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On Bäcklund and Ribaucour Transformations for Hyperbolic Linear Weingarten Surfaces *

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abstract: We consider Bäcklund transformations for hyperbolic linear Weingarten surfaces in Euclidean 3-space. The composition of these transformations is obtained in the Permutability Theorem that generates a 4-parameter family of surfaces of the same type. The analytic interpretation of the geometric results is given in terms of solutions of the sine-Gordon equation. Since a Ribaucour transformation of a hyperbolic linear Weingarten surface also gives a 4-parameter family of such surfaces, one has the following natural question. Are these two methods equivalent, as it occurs with surfaces of constant positive Gaussian curvature or constant mean curvature? In this paper, we obtain necessary and sufficient conditions for the surfaces given by the two procedures to be congruent.

Key Words: Bäcklund transformations, Ribaucour transformations, Hyperbolic linear Weingarten surfaces, Sine-Gordon equation.

Contents

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1. Introduction

A surface M contained in the Euclidean space $\mathbb{R}^3$ whose mean and Gaussian curvatures, H and K, satisfy a relation of the form $\alpha + 2\beta H + \gamma K = 0$, $\alpha, \beta, \gamma \in \mathbb{R}$, is called a linear Weingarten surface. The development of the theory of these surfaces started in the early 19 hundreds. More recent results obtained by several authors can be found in [1], [8], [10], [15]-[17]. If the linear Weingarten surface satisfies $\beta^2 - \alpha\gamma < 0$, then it is said to be hyperbolic. In this case, without loss of generality, we may assume that $\alpha = 1$. Moreover, if $\beta = 0$, and $\gamma = 1$ then M is a pseudo-spherical surface, i.e., $K = -1$, and there is a well known theory on Bäcklund transformations for pseudo-spherical surfaces studied by Bäcklund [2,3] and on composition of such transformations called Permutability theorem obtained by Bianchi [4] .

In this paper, we study an extension of the concept of pseudo-spherical line congruence, called a hyperbolic linear Weingarten congruence. Namely, we consider a diffeomorphism between surfaces M and M' such that at corresponding points $p \in M$ and $p' \in M'$, the straight line determined by these points has a constant angle ϕ with the normal N_p and a constant angle ρ with the normal $N'_{p'}$. Moreover, we assume that the segment pp' has constant length r and N_p has a constant angle θ with $N'_{p'}$. Then M and M' are hyperbolic linear Weingarten surfaces satisfying, respectively, $1 + 2\beta H + \gamma K = 0$ and $1 + 2\beta' H' + \gamma' K' = 0$, where $\beta^2 - \gamma = (\beta')^2 - \gamma' < 0$. We observe that whenever $\phi = \rho = \pi/2$, then the theory coincides with the classical results for pseudo-spherical surfaces.

The Integrability Theorem shows that given such a surface M there exists a 3-parameter family of surfaces M', satisfying $1 + 2\beta' H' + \gamma' K' = 0$, associated to M by a hyperbolic linear Weingarten congruence. The surfaces M' are said to be associated to M by a Bäcklund transformation for hyperbolic linear Weingarten surfaces.

The Permutability theorem shows that the composition of such transformations is commutative when one chooses the parameters apropriately. In this case, starting with a hyperbolic linear Weingarten surface M satisfying $1 + 2\beta H + \gamma K = 0$, one gets a 4-parameter family of surfaces M^*, satisfying $1 + 2\beta H^* + \gamma K^* = 0$, with the same constants β, γ of the surface M.

Another transformation, called Ribaucour transformation, also relates linear Weingarten surfaces in space forms (see [8], [20]). Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, and for surfaces with constant Gaussian or mean curvatures (including minimal surfaces) were classically studied by Bianchi [5]. More recent applications of this transformation can be found for example in [6]-[9], [14], [20] and [21]. In [8], Corro-Ferreira-Tenenblat proved that starting with a linear Weingarten surface M in $\mathbb{R}^3$, satisfying $\alpha + 2\beta H + \gamma K = 0$, with $\beta^2 - \alpha\gamma \neq 0$, then appropriate Ribaucour transformations provide a 4-parameter family of surfaces $\tilde{M}$ satisfying $\alpha + 2\beta \tilde{H} + \gamma \tilde{K} = 0$. In particular, this holds for hyperbolic linear Weingarten surfaces.

