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On Some Variant of a Whittaker Integral Operator and its Representative in a Class of Square Integrable Boehmians

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ABSTRACT: This paper investigates some variant of Whittaker integral operators on a class of square integrable Boehmians. We define convolution products and derive the convolution theorem which substantially satisfy the axioms necessary for generating the Whittaker spaces of Boehmians. Relied on this analysis, we give a definition and properties of the Whittaker integral operator in the class of square integrable Boehmians. The extended Whittaker integral operator, is well-defined, linear and coincides with the classical integral in certain properties.

Key Words: Whittaker integral operator, Whittaker function, Laplace transform, Mellin transform, Hypergeometric function, Boehmian space.

Contents

1. Introduction

The Whittaker functions $\boldsymbol{m}_{k,\mu}$ and $\boldsymbol{w}_{k,\mu}$ (of first and second order, respectively) have acquired an increasing significance due to their frequent use in applications of mathematics and some physical and technical problems. They are closely related to the confluent hypergeometric function which fairly play an important role in various branches of applied mathematics and theoretical physics; this is the case in fluid mathematics, electromagnetic diffraction theory and atomic structure theory, which indeed justifies a continuous effort in studying properties of these functions, as well as those integral operators generated by them.

Boehmians are a motivation of regular operators [18] and contain all distributions and some objects which are neither operators nor distributions. An abstract construction of Boehmian spaces with two notions of convergence is given by [17]. Various integral transforms for various Boehmians spaces are defined in the recent past and their properties are developed. In this article , we define a Whittaker integral operator in a class of Boehmians and study some operational properties. We further in section define convolution products and recall some auxiliary results from literature. Throughout Section 2 we derive requested axioms for generating

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the Boehmian spaces. In Section 3, we prove the convolution theorem and give definition and properties of the generalized integral.

The Whittaker integral operator with a kernel involving confluent hypergeometric functions is a generalization of the classical Laplace transform

$$\boldsymbol{l}(p) = \int_0^\infty e^{-px} f(x)\,\mathtt{d}x \tag{1.1}$$

given by the integral equation [26]

$$\phi_{k,\mu}^{v}(p) = \int_0^\infty (px)^v e^{-1/2px} \boldsymbol{w}_{k,\mu}(px) f(x)\,\mathtt{d}x, \tag{1.2}$$

where the kernel function is expressed in a Mellin type representation,

$$\frac{\Gamma\left(v+z-\mu+\frac{1}{2}\right)\Gamma\left(v+z+\mu+\frac{1}{2}\right)}{\Gamma(v+z-k+1)}, \tag{1.3}$$

where $z = \sigma + it$.

A generalization, varying from those given in [1-4], which generalizes (1.2) and the Laplace integral, for $v = \mu, k + \mu = \frac{1}{2}$ and $r = q = 1$ is, due to Srivastava [27], given as

$$\phi_{k,\mu}^{v}(p) = \int_0^\infty (px)^{v-\frac{1}{2}} \exp\left(-\frac{1}{2}qpx\right) \boldsymbol{w}_{k,\mu}(rpx) f(x)\,\mathtt{d}x. \tag{1.4}$$

An inversion formula can be recovered from (1.2) as follows.
Theorem 1.1 (Inversion Theorem) Let $p > 0, 1 + v > \max\left(|\mu|, k - \frac{1}{2}\right), \mu + k > \frac{1}{2}, f(x) \in \boldsymbol{l}^2(0,\infty),\ x^{v+\mu+\frac{1}{2}} f(x) \in \boldsymbol{l}^2(0,\infty)$, then an inversion formula of the Whittaker integral (1.2) is given as

$$f(x) = x^{-\left(v+\mu+\frac{1}{2}\right)} \boldsymbol{l}^{-1}\left(p^{-\left(v+\mu+\frac{1}{2}\right)} R - \left(v+\mu-\frac{1}{2}, v-k+1 : 1\right) \phi_{k,\mu}^{v}(p)\right), \tag{1.5}$$

where $\boldsymbol{l}^{-1}$ is the inverse Laplace transform and R is the fractional integration operator

$$R(\alpha,\beta:1) f(x) = \frac{x^\beta}{\Gamma(\alpha)} \int_x^\infty (v+x)^{\alpha-1} v^{-\beta-1} f(v)\,\mathtt{d}v, \tag{1.6}$$

whereas, the series representation of a hypergeometric function $\phi(b;c;z)$ is given as

$$\phi(b;c:z) = \sum_{n=0}^\infty \frac{ß(b+n, c-b)}{ß(b, c-b)} \frac{z^n}{n!},$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, ß is the classical beta function.
The product we recall here is given by the following definition.

