1), \cdots, \mathcal{A}_{p-1} = M_k(\mathcal{D}_{p-1}).$$

**Example 2.4.** Let $p$ be any prime number. Let $\omega$ be a fixed primitive $p$-th root of unity. For $a, b \in K^\times$, let $\mathcal{A} =< a, b >_\omega$ be the $K$-algebra which is generated by $\{i, j\}$ which satisfy

$$\{i^p = a\ ,\ j^p = b\ ,\ ij = \omega ji\}.$$

Then $\mathcal{A}$ is a vector space over $K$ with basis

$$\{i^r j^s\ :\ 0 \leq r, s < p\}.$$

So $\mathcal{A}$ has dimension $p^2$ as a $K$-algebra. (See [11, section 15.4] and [6, Exercise 4.28]). This is a generalization of the quaternion algebras. We can make $\mathcal{A}$ into $\mathbb{Z}_p$-algebra by setting

$$\mathcal{A}_l =< i^k j^m\ :\ k + m \equiv l( \mod p) >_K .$$

**Example 2.5.** *Let $\mathcal{D}$ be a central division algebra over $K$ and let $\mathcal{A} = M_3(\mathcal{D})$. If*

$$\mathcal{A}_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \mathcal{A}_1 = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}.$$

*Then $\mathcal{A}$ is a $\mathbb{Z}_3$-algebra, written by $\mathcal{A} = M_{1+1+1}(\mathcal{D})$, since $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ where the subscripts are taken modulo 3.*

In the last example we can generalize the above example as follows, where $p$ is any prime number.

**Example 2.6.** Let $p$ be any prime number. Let $\mathcal{D}$ be a central division algebra over a field $K$ and let $\mathcal{A} = M_p(\mathcal{D})$, then $\mathcal{A}$ can be made into $\mathbb{Z}_p$-algebra by setting

$$\mathcal{A}_0 = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix},\ \mathcal{A}_1 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & * \\ * & 0 & \cdots & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & * & 0 \\ 0 & 0 & \cdots & \cdots & 0 & * \\ * & 0 & 0 & \cdots & \cdots & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 & 0 \end{bmatrix}, \cdots\cdots,\ \mathcal{A}_{p-1} = \begin{bmatrix} 0 & * & 0 & \cdots & 0 \\ \vdots & 0 & * & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & * \\ * & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In the next theorem we give an example of a simple associative $\mathbb{Z}_3$-algebra but not simple as an algebra.

**Theorem 2.7.** *Let $R$ be a simple associative algebra, then the associative $\mathbb{Z}_3$-algebra*

$$\mathcal{A} = \{\begin{pmatrix} a & y & x \\ x & a & y \\ y & x & a \end{pmatrix} \ : \ a, x, y \in R\}$$

*is simple as a $\mathbb{Z}_3$-algebra but not as an algebra.*

**Proof:** Let $J \neq 0$ be a $\mathbb{Z}_3$-ideal in $\mathcal{A}$. Then there exists $\begin{pmatrix} a & y & x \\ x & a & y \\ y & x & a \end{pmatrix} \neq 0 \in J$, which implies that $a \neq 0$ or $y \neq 0$ or $x \neq 0$. If $a \neq 0$, then $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \neq 0 \in J_0$ which implies that the identity matrix $I_3$ is in $J_0$ and hence $J = \mathcal{A}$. similarly, if $x \neq 0$ (or $y \neq 0$), then $\begin{pmatrix} 0 & 0 & x \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \neq 0 \in J_1$ (or $\begin{pmatrix} 0 & y & 0 \\ 0 & 0 & y \\ y & 0 & 0 \end{pmatrix} \neq 0 \in J_2$) and $\begin{pmatrix} 0 & \frac{1}{x} & 0 \\ 0 & 0 & \frac{1}{x} \\ \frac{1}{x} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & x \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix} = I_3$ (or $\begin{pmatrix} 0 & 0 & \frac{1}{y} \\ \frac{1}{y} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \end{pmatrix} \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & y \\ y & 0 & 0 \end{pmatrix} = I_3$) which implies that the identity matrix $I_3$ is in $J_0$ and hence $J = \mathcal{A}$. Therefore $\mathcal{A}$ is simple $\mathbb{Z}_3$-algebra.
Let $J = \{\begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix} \ : \ a \in R\}$, then $J \neq \mathcal{A}$ is an ideal in $\mathcal{A}$, which implies that $\mathcal{A}$ is not simple as an algebra. $\square$