Therefore, starting with a hyperbolic linear Weingaten surface M in $\mathbb{R}^3$, satisfying $1 + 2\beta H + \gamma K = 0$, one gets a 4-parameter family of hyperbolic linear

Weingarten surfaces with the same constants β, γ, either by the composition of Bäcklund transformations or by Ribaucour transformations. Hence, it is natural to ask if these two methods are equivalent, i.e., if the surfaces obtained by these two methods are congruent. In this paper, we will show that in general the families obtained by these procedures are distinct, in contrast with what happens in the case of constant positive Gaussian curvature (see for example Tenenblat [18]) and surfaces of nonzero constant mean curvature (Jeromin-Pedit [12]). In the particular case $K=-1$, necessary and sufficient conditions were established by Goulart-Tenenblat [11], for a composition of Bäcklund transformations to be congruent to a Ribaucour transformation.

The analytic interpretation of the above geometric results are given in terms of the sine-Gordon equation. A hyperbolic linear Weingarten surface, parametrized by lines of curvature and satisfying $1+2\beta H+\gamma K=0$ is determined by its first fundamental form $ds^2=\gamma(\cos^2\frac{\psi}{2}dx_1^2+\sin^2\frac{\psi}{2}dx_2^2)$, where ψ satisfies the sine-Gordon equation $\psi_{x_1x_1}-\psi_{x_2x_2}=\sin(\psi+C_{\beta\gamma})$ and $C_{\beta\gamma}$ is a constant defined in terms of β and γ. In particular, when $\beta=0$ and $\gamma=1$ i.e., $K=-1$, then $C_{01}=0$. Although Hilbert theorem says that there are no complete surfaces with $K=-1$ in $\mathbb{R}^3$, there are many complete hyperbolic linear Weingarten surfaces in the Euclidean 3-space (see [8], [16]).

The analytic interpretation of the Bäcklund transformation gives an integrable system of equations, in terms of ψ and 2 parameters, whose solutions ψ' give new solutions of the sine-Gordon equation. By considering ψ' and ψ'' two distinct such solutions, the analytic permutability theorem gives a superposition formula that provides an algebraic expression for new solutions ψ^*, which depend on 4 parameters. Moreover, the Ribaucour transformation gives an integrable linear system in terms of ψ and a constant C_R, whose solutions $\tilde{\psi}$ depend also in 4-parameters and satisfy the sine-Gordon equation. The solutions ψ^* and $\tilde{\psi}$ obtained by these procedures are distinct.

The paper is organized as follows: In Section 2, we introduce the hyperbolic linear Weingarten congruence and we prove Bäcklund Theorem for hyperbolic linear Weingaten surfaces, the Geometric Integrability Theorem and the Geometric Permutability Theorem. In Section 3, considering the correspondence between such surfaces and solutions of the sine-Gordon equation, we prove the Analytic Integrability Theorem and we state the Analytic Permutability Theorem, whose proof is given in the Appendix. In Section 4, we start recalling some results on Ribaucour transformation. Then we obtain necessary and sufficient conditions for the hyperbolic linear Weingarten surfaces, obtained by the composition of Bäcklund transformations, to be congruent to those obtained by Ribaucour transformations. These conditions are given in terms of the first fundamental forms i.e., in terms of the corresponding solutions of the sine-Gordon equation.

2. Bäcklund transformations for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$ - Geometric Theory

In this section, we introduce the concept of hyperbolic linear Weingarten congruence and we study a Bäcklund transformation for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$. Moreover, we also prove the integrability and the permutability theorems for these transformations.