Definition 1.2 The Mellin type convolution product of two integrable functions f and g is defined by [15]

$$(f * g)(y) = \int_0^\infty x^{-1} f\left(yx^{-1}\right) g(x)\, \mathtt{d}x \tag{1.7}$$

whose properties that we recite here are as follows

(i) $g_1 * g_2 = g_2 * g_1$; (ii) $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$; (iii) $(\alpha g_1) * g_2 = \alpha (g_1 * g_2)$ and that

$$(iv) \;\; g_1 * (g_1 + g_3) = g_1 * g_2 + g_1 * g_3.$$

The major product we request here can be introduced as follows.

Definition 1.3 Let f and g be integrable functions defined on $(0, \infty)$; then for f and g we define a product given by

$$(f \otimes g)(y) = \int_0^\infty f(yx)\, g(x)\, \mathtt{d}x, \tag{1.8}$$

provided the integral is finite. By $\boldsymbol{l}^2(0, \infty)$ we denote the space of square Lebesgue integrable functions defined on $(0, \infty)$.

Lemma 1.4 (Fox's Lemma)[28] Let the following hold.

(i) $x > 0$;

(ii) $f(z), g(z) \in \boldsymbol{l}^2(0, \infty)$;

(iii) $\boldsymbol{\mu}(f(x)), \boldsymbol{\mu}(g(x))$; are the Mellin transforms of f and g, respectively.

(iv) $\boldsymbol{\mu}(g(x))$ is bounded on the line $s = \frac{1}{2} + it, t \in (0, \infty)$.

Then, we have

$$\text{(i)} \int_0^\infty g(xz)\, f(z)\, \mathtt{d}z \in \boldsymbol{l}^2(0, \infty) \tag{1.9}$$

$$\text{(ii)}\, \boldsymbol{\mu}\left(\int_0^\infty g(xz)\, f(z)\, \mathtt{d}z\right) = \boldsymbol{\mu}(g(x))(s)\, \boldsymbol{\mu}(f(x))(1-s). \tag{1.10}$$

Following lemma is justified by the Fox's lemma.

Lemma 1.5 [27, p.310] Let $p > 0, v > |\mu| - 1, f(x) \in \boldsymbol{l}^2(0, \infty)$; then we have

$$\int_0^\infty (px)^v e^{-\frac{1}{2}px} \boldsymbol{w}_{k,\mu}(px)\, f(x)\, \mathtt{d}x \in \boldsymbol{l}^2(0, \infty)$$

and

$$\boldsymbol{\mu}\left(\int_0^\infty (px)^v e^{-\frac{1}{2}px} \boldsymbol{w}_{k,\mu}(px)\, f(x)\, \mathtt{d}x\right) = \frac{\Gamma\left(v + s + \mu + \frac{1}{2}\right) \Gamma\left(v + s - \mu + \frac{1}{2}\right)}{\Gamma(v + s - k + 1)} F(1-s).$$

More information on Whittaker integral operators are given by [1, 4, 5, 27] are references cited therein.

2. Boehmians

Boehmians were introduced by Mikusinski and Mikusinski [8] as quotients of sequences to generalize functions and distributions . From the remarkable work on the convergence of Boehmians , a lot of works on Boehmians and integral transforms have been carried out by many researchers with different perspectives such as [6, 9−25]. In this article, we extend the Whittaker transform (1.2) to a space of square-integrable Boehmians, which is properly larger than the space $\boldsymbol{l}^2(0,\infty)$ of square-integrable functions defined on $\mathbb{R}$. Then we investigate some properties of the extended transform.

Now we predicate the spaces of extension.

Let $\boldsymbol{k}(0,\infty)$ denote the space of test functions of compact support over $(0,\infty)$, and Δ be the subset of $\boldsymbol{k}(0,\infty)$ of sequences satisfying

$$\text{(i)} \int_0^\infty (\delta_n)(x)\,\mathtt{d}x = 1, \tag{2.1}$$

$$\text{(ii)}\, |\delta_n(x)| < m^*, m^* \in \mathbb{R}, m^* > 0, \tag{2.2}$$

$$\text{(iii)}\, \operatorname{supp}\delta_n \subseteq (a_n, b_n), a_n, b_n \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Each (δ_n) in Δ is called delta sequence or an approximating identity to corresponds with the delta distribution.