## 3. Primitive $\mathbb{Z}_p$-rings

We first start by establishing the elementary results for Primitive $\mathbb{Z}_p$-rings analogous to those for rings [10, Chaps. II and III].
If $R = \bigoplus_{i=0}^{p-1} R_i$ is an associative $\mathbb{Z}_p$-ring, a (right) $\mathbb{Z}_p$-module $M$ over $R$ is a right $R$-module with grading $M = \bigoplus_{i=0}^{p-1} M_i$ as $R_0$-modules such that

$$m_i r_j \in M_{i+j} \quad \text{for any} \quad m_i \in M_i, r_j \in R_j, i, j \in \mathbb{Z}_p.$$

If $N = \bigoplus_{i=0}^{p-1} N_i$ is also a $\mathbb{Z}_p$-module over $R$, then a $\mathbb{Z}_p$-module homomorphism over $R$ from $M$ to $N$ is an $R_0$-module homomorphism $h_j, j \in \mathbb{Z}_p$, such that

$$M_i h_j \subseteq N_{i+j}.$$

Given a $\mathbb{Z}_p$-module $M$ over $R$, $\mathrm{End}(M)$ (action of $\mathrm{End}(M)$ on the right), the ring of $\mathbb{Z}_p$-module endomorphism of $M$ over $R$, is a $\mathbb{Z}_p$-ring. For $i \in \mathbb{Z}_p$, let

$$\mathrm{End}_i(M) := \{h \in \mathrm{End}(M) \ : \ M_j h \subseteq M_{i+j}\}.$$

The *Commuting $\mathbb{Z}_p$-ring* $\mathcal{C}$ of $R$ on $M$ is defined to be $\mathcal{C} = \bigoplus_{i=0}^{p-1} \mathcal{C}_i$, where

$$\mathcal{C}_k := \{c_k \in \mathrm{End}_k(M) \ : \ c_k r_i = (-1)^{ik} r_i c_k, \ i \in \mathbb{Z}_p\}.$$

Thus the elements of $\mathcal{C}$ are $\mathbb{Z}_p$-commuting with those of $R$ acting on $M$.

A $\mathbb{Z}_p$-module $M$ over $R$ is *irreducible* if $MR \neq \{0\}$ and $M$ has no proper $\mathbb{Z}_p$-submodule. The next two results are standard and are included for completeness'sake.

**Lemma 3.1** (Generalized Schur's Lemma)**.** *Let* $M = \bigoplus_{i=0}^{p-1} M_i$, $N = \bigoplus_{i=0}^{p-1} N_i$ *be irreducible* $\mathbb{Z}_p$*-modules over* $R = \bigoplus_{i=0}^{p-1} R_i$ *and let* $f_j$ *be a* $\mathbb{Z}_p$*-module homomorphism of* $M$ *into* $N$*. If* $f_j \neq 0$*, then* $f_j$ *is invertible.*

**Proof:** Since $f_j \neq 0$, $Mf_j = \bigoplus_{i=0}^{p-1}(M_i f_j)$ is a nonzero $\mathbb{Z}_p$-submodule of $N$. By the irreducibility of $N$, $Mf_j = N$. Let $\ker_i(f_j) = \{m_i \in M_i \ : \ m_i f_j = 0\}$. Then $\ker(f_j) = \bigoplus_{i=0}^{p-1} \ker_i(f_j)$ is a $\mathbb{Z}_p$-submodule of $M$ properly contained in $M$ ($f_j \neq 0$). By irreducibility of $M$, $\ker(f_j) = \{0\}$ and $f_j$ is invertible. □