2.1. Bäcklund Theorem for hyperbolic linear Weingarten surfaces

Definition 2.1. *We say that $M \subset R^3$ is a Weingarten surface if there exists a differentiable function relating the mean and Gaussian curvatures H and K of M. A surface M is said to be linear Weingarten if H and K satisfy a linear relation, i.e., there exist real constants α, β, γ such that $\alpha + 2\beta H + \gamma K = 0$. Moreover M is hyperbolic, if $\beta^2 - \alpha\gamma < 0$, elliptic, if $\beta^2 - \alpha\gamma > 0$ and tubular, if $\beta^2 - \alpha\gamma = 0$.*

Remark 2.2. *If M is a hiperbolic linear Weingarten surface satisfying $\alpha + 2\beta H + \gamma K = 0$ then, without loss of generality, we can assume that $\alpha = 1$ and $\gamma > 0$.*

Definition 2.3. *Let $l : M \longrightarrow M'$ be a diffeomorphism between surfaces $M, M' \subset \mathbb{R}^3$. For each $p \in M$ and $p' = l(p) \in M'$ with $p' \neq p$, denote by $v = v(p)$ the unit vector in the direction of the straight line passing through p and p'. Let N_p (resp. $N'_{p'}$) be the unit vector normal to M (resp. M') in p (resp. p'). We say that l is a hyperbolic linear Weingarten congruence with constants (r, θ, ϕ, ρ), where $r > 0$, $0 < \theta < \pi$, $0 < \phi, \rho \leq \frac{\pi}{2}$, if the distance between p and p' is constant equal to r, the angle between N_p and $N'_{p'}$ is θ, the angle between N_p and v is ϕ and the angle between $N'_{p'}$ and $(-v)$ is equal to ρ.*

Remark 2.4. *When $\phi = \rho = \pi/2$, then the direction of the line congruence is tangent to both surfaces M and M' and it reduces to the so called pseudo-spherical line congruence of surfaces in $\mathbb{R}^3$.*

The following theorem justifies the definition of a *hyperbolic linear Weingarten congruence*, for a diffeomorphism l as in Definition 2.3. Moreover, it reduces to the classical Bäcklund Theorem between pseudo-spherical surfaces when $\phi = \rho = \pi/2$.

Theorem 2.5. (Bäcklund Theorem for hyperbolic linear Weingarten surfaces) *Let M and M' be two surfaces imersed in $\mathbb{R}^3$. Suppose there exists a hyperbolic linear Weingarten congruence $l : M \longrightarrow M'$ with constant (r, θ, ϕ, ρ) as in Definition 2.3. For any $p \in M$ and $p' = l(p) \in M'$, suppose that the normal vectors N_p and $N'_{p'}$ and the vector $v = v(p)$ are not coplanar. Then M and M' are hyperbolic linear Weingarten surfaces. More precisely, the Gaussian curvature K (resp. K') and mean curvature H (resp. H') of M (resp. M') satisfy the relation $1 + 2\beta H + \gamma K = 0$ (resp. $1 + 2\beta' H' + \gamma' K' = 0$), where*

$$\beta = \frac{-r(\cos\phi + \cos\rho\cos\theta)}{\sin^2\theta}, \qquad \gamma = \frac{r^2\sin^2\rho}{\sin^2\theta}, \tag{2.1}$$

$$\beta' = \frac{-r(\cos\rho + \cos\phi\cos\theta)}{\sin^2\theta}, \qquad \gamma' = \frac{r^2\sin^2\phi}{\sin^2\theta}. \tag{2.2}$$

Moreover, we have that $(\beta')^2 - \gamma' = \beta^2 - \gamma < 0$.

Proof: Let $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ be orthonormal frames adapted to M and M', respectively such that, for every $p \in M$, $e_3(p) = N_p$, $e'_3(p') = N'_{p'}$ and the sets $\{v, e_1, e_3\}$ and $\{v, e'_1, e'_3\}$ are linearly dependent. If X is a local parametization of M in p then

$$X' = X + r\sin\phi e_1 + r\cos\phi e_3 \tag{2.3}$$

is a local parametization of M' in p'.