Theorem 2.1 Let $f \in \boldsymbol{l}^2(0,\infty)$ and $g \in \boldsymbol{k}(0,\infty)$ then $f \otimes g \in \boldsymbol{l}^2(0,\infty)$.

Proof of this theorem is an immediate result of Equation 9 of Fox's lemma .

Lemma 2.2 Let $f_1, f_2 \in \boldsymbol{l}^2(0,\infty)$ and $g_1, g \in \boldsymbol{k}(0,\infty)$; then the following hold.

$(i)\ ((f_1+f_2)\otimes g)(x) = (f_1 \otimes g)(x) + (f_2 \otimes g)(x)$.

$(ii)\ (f_1 \otimes (g_1 \times g))(y) = ((f_1 \otimes g_1)\otimes g)(y)$.

Proof of the identity (i) follows from simple integration. To prove the second identity, we start from Definitions 1.2 and 1.3 to reach

$$(f_1 \otimes (g_1 * g))(y) = \int_0^\infty f_1(yx) \int_0^\infty t^{-1} g_1\left(xt^{-1}\right) g(t)\,\mathtt{d}t\mathtt{d}x. \tag{2.4}$$

Fubini's theorem and change of variables $xt^{-1} = z$ puts (2.4) into the form

$$(f_1 \otimes (g_1 * g))(y) = \int_0^\infty g(t) \int_0^\infty f_1(yzt)\, g_1(z)\,\mathtt{d}z\mathtt{d}t. \tag{2.5}$$

Hence, (2.5) is reduced to give the integral equation

$$(f_1 \otimes (g_1 * g))(y) = \int_0^\infty g(t)\,(f_1 \otimes g_1)(yt)\,\mathtt{d}t.$$

Therefore, the theorem is completely proved.

The proof of the following theorem is straightforward from simple integration.

Theorem 2.3 Let $(f_n), f \in \boldsymbol{l}^2(0,\infty)$ be such that $f_n \to f$ as $n \to \infty$, and $g \in \boldsymbol{k}(0,\infty)$; then $f_n \otimes g \to f \otimes g$ as $n \to \infty$.

Theorem 2.4 (i) Let $f \in \boldsymbol{l}^2(0,\infty)$ and $(\delta_n) \in \Delta$; then $f \otimes \delta_n \to f$ as $n \to \infty$.
(ii) Let $\alpha \in \mathbb{C}$; then $\alpha(f \otimes g) = (\alpha f) \otimes g$, for $f \in \boldsymbol{l}^2(0,\infty)$ and $g \in \boldsymbol{k}(0,\infty)$.
Proof Under the assumption that $f \in \boldsymbol{l}^2(0,\infty)$ and that $(\delta_n) \in \Delta$, it follows, by (1.8), that

$$\begin{aligned} \|(f \otimes g - f)(y)\|_{\boldsymbol{l}^2}^2 &= \int_0^\infty |(f \otimes g - f)(y)|^2 \,\mathtt{d}t \\ &= \int_0^\infty \left| \int_0^\infty f(yx)\,\delta_n(x)\,\mathtt{d}x - f(y) \right|^2 \mathtt{d}t. \end{aligned} \tag{2.6}$$

Hence, by (2.1) and Jensen's inequality, the integral equation (2.6) can be read as

$$\|(f \otimes g - f)(y)\|_{\boldsymbol{l}^2}^2 \le \int_0^\infty \int_0^\infty |f(yx) - f(y)|^2 \,|\delta_n(x)|\,\mathtt{d}x\mathtt{d}t.$$

If $[a_n, b_n]$ is an interval such that $\operatorname{supp}\delta_n(x) \subseteq [a_n, b_n]$, $a_n, b_n > 0, a_n < b_n$, then, we write

$$\|(f \otimes \delta_n - f)(y)\|_{\boldsymbol{l}^2}^2 \le \int_0^\infty \int_{a_n}^{b_n} |f(yx) - f(y)|^2 \,|\delta_n(x)|\,\mathtt{d}x\mathtt{d}t.$$

By (2.2) we get

$$\|(f \otimes \delta_n - f)(y)\|_{\boldsymbol{l}^2}^2 \le M^*(a_n, b_n) \int_0^\infty |f(yx) - f(y)|^2 \,\mathtt{d}x.$$

Hence, (2.3) yields

$$\|(f \otimes \delta_n - f)(y)\|_{\boldsymbol{l}^2}^2 \to 0 \text{ as } n \to \infty.$$