**Corollary 3.2.** *Let* $M = \bigoplus_{i=0}^{p-1} M_i$ *be an irreducible* $\mathbb{Z}_p$*-module over* $R = \bigoplus_{i=0}^{p-1} R_i$*. Then the commuting* $\mathbb{Z}_p$*-ring* $\mathcal{C}$ *of* $R$ *on* $M$ *is a division* $\mathbb{Z}_p$*-ring.*

**Proof:** If $c_j \neq 0 \in \mathcal{C}_j$, then $m_i c_j \neq 0$ for some $m_i \in M_i$. By generalized Schur's Lemma, $c_j$ is invertible in $\mathrm{End}(M)$ and hence in $\mathcal{C}$. Thus $\mathcal{C}$ is a division $\mathbb{Z}_p$-ring. □

The following lemma is the key to the proof of the density theorem for associative $\mathbb{Z}_p$-algebras.

**Lemma 3.3.** *Let* $M = \bigoplus_{i=0}^{p-1} M_i$ *be an irreducible* $\mathbb{Z}_p$*-module over the* $\mathbb{Z}_p$*-ring* $R = \bigoplus_{i=0}^{p-1} R_i$*. If* $M_i \neq \{0\}$*, then* $M_i$ *is an irreducible* $R_0$*-module and for any nonzero* $m_i \in M_i$*,* $m_i R_j = M_{i+j}$*. If* $M_k \neq \{0\}$ *for all* $0 \leq k \leq p-1$*, then the commuting ring of* $R_0$ *on* $M_k$ *can be identified with* $\mathcal{C}_0$*, the zero part of the commuting* $\mathbb{Z}_p$*-ring* $\mathcal{C}$ *of* $R$ *on* $M$*.*

**Proof:** If $N_i$ is a nonzero $R_0$-submodule of $M_i$, then $N_i + \sum_{j=1}^{p-1} N_i R_j$ is a nonzero $\mathbb{Z}_p$-submodule of $M$. Therefore $N_i + \sum_{j=1}^{p-1} N_i R_j = M$. So $N_i = M_i$ and $M_i$ is an irreducible $R_0$-module.
If $m_i R_0 = \{0\}$ for some $m_i \neq 0 \in M_i$, let $N_i = \{n_i \in M_i \ : \ n_i R_0 = \{0\}\}$. Since $N_i$ is a nonzero $R_0$-submodule of $M_i$, $N_i = M_i$. So $M_i R_0 = \{0\}$. If $M_i R_\alpha =$

$\{0\}$ $\forall\alpha \in \mathbb{Z}_p$, then $M_iR = \{0\}$ and hence $M_i$ is a proper $\mathbb{Z}_p$-submodule of $M$, a contradiction. Therefore $M_iR_\alpha \neq \{0\}$ for some $\alpha \in \mathbb{Z}_p$ which implies that $\sum_{j=1}^{p-1} M_iR_j$ is a proper $\mathbb{Z}_p$-submodule of $M$, a contradiction. Hence if $m_i \neq 0$, then $m_iR_0 \neq \{0\}$ and $m_iR_0 = M_i$. Also $m_iR_j \supseteq m_iR_0R_j = M_iR_j$ is an $R_0$-submodule of $M_{i+j}$, for $1 \leq j \leq p-1$. Now let $H = \{\alpha \in \mathbb{Z}_p : M_iR_\alpha = 0\}$. Then $M_i + \sum_{1\leq j\leq p-1\notin H} M_iR_j$ is a proper $\mathbb{Z}_p$-submodule of $M$ over $R$, a contradiction. Therefore $m_iR_j = M_iR_j = M_{i+j}$ and $\forall j \in \mathbb{Z}_p$. Let $\mathcal{D}$ be the commuting ring of $R_0$ on $M_k$ considered an $R_0$-module. So for all $d \in \mathcal{D}$, $r_0 \in R_0$, and $m_k \in M_k$,

$$m_kr_0d = m_kdr_0.$$

Given $d \in \mathcal{D}$ we wish to extend it's action to $M_{k+\alpha}$ for each $1 \leq \alpha \leq p-1$. Fix a nonzero $m_k \in M_k$. Since $m_kR_\alpha = M_{k+\alpha}$ for each $1 \leq \alpha \leq p-1$, define an action of $\mathcal{D}$ on $M_{k+\alpha}$ by

$$m_kr_\alpha d := m_kdr_\alpha \quad \text{for any } d \in \mathcal{D} \text{ and } r_\alpha \in R_\alpha, 1 \leq \alpha \leq p-1.$$