We consider $a_{ij},\ 1 \le i, j \le 3$ satisfying $e'_i = \sum_{j=1}^{3} a_{ij} e_j$. Since $l : M \to M'$ is a hyperbolic linear Weingarten congruence with constants (r, θ, ϕ, ρ) it follows that

$$\begin{aligned}
a_{31} &= \frac{-\cos\rho - \cos\phi\cos\theta}{\sin\phi}, & a_{32} &= -\sqrt{\sin^2\theta - a_{31}^2},\\
a_{33} &= \cos\theta, & a_{11} &= \frac{-\sin\phi - a_{31}\cos\rho}{\sin\rho},\\
a_{12} &= \frac{-a_{32}\cos\rho}{\sin\rho}, & a_{13} &= \frac{-\cos\phi - \cos\theta\cos\rho}{\sin\rho},\\
a_{21} &= \frac{-a_{32}\cos\phi}{\sin\rho}, & a_{22} &= \frac{a_{31}\cos\phi - \cos\theta\sin\phi}{\sin\rho},\\
a_{23} &= \frac{a_{32}\sin\phi}{\sin\rho}.
\end{aligned} \tag{2.4}$$

Let $\omega_1, \omega_2, \omega_{12}, \omega_{13}, \omega_{23}$ (resp. $\omega'_1, \omega'_2, \omega'_{12}, \omega'_{13}, \omega'_{23}$) be the dual and the conection forms associated to the orthonormal frame $\{e_1, e_2, e_3\}$ (resp. $\{e'_1, e'_2, e'_3\}$). Differentiating (2.3) and using the structure equations, we have

$$\begin{cases}
a_{11}\omega'_1 + a_{21}\omega'_2 = \omega_1 - r\cos\phi\omega_{13},\\
a_{12}\omega'_1 + a_{22}\omega'_2 = \omega_2 + r\sin\phi\omega_{12} - r\cos\phi\omega_{23},\\
a_{13}\omega'_1 + a_{23}\omega'_2 = r\sin\phi\omega_{13}.
\end{cases} \tag{2.5}$$

Since the vectors e'_3, e_3, v are not coplanar then $a_{32} \neq 0$. Using (2.4), we obtain

$$\omega'_1 = \left(\frac{-a_{23}}{a_{32}}\right)\omega_1, \qquad \omega'_2 = \frac{1}{a_{32}}\{a_{13}\omega_1 + r\sin\rho\omega_{13}\}.$$

Therefore, it follows from the second equation of (2.5) that

$$\omega_{12} = c_1\omega_1 + c_2\omega_2 + c_3\omega_{13} + c_4\omega_{23}, \tag{2.6}$$

where

$$\begin{aligned}
c_1 &= \left(\frac{-a_{31}}{r a_{32}\sin\phi}\right), & c_2 &= \left(\frac{-1}{r\sin\phi}\right),\\
c_3 &= \left(\frac{a_{22}\sin\rho}{a_{32}\sin\phi}\right), & c_4 &= \left(\frac{\cos\phi}{\sin\phi}\right).
\end{aligned} \tag{2.7}$$

Differentiating (2.6) it follows from the structure equations, the definition of mean and Gaussian curvatures and the Gauss equation that

$$d\omega_{12} = [(c_1^2 + c_2^2) + 2(c_1c_3 + c_2c_4)H + (c_3^2 + c_4^2)K](\omega_1 \wedge \omega_2).$$