Proof of the second part is straightforward from usual properties of simple integrations.
This completes the proof of the theorem.
Theorem 2.5 Let $(\delta_n), (\epsilon_n) \in \Delta$; then for every natural n, $\delta_n * \epsilon_n \in \Delta$.
Proof of this theorem can be easily inspected from (1.7). We prefer to omit the details .
Hence, the space $\beta_1 := \beta\left(\boldsymbol{l}^2, (\boldsymbol{k}, *), \otimes, \Delta\right)$ is regarded as a Boehmian space.
The sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] + \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{f_n \otimes \tau_n\} + \{g_n \otimes \epsilon_n\}}{\{\epsilon_n * \tau_n\}}\right], \lambda \left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{\lambda f_n\}}{\{\epsilon_n\}}\right],$$

λ being complex number.
The operation $\otimes$ and differentiation are defined by

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] \otimes \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{f_n \otimes g_n\}}{\{\epsilon_n * \tau_n\}}\right] \text{ and } \mathcal{D}^\alpha \left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{\mathcal{D}^\alpha f_n\}}{\{\epsilon_n\}}\right].$$

The operation $\otimes$ is extended to $\beta_1 \times \boldsymbol{k}$ as follows: If $\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] \in \beta_1$ and $\phi \in \boldsymbol{k}$, then

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] \otimes \phi = \left[\frac{\{f_n \otimes \phi\}}{\{\epsilon_n\}}\right].$$

In β_1, two types of convergence, δ and Δ-convergence, are defined as follows :
A sequence of Boehmians (β_n) in β_1 is said to be δ-convergent to a Boehmian β in β_1, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ϵ_k) such that $(\beta_n \otimes \epsilon_k), (\beta \otimes \epsilon_k) \in \boldsymbol{l}^2, \forall k, n \in \mathbb{N}$, and

$$(\beta_n \otimes \epsilon_k) \rightarrow (\beta \otimes \epsilon_k) \text{ as } n \rightarrow \infty, \text{ in } \boldsymbol{l}^2, \text{ for every } k \in \mathbb{N}.$$

The following is equivalent for the statement of δ-convergence
The sequence $\beta_n \xrightarrow{\delta} \beta$ $(n \rightarrow \infty)$ in β_1 if and only if there is $f_{n,k}, f_k \in \boldsymbol{l}^2$ and $\epsilon_k \in \Delta$ such that $\beta_n = \left[\frac{\{f_{n,k}\}}{\{\epsilon_k\}}\right], \beta = \left[\frac{\{f_k\}}{\{\epsilon_k\}}\right]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \rightarrow f_k \text{ as } n \rightarrow \infty \text{ in } \boldsymbol{l}^2.$$

A sequence of Boehmians (β_n) in β_1 is said to be Δ-convergent to a Boehmian β in β_1, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \otimes \epsilon_n \in \boldsymbol{l}^2, \forall n \in \mathbb{N}$, and $(\beta_n - \beta) \otimes \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ in $\boldsymbol{l}^2$.
Construction of the space $\beta_2 := \beta\left(\boldsymbol{l}^2, \boldsymbol{k}, *, \Delta\right)$ can be similarly checked out by the properties of $*$ given above.
The sum and multiplication by a scalar of two Boehmians in $\beta_2 := \beta\left(\boldsymbol{l}^2, \boldsymbol{k}, *, \Delta\right)$ can be defined in a natural way

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] + \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{f_n * \tau_n\} + \{g_n * \epsilon_n\}}{\{\epsilon_n * \tau_n\}}\right] \text{and } \lambda \left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{\lambda f_n\}}{\{\epsilon_n\}}\right], \quad (2.7)$$

λ being complex number. The operation $*$ and differentiation are defined by

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] * \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{f_n * g_n\}}{\{\epsilon_n * \tau_n\}}\right] \text{and } \mathcal{D}^\alpha \left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{\mathcal{D}^\alpha f_n\}}{\{\epsilon_n\}}\right].$$

The operation $*$ is extended to $\beta_2 \times \boldsymbol{k}$ by : If $\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] \in \beta_2$ and $\phi \in \boldsymbol{k}$, then we have

$$\left[\frac{\{f_n\}}{\{\epsilon_n\}}\right] * \phi = \left[\frac{\{f_n * \phi\}}{\{\epsilon_n\}}\right].$$

3. The generalized Whittaker integral operator

Before we get our transform be defined , we request the following convolution theorem to be established .