We must show that this action is well-defined, namely, that if $m_kr_j = 0$, then $n_{k+j} = m_kdr_j = 0$ where $1 \leq j \leq p-1$. If $n_{k+j} \neq 0$ for some $1 \leq j \leq p-1$, then

$$n_{k+j}R_\alpha = M_{k+j+\alpha} \quad \text{for each} \quad 1 \leq \alpha \leq p-1.$$

Now let $1 \leq j \leq p-1$ with $m_kr_j = 0$, then $n_{k+j}R_{p-j} = M_k$ and $m_k = n_{k+j}s_{p-j}$ for some $s_{p-j} \in R_{p-j}$. Therefore $m_k = n_{k+j}s_{p-j} = (m_kdr_j)s_{p-j} = m_k(r_js_{p-j})d = (m_kr_j)s_{p-j}d = 0$, a contradiction. Thus the action is well-defined. Note that the computation (for $1 \leq j \leq p-1$) shows that $d$ commutes with all $s_{p-j} \in R_{p-j}$ on $M_{k+j}$. To complete the proof we show that $d$ commutes with $R_\alpha$ on $M_{k+j}$ for any $\alpha \neq p-j$. By definition, $d$ commutes with all of $R_\alpha$ on $M_k$ for all $1 \leq \alpha \leq p-1$. First we prove that $d$ commutes with $R_0$ on $M_{k+j}$. For all $r_0 \in R_0$, $r_j \in R_j$, and $d \in \mathcal{D}$

$$\begin{aligned}(m_kr_j)dr_0 = (m_kr_jd)r_0 &= (m_kd)(r_jr_0)\\ &= m_k(r_jr_0)d\\ &= (m_kr_j)r_0d,\end{aligned}$$

so $d$ commutes with $R_0$ on $M_{k+j}$, and for all $r'_\alpha \in R_\alpha$, $r_j \in R_j$, and $d \in \mathcal{D}$ where $\alpha \neq p-j$ we have

$$\begin{aligned}(m_kr_j)dr'_\alpha = (m_kr_jd)r'_\alpha &= (m_kd)(r_jr'_\alpha)\\ &= m_k(r_jr'_\alpha)d\\ &= (m_kr_j)r'_\alpha d,\end{aligned}$$

so also $d$ commutes with all of $R_\alpha$ on $M_{k+j}$ for any $\alpha \neq p-j$, and hence $\mathcal{D}$ commutes with $R$ on $M$. Thus we can identify $\mathcal{D}$ with $\mathcal{C}_0$. □

Following [10] we prefer to have the commuting $\mathbb{Z}_p$-ring act on the left and the endomorphism $\mathbb{Z}_p$-ring act on the right. We do this by letting the opposite $\mathbb{Z}_p$-ring of $\mathcal{C}$ ($\mathcal{C}^{op}$) act on the left via

$$c_i v_k = (-1)^{ik} v_k c_i.$$

Let $M$ be a $\mathbb{Z}_p$-module over a $\mathbb{Z}_p$-ring $R$, then the annihilator of $M$ over $R$, which is denoted by $\mathrm{Ann}_R M$, is $\mathrm{Ann}_R M = \{r \in R \ : \ Mr = 0\}$. One can easily check that $\mathrm{Ann}_R M$ is a $\mathbb{Z}_p$-ideal of $R$.