On the other hand, we kwon that $d\omega_{12} = -K(\omega_1 \wedge \omega_2)$. Therefore, the mean and Gaussian curvature of M satisfy

$$(c_1^2 + c_2^2) + 2(c_1c_3 + c_2c_4)H + (c_3^2 + c_4^2 + 1)K = 0. \tag{2.8}$$

The constants c_1, c_2, c_3, c_4 defined in (2.7) imply that

$$\begin{aligned} c_1^2 + c_2^2 &= \frac{\sin^2\theta}{r^2 a_{32}^2 \sin^2\phi}, \quad c_3^2 + c_4^2 + 1 = \frac{r^2\sin^2\rho}{r^2 a_{32}^2\sin^2\phi} \\ c_1c_3 + c_2c_4 &= \frac{-r(\cos\phi + \cos\rho\cos\theta)}{r^2 a_{32}^2 \sin^2\phi}. \end{aligned} \tag{2.9}$$

In other words, M is a linear Weingarten surface satisfying $1 + 2\beta H + \gamma K = 0$, where β and γ are given by (2.1). Interchanging ϕ and ρ in the previous computations, we obtain that M' is also a linear Weingarten surface satisfying $1 + 2\beta' H' + \gamma' K' = 0$, where β' and γ' are given by (2.2). Moreover, using the constants a_{31} and a_{32} defined by (2.4), we have

$$(\beta')^2 - \gamma' = \beta^2 - \gamma = -\frac{r^2\sin^2\phi a_{32}^2}{\sin^4\theta} < 0. \tag{2.10}$$

Hence, M and M' are hyperbolic linear Weingarten surfaces. □

Remark 2.6. *The Equation (2.6) is called Bäcklund transformation for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$ and it is denoted by $BT(r, \theta, \phi, \rho)$, where r, θ, ϕ, ρ are the constants introduced in Definition 2.3.*

We conclude this section by establishing some notation and some identities that will be used throughout this paper.

Remark 2.7. *Given real numbers β, γ such that $\gamma - \beta^2 > 0$, we consider constants $r > 0$, $0 < \theta < \pi$ and $0 < \phi, \rho \leq \frac{\pi}{2}$ satisfying (2.1).We denote by b_1, b_2, b_3 the real constants*

$$\begin{aligned} b_1 &= \frac{-(\cos\rho + \cos\phi\cos\theta)}{\sin\phi}, \qquad b_2 = -\sqrt{\sin^2\theta - b_1^2}, \\ b_3 &= \frac{b_1\cos\phi - \sin\phi\cos\theta}{\sin\rho}. \end{aligned} \tag{2.11}$$

We observe that the hypothesis $\gamma - \beta^2 > 0$ ensures that

$$\sin^2\theta - b_1^2 = \sin^2\theta - \frac{(\cos\rho + \cos\phi\cos\theta)^2}{\sin^2\phi} > 0. \tag{2.12}$$

Thus, the constants c_1, c_2, c_3, c_4 given by (2.7) are written in the form

$$c_1 = \frac{-b_1}{rb_2\sin\phi}, \qquad c_2 = \frac{-1}{r\sin\phi}, \qquad c_3 = \frac{b_3\sin\rho}{b_2\sin\phi}, \qquad c_4 = \frac{\cos\phi}{\sin\phi}. \tag{2.13}$$

Analogous to (2.9), we have that

$$\begin{aligned} c_1^2 + c_2^2 &= \frac{\sin^2\theta}{r^2\sin^2\phi b_2^2}, \qquad c_3^2 + c_4^2 = \frac{r^2\sin^2\rho}{r^2\sin^2\phi b_2^2} - 1 \\ c_1c_3 + c_2c_4 &= \frac{-r(\cos\phi + \cos\rho\cos\theta)}{r^2\sin^2\phi b_2^2}, \end{aligned} \tag{2.14}$$