Theorem 3.1 Let $\{f_n\} \in \boldsymbol{l}^2(0,\infty)$ and $\{\delta_n\} \in \Delta$; then we have

$$(f_n * \delta_n)^v_{k,\mu}(p) = \left((f_n)^v_{k,\mu} \otimes \delta_n\right)(p). \tag{3.1}$$

Proof For $f_n \in \boldsymbol{l}^2(0,\infty)$ and $\{\delta_n\} \in \Delta$, we have

$$\begin{aligned}
(f_n * \delta_n)^v_{k,\mu}(p) &= \int_0^\infty (px)^v e^{-\frac{1}{2}px} \boldsymbol{w}_{k,\mu}(px)(f_n * \delta_n)(x)\,\mathtt{d}x \\
&= \int_0^\infty (px)^v e^{-\frac{1}{2}px} \boldsymbol{w}_{k,\mu}(px) \int_0^\infty f_n\left(xt^{-1}\right) t^{-1}\delta_n(t)\,\mathtt{d}t\mathtt{d}x \\
&= \int_0^\infty \delta_n(t) \int_0^\infty (px)^v e^{-\frac{1}{2}px} \boldsymbol{w}_{k,\mu}(px) f_n\left(xt^{-1}\right) t^{-1}\mathtt{d}x\mathtt{d}t.
\end{aligned}$$

By Fubini's theorem and change of variables we get

$$(f_n * \delta_n)^v_{k,\mu}(p) = \int_0^\infty (px)^v \int_0^\infty (ptz)^v e^{-\frac{1}{2}ptz} \boldsymbol{w}_{k,\mu}(ptz)\,\mathtt{d}z\mathtt{d}t. \tag{3.2}$$

Hence, (3.2) gives

$$(f_n * \delta_n)^v_{k,\mu}(p) = \int_0^\infty (f_n)^v_{k,\mu}(pt)\, g(t)\,\mathtt{d}t.$$

This completes the proof of the theorem.

Let $\beta = \left[\frac{\{f_n\}}{\{\delta_n\}}\right] \in \beta_1$; then in view of Theorem 3.1, we extend the Whittaker integral operator to the space β as

$$\breve{\boldsymbol{w}}^{v,ex}_{k,\mu}(\beta) := \left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)^{v,ex}_{k,\mu} = \left[\frac{(f_n)^v_{k,\mu}}{\{\delta_n\}}\right],$$

which is, indeed, a member of the space β_2.

Theorem 3.2 The operator $\breve{\boldsymbol{w}}^{v,ex}_{k,\mu}(.) : \beta_{\mathbf{1}} \to \beta_2$ is well- defined.

Proof Let $\left[\frac{\{f_n\}}{\{\delta_n\}}\right] = \left[\frac{(g_n)}{\{\varepsilon_n\}}\right] \in \beta_{\mathbf{1}}$; then by the concept of quotients of the space $\beta_{\mathbf{1}}$ we get

$$f_n * \varepsilon_m = g_m * \delta_n, m, n \in \mathbb{N}.$$

Hence,

$$(f_n * \varepsilon_m)^v_{k,\mu} = (g_m * \delta_n)^v_{k,\mu}, m, n \in \mathbb{N}.$$

Therefore, Theorem 3.1 gives

$$\left((f_n)^v_{k,\mu} \otimes \varepsilon_m\right)(p) = \left((v_m)^v_{k,\mu} \otimes \delta_n\right)(p), \forall p, m, n \in \mathbb{N}.$$

Concept of quotients and equivalent classes in β_2 imply

$$\left[\frac{\left\{(f_n)^v_{k,\mu}\right\}}{\{\delta_n\}}\right] = \left[\frac{\left\{(g_n)^v_{k,\mu}\right\}}{\{\varepsilon_n\}}\right], m, n \in \mathbb{N}.$$

That is
$$\left(\left[\frac{(f_n)}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}=\left(\left[\frac{(g_n)}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex},m,n\in\mathbb{N}.$$
This completes the proof of the theorem.

Theorem 3.3 The operator $\breve{\boldsymbol{w}}_{k,\mu}^{v,ex}(.):\beta_{\mathbf{1}}\to\beta_2$ is linear.