**Definition 3.4.** *Let $M$ be a $\mathbb{Z}_p$-module over a $\mathbb{Z}_p$-ring $R$, then $M$ is a faithful module over $R$ if $\mathrm{Ann}_R M = \{0\}$.*

The $\mathbb{Z}_p$-ring $R$ is a (right) *primitive* if it has a faithful irreducible $\mathbb{Z}_p$-module. If $M$ is a faithful irreducible (right) $\mathbb{Z}_p$-module over $R$, we may consider $M$ as a left $\mathbb{Z}_p$-module over $\Delta = \mathrm{End}_R M$, then $\Delta \cong \mathcal{C}^{op}$, where $\mathcal{C}$ is the commuting $\mathbb{Z}_p$-ring of $R$ on $M$. Now $R$ is said to be *dense* on $M$ if for every positive integer $n$ and choice of $v_{(1,i)}, v_{(2,i)}, \cdots, v_{(n,i)} \in M_i$ linearly independent over $\Delta$ and $w_{(1,k)}, w_{(2,k)}, \cdots, w_{(n,k)} \in M_k$ there is an element $r_{(p-1)i+k} \in R_{(p-1)i+k}$ such that $v_{(l,i)} r_{(p-1)i+k} = w_{(l,k)}$ for all $l = 1, 2, ..., n$.

**Lemma 3.5.** *Let $R = \bigoplus_{i=0}^{p-1} R_i$ be a primitive $\mathbb{Z}_p$-ring, let $M = \bigoplus_{i=0}^{p-1} M_i$ be a faithful irreducible $\mathbb{Z}_p$-module over $R$, let $\Delta = \mathrm{End}_R M$ (action of $\Delta$ on the left). Then for any $\Delta$-linearly independent elements $v_{(1,i)}, v_{(2,i)}, \cdots, v_{(n,i)} \in M_i$ there is a homogeneous element $r \in R$ such that $v_{(1,i)} r \neq 0$, $v_{(2,i)} r = \cdots = v_{(n,i)} r = 0$.*

**Proof:** We prove it by induction on $n$, the case $n = 1$ is trivial. Assuming the result is proven for $n-1$, let $J = \{r \in R \ : \ v_{(3,i)} r = \cdots = v_{(n,i)} r = 0\}$ be the right annihilator of $v_{(3,i)}, ..., v_{(n,i)}$. By induction hypothesis $v_{(1,i)} J \neq 0$ and $v_{(2,i)} J \neq 0$, and by irreducibility of $M$, $M = v_{(1,i)} J = v_{(2,i)} J$. If there is an homogeneous element $r \in J$ with $v_{(2,i)} r = 0 \neq v_{(1,i)} r$, we are done. Otherwise, the map $\Psi : M = v_{(2,i)} J \to M v_{(1,i)} J$ such that $\Psi(v_{(2,i)} r) = v_{(1,i)} r$ for any $r \in J$ is well defined and belongs to $\Delta$, so that $\Psi = d$ for some homogeneous element $d \in \Delta$. Then $(v_{(1,i)} - d v_{(2,i)}) J = 0$, so by induction hypothesis, $v_{(1,i)} - d v_{(2,i)} \in \Delta v_{(3,i)} + \cdots + \Delta v_{(n,i)}$, a contradiction. □

**Corollary 3.6.** *(DENSITY THEOREM)*
*Let $R = \bigoplus_{i=0}^{p-1} R_i$ be a primitive $\mathbb{Z}_p$-ring, let $M = \bigoplus_{i=0}^{p-1} M_i$ be a faithful irreducible $\mathbb{Z}_p$-module over $R$, let $\Delta = \mathrm{End}_R M$ (action of $\Delta$ on the left). Then for any $\Delta$-linearly independent elements $v_{(1,i)}, v_{(2,i)}, \cdots, v_{(n,i)} \in M_i$ and for any elements*

$$w_{(1,k)}, w_{(2,k)}, \cdots, w_{(n,k)} \in M_k$$

*there is an element $r_{(p-1)i+k} \in R_{(p-1)i+k}$ such that $v_{(q,i)} r_{(p-1)i+k} = w_{(q,k)}$ for all $q = 1, 2, ..., n$.*

A (right) $\mathbb{Z}_p$-ideal is a (right) $\mathbb{Z}_p$-submodule of the $\mathbb{Z}_p$-ring $R$ considered as a (right) $\mathbb{Z}_p$-submodule over $R$. An associative $\mathbb{Z}_p$-ring is (right) *Artinian* if it satisfies the descending condition on right $\mathbb{Z}_p$-ideals. Now, we have the following result about Artinian simple associative $\mathbb{Z}_p$-rings.

**Theorem 3.7.** *If $\mathcal{A} = \bigoplus_{i=0}^{p-1} \mathcal{A}_i$ is an artinian simple associative $\mathbb{Z}_p$-ring, then (as a $\mathbb{Z}_p$-ring) $\mathcal{A} \cong \mathrm{End}_{\mathcal{D}} V$, where $V$ is a finite dimensional $\mathbb{Z}_p$-space over an associative $\mathbb{Z}_p$-algebra $\mathcal{D}$.*

**Proof:** Let $I = \bigoplus_{i=0}^{p-1} I_i$ be a nonzero minimal right $\mathbb{Z}_p$-ideal of $\mathcal{A}$. By minimality, $I$ is an irreducible $\mathbb{Z}_p$-module of $\mathcal{A}$, that is $I\mathcal{A} = I$. Since $\mathcal{A}$ is simple, $I$ is a faithful $\mathbb{Z}_p$-module over $\mathcal{A}$. Therefore $\mathcal{A}$ is a primitive $\mathbb{Z}_p$-ring with faithful irreducible $\mathbb{Z}_p$-module $M = I$. $M$ is a left $\mathbb{Z}_p$-module over $\mathcal{D} = \mathcal{C}^{op} \cong \mathrm{End}_{\mathcal{A}} M$, where $\mathcal{C}$ is the commuting $\mathbb{Z}_p$-ring of $\mathcal{A}$ on $M$. Thus by DENSITY THEOREM, $\mathcal{A}$ is isomorphic to a dense $\mathbb{Z}_p$-subring of the $\mathbb{Z}_p$-ring $\mathrm{End}_{\mathcal{D}} M$. If $M$ is infinite dimensional over $\mathcal{D}_0$, then so must $M_i$ for at least one $i \in \mathbb{Z}_p$. Let $v_{(1,i)}, v_{(2,i)}, ..., v_{(n,i)}, ...$ be an infinite sequence of linearly independent elements of $M_i$. For $V_j = \sum_{k=1}^{j} \mathcal{D} v_{(k,i)}$ the annihilators $\mathrm{Ann} V_j = \bigoplus_{k=0}^{p-1} \mathrm{Ann}_k V_j$, where $\mathrm{Ann}_k V_j = \{b_k \in \mathcal{A}_k \ : \ V_j b_k = 0\}$, form a properly infinite descending chain of right $\mathbb{Z}_p$-ideals of $\mathcal{A}$, a contradiction. Therefore $\dim_{\mathcal{D}_0} M$ is finite, say $n$, and, by density theorem, $\mathcal{A} \cong \mathrm{End}_{\mathcal{D}} M = \bigoplus_{i=0}^{p-1} \mathrm{End}_i M$. $\square$

So by Theorem 3.7, if $\mathcal{A}$ is an artinian simple associative $\mathbb{Z}_p$-ring, then (as a ring) $\mathcal{A} \cong M_n(\mathcal{D})$, where $\mathcal{D}$ is an associative division $\mathbb{Z}_p$-algebra.

In the next theorem we show that any two faithful irreducible (right) $\mathbb{Z}_p$-modules over a primitive $\mathbb{Z}_p$-ring $R$ are isomorphic.