Proof Let $\left[\frac{\{f_n\}}{\{\delta_n\}}\right],\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\in\beta_1$ be given; then
$$\left[\frac{\{f_n\}}{\{\delta_n\}}\right]+\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]=\left[\frac{\{f_n*\varepsilon_n+g_n*\delta_n\}}{\{\delta_n*\varepsilon_n\}}\right]. \tag{3.3}$$
By using (2.3), Theorem 3.1 and linearity of Whittaker integral operators, (3.3) reveals
$$\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]+\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}=\left[\frac{\left\{(f_n)_{k,\mu}^{v}\otimes\varepsilon_n+(g_n)_{k,\mu}^{v}\otimes\delta_n\right\}}{\{\delta_n*\varepsilon_n\}}\right].$$
Hence, addition of Boehmians in β_2 leads to
$$\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]+\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}=\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}+\left(\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}.$$
Moreover, for given $\alpha^*\in\mathbb{C}$; it easy to see that
$$\left(\alpha^*\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}=\alpha^*\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}.$$
This completes the proof of the theorem.

Theorem 3.4 Let $\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\in\beta_{\mathbf{1}}$, $\left[\frac{\{f_n\}}{\{\delta_n\}}\right]=0$, then $\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}=0.$

Proof of this theorem is straightforward. Details are, therefore, omitted.

Theorem 3.5 Let $\left[\frac{\{f_n\}}{\{\delta_n\}}\right],\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\in\beta_{\mathbf{1}}$; then
$$\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]*\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}=\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex}\otimes\left(\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}$$
in the space $\beta_{\mathbf{2}}$.

Proof Let $\left[\frac{\{f_n\}}{\{\delta_n\}}\right],\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\in\beta_{\mathbf{1}}$. Then, applying $*$ to $\beta_{\mathbf{1}}$ yields
$$\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]*\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}=\left(\left[\frac{\{f_n*g_n\}}{\{\delta_n*\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}.$$

Hence, by Theorem 3.1, we write

$$\begin{aligned}\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right] * \left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex} &= \left[\frac{\left\{(f_n * g_n)_{k,\mu}^{v}\right\}}{\{\delta_n * \varepsilon_n\}}\right] \\ &= \left[\frac{\left\{(f_n)_{k,\mu}^{v}\right\}}{\{\delta_n\}}\right] \otimes \left[\frac{\left\{(g_n)_{k,\mu}^{v}\right\}}{\{\varepsilon_n\}}\right].\end{aligned}$$

That is

$$\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right] * \left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex} = \left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex} \otimes \left(\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}.$$

The theorem has been completely proved.

Theorem 3.6 The operator $\breve{\boldsymbol{w}}_{k,\mu}^{v,ex}(.)$ is compatible with the classical integral operator .

Proof Let $f \in \boldsymbol{l}^2$ and β be its representative in $\beta_{\mathbf{1}}$; then $\beta = f * \left[\frac{\{\delta_n\}}{\{\delta_n\}}\right] = \left[\frac{f * \{\delta_n\}}{\{\delta_n\}}\right]$, where $\{\delta_n\} \in \Delta, \forall n \in \mathbb{N}$. Its clear that $\{\delta_n\}$ is independent from the representative, $\forall n \in \mathbb{N}$. On the other hand,

$$\breve{\boldsymbol{w}}_{k,\mu}^{v,ex}(\beta) = \left(f * \left[\frac{\{\delta_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex} = \left[\frac{(f * \{\delta_n\})_{k,\mu}^{v}}{\{\delta_n\}}\right] = \left[\frac{(f)_{k,\mu}^{v} * \{\delta_n\}}{\{\delta_n\}}\right]$$

as the representative of the classical operator $(f)_{k,\mu}^{v}$.

Hence the proof is completed.

Theorem 3.7 $\breve{\boldsymbol{w}}_{k,\mu}^{v,ex}(.) : \beta_{\mathbf{1}} \to \beta_2$ is injective.

Proof Assume that $\left(\left[\frac{\{f_n\}}{\{\delta_n\}}\right]\right)_{k,\mu}^{v,ex} = \left(\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)_{k,\mu}^{v,ex}$ in β_2. By Theorem 3.1 we have $(f_n)_{k,\mu}^{v} \otimes \varepsilon_m = (g_n)_{k,\mu}^{v} \otimes \delta_n$. Therefore, Theorem 3.1 implies

$$(f_n * \varepsilon_m)_{k,\mu}^{v} = (g_m * \delta_n)_{k,\mu}^{v}.$$

Hence, $f_n * \varepsilon_m = g_m * \delta_n$. Therefore, the concept of equivalent classes of $\beta_{\mathbf{1}}$ suggests

$$\left[\frac{\{f_n\}}{\{\delta_n\}}\right] = \left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right].$$

This completes the proof of the theorem.

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