**Theorem 3.8.** *Let $R$ be a primitive $\mathbb{Z}_p$-ring having a minimal right $\mathbb{Z}_p$-ideal. Then any two faithful irreducible (right) $\mathbb{Z}_p$-modules over $R$ are isomorphic.*

**Proof:** Let $I$ be a minimal right $\mathbb{Z}_p$-ideal of $R$ and let $M$ be a faithful irreducible $\mathbb{Z}_p$-module over $R$, then the faithfulness of $M$ ensures that $m_i I \neq \{0\}$ for some $m_i \in M_i$. Since $m_i I$ is a nonzero $\mathbb{Z}_p$-submodule of the irreducible $\mathbb{Z}_p$-module $M$, it must be all of $M$. Since the annihilator of $m_i$ in $I$ is a right $\mathbb{Z}_p$-ideal of $R$ properly contained in $I$, it is $\{0\}$ and the map $b \mapsto m_i b$, $b \in I$, is a $\mathbb{Z}_p$-isomorphism over $R$ of $I$ onto $M$. Thus every faithful irreducible $\mathbb{Z}_p$-module over $R$ is isomorphic to $I$. $\square$

Note that by [15, Theorem 4], if $\mathcal{A} \cong M_n(\mathcal{D})$, then $n$ and $\mathcal{D}$ are unique up to isomorphism.

We say that a $\mathbb{Z}_p$-ring $R$ is *semiprime* if it has no nonzero nilpotent $\mathbb{Z}_p$-ideals, and that it is *prime* if for any nonzero $\mathbb{Z}_p$-ideals $I$, $J$, the product $IJ \neq \{0\}$. Easy computations imply the following lemma.

**Lemma 3.9.** *Let $R$ be a $\mathbb{Z}_p$-ring, and let $i, j \in \mathbb{Z}_p$. Then*
*(1) If $R$ is a primitive, then $R$ is a prime.*
*(2) If $R$ is a prime with a minimal one sided $\mathbb{Z}_p$-ideal, then it is a primitive.*
*(3) $R$ is semiprime if and only if $a_iRa_i \neq \{0\}$ for all $0 \neq a_i \in R_i$.*
*(4) $R$ is prime if and only if $a_iRb_j \neq \{0\}$ for all $0 \neq a_i \in R_i$, $0 \neq b_j \in R_j$.*

Now, we have the following result about primitive $\mathbb{Z}_p$-rings with a minimal one sided $\mathbb{Z}_p$-ideal.

**Theorem 3.10.** *Let $R$ be a semiprime $\mathbb{Z}_p$-ring. Then*
*(i) If $I$ is a minimal right $\mathbb{Z}_p$-ideal of $R$, then there is an idempotent $e \in I_0$ such that $I = eR$. Moreover, for any homogeneous element $x \in I$ with $xI \neq \{0\}$, there exists an idempotent $e = e^2 \in I_0$ such that $I = eR$ and $ex = xe = x$.*
*(ii) If $e$ is a nonzero idempotent element of $R_0$ and $eR = I$ is a minimal right $\mathbb{Z}_p$-ideal of $R$, then $eRe$ is a division $\mathbb{Z}_p$-ring, which is isomorphic to the $\mathbb{Z}_p$-algebra $\Delta = \mathrm{End}_R I$ (acting from the left).*
*(iii) If $e$ is a nonzero idempotent element of $R_0$ such that $eRe$ is a division $\mathbb{Z}_p$-ring, then $eR$ is a minimal right $\mathbb{Z}_p$-ideal of $R$.*
*(iv) If $a$ is a homogeneous element in $R$ such that $aR$ is a minimal right $\mathbb{Z}_p$-ideal of $R$, then $Ra$ is a minimal left $\mathbb{Z}_p$-ideal of $R$.*

**Proof:** (i) Since $R$ is semiprime $I^2 \neq \{0\}$, so by minimality $I^2 = I$ and there is a nonzero homogeneous element $x$ in $I$ such that $xI \neq \{0\}$. Again, since $I$ is minimal $xI = I$, and hence there is an element $e \in I_0$ such that $x = xe$. Take $J = \{a \in I \ : \ xa = 0\}$. Then $J$ is a right $\mathbb{Z}_p$-ideal of $R$ strictly contained in $I$, so $J = \{0\}$. Since $e^2 - e \in J$, we conclude that $e$ is a nonzero idempotent of $I_0$, and since $0 \neq eI \subseteq I$, the minimality of $I$ forces $I = eI$, as desired. In particular $ey = y$ for any $y \in I$. That is $I = eR$.
(ii) Note that if $x$ is an homogeneous element of $R$ such that $exe \neq 0$, then $0 \neq exeR \subseteq eR$ so $exeR = eR$ by minimality. Therefore there exists an homogeneous element $y \in R$ such that $exey = e$, so $(exe)(eye) = e$, which is the unity of the $\mathbb{Z}_p$-algebra $eRe$. Therefore, any nonzero homogeneous element of $eRe$ has a right inverse. This is enough to insure that $eRe$ is a division $\mathbb{Z}_p$-algebra. Besides, the linear map $eRe \to \mathrm{End}_R(I)$ given by $exe \mapsto \lambda_{exe}$, where $\lambda_{exe} : I \to I$ is defined by $\lambda_{exe}(z) = exez$ for any $z \in I$, is easily shown to be a $\mathbb{Z}_p$-isomorphism, since for any homogeneous element $f$ in $\mathrm{End}_R(I)$, $ea = f(e) = f(e^2) = eae$ for some homogeneous element $a \in R$, and so for any $z = ez \in I$, $f(z) = f(ez) = f(e)z = eaez = \lambda_{eae}(z)$. Note that this is valid even if $R$ is not semiprime.
(iii) Suppose $e = e^2 \in R_0$ such that $eRe$ is a division $\mathbb{Z}_p$-algebra, and let $I$ be a nonzero right $\mathbb{Z}_p$-ideal contained in $eR$. Let $x$ be a nonzero homogeneous element of $I$, so $x = ex$. Let $y \in R$ such that $exye \neq 0$ (note that if $exRe = \{0\}$, then $(RexR)^2 = \{0\}$, contradicting the semiprimeness of $R$), since $eRe$ is a division

$\mathbb{Z}_p$-algebra, there is another homogeneous element $z$ such that $xyeze = exyeze = (exye)(eze) = e$. In particular, $e \in xR$ and $eR \subseteq xR \subseteq I$, so $I = eR$. This shows that $eR$ is a minimal right $\mathbb{Z}_p$-ideal.
(iv) Suppose that $a$ is an homogeneous element such that $aR$ is a minimal right $\mathbb{Z}_p$-ideal. As in (i), let $e$ be an idempotent such that $ae = ea = a$ and $aR = eR$. By (ii) $eRe$ is a division $\mathbb{Z}_p$-algebra, and by symmetry, item (iii) shows that $Re$ is a minimal left $\mathbb{Z}_p$-ideal. But $R^2a \neq \{0\}$, and $Ra = (Re)a$ is a homomorphic image of the irreducible left $\mathbb{Z}_p$-module $Re$, so it is irreducible too. That is, $Ra$ is a minimal left $\mathbb{Z}_p$-ideal. □

**Corollary 3.11.** *Let $R$ be a semiprime $\mathbb{Z}_p$-ring, and let $a$ be a homogeneous element in $R$. Then $aR$ is a minimal right $\mathbb{Z}_p$-ideal of $R$ if and only if $Ra$ is a minimal left $\mathbb{Z}_p$-ideal of $R$.*

A $\mathbb{Z}_p$-involution of a central simple associative $\mathbb{Z}_p$-algebra $\mathcal{A}$ is a graded additive map $* : \mathcal{A} \to \mathcal{A}$ such that

$$a^{**} = a \quad \text{and} \quad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta} b_\beta^* a_\alpha^* \quad \forall a_\alpha \in \mathcal{A}_\alpha,\ b_\beta \in \mathcal{A}_\beta.$$

In [5] we classified the properties of $\mathbb{Z}_p$-involution defined on primitive $\mathbb{Z}_p$-algebras having a minimal one sided $\mathbb{Z}_p$-ideal.

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