We can also prove that

$$\begin{aligned} -c_1^2 + c_2^2 &= \frac{\sin^2\theta - 2b_1^2}{r^2\sin^2\phi b_2^2}, \\ -c_1c_3 + c_2c_4 &= \frac{(2b_1^2 - \sin^2\theta)\cos\phi - b_1\sin\phi\cos\theta}{r\sin^2\phi b_2^2}, \\ -c_3^2 + c_4^2 &= \frac{(\sin^2\theta - 2b_1^2)cos^2\phi + 2b_1\sin\phi\cos\phi\cos\theta - cos^2\theta\sin^2\phi}{\sin^2\phi b_2^2}, \\ -c_1c_2 &= \frac{-b_1}{r^2\sin^2\phi b_2}, \\ c_1c_4 + c_2c_3 &= \frac{-2b_1\cos\phi + \sin\phi\cos\theta}{r\sin^2\phi b_2}, \\ -c_3c_4 &= \frac{-b_1cos^2\phi + \sin\phi\cos\phi\cos\theta}{\sin^2\phi b_2}. \end{aligned} \tag{2.15}$$

and

$$(\beta')^2 - \gamma' = \beta^2 - \gamma = -\frac{r^2\sin^2\phi b_2^2}{\sin^4\theta}, \tag{2.16}$$

$$\gamma + 2\beta r\cos\phi + r^2\cos^2\phi = \frac{r^2\sin^2\phi(1 - b_1^2)}{\sin^2\theta}, \tag{2.17}$$

$$\beta + r\cos\phi = \frac{r\cos\theta\sin\phi b_1}{\sin^2\theta}. \tag{2.18}$$

where β', γ' and b_1, b_2 are given by (2.2) and (2.11), respectively.

2.2. The Geometric Integrability Theorem

The Geometric Integrability Theorem, that we prove below, shows that given a hyperbolic linear Weingarten surface M satisfying (2.6) there exists a family of surfaces M' associated to M by a hyperbolic linear Weingarten congruence.

Theorem 2.8. (Geometric Integrability Theorem) *Let $M \subset \mathbb{R}^3$ be a hyperbolic linear Weingarten surface with Gaussian curvature K and mean curvature H satisfying $1 + 2\beta H + \gamma K = 0$. We consider real numbers $r > 0$, $0 < \theta < \pi$ and $0 < \phi, \rho \leq \frac{\pi}{2}$ satisfying (2.1). Let $p_0 \in M$ and let $v_0 \in \mathbb{R}^3$ be a unit vector whose angle with N_{p_0} (normal to M at p_0) is ϕ. Suppose that v_0^T, the tangential component of v_0, is not a principal direction. Then there exists a linear Weingarten* *surface $M' \subset \mathbb{R}^3$ with Gaussian curvature K' and mean curvature H', satisfying $1+2\beta' H' + \gamma' K' = 0$, where β', γ' satisfy (2.2) and a hyperbolic linear Weingarten congruence l with constants (r, θ, ϕ, ρ) between neighborhoods of p_0 in M and $l(p_0)$ in M', such that the straight line connecting p_0 to $l(p_0)$ is in the direction of v_0.*

Proof: Since M is a hyperbolic linear Weingarten surface satisfying $1 + 2\beta H + \gamma K = 0$ then, taking real numbers $r > 0$, $0 < \theta < \pi$ e $0 < \phi, \rho \leq \frac{\pi}{2}$ such that (2.1) is verified, we have

$$\sin^2\theta - \left(\frac{\cos\rho + \cos\phi\cos\theta}{\sin\phi}\right)^2 = \frac{\sin^4\theta}{r^2\sin^2\phi}(\gamma - \beta^2) > 0. \tag{2.19}$$

Thus, we can consider the real constants b_1, b_2, b_3 and $c_1,\ c_2,\ c_3,\ c_4$ defined by (2.11) and (2.13), respectively. The idea is to apply Frobenius theorem to construct an orthonormal frame $\{e_1, e_2, e_3\}$ adapted to M, defined in a neighborhood of p_0, whose dual and connection forms satisfy

$$\omega_{12} = c_1\omega_1 + c_2\omega_2 + c_3\omega_{13} + c_4\omega_{23}, \tag{2.20}$$

such that $e_1(p_0) = \dfrac{v_0^T}{\mid v_0^T \mid}$. Let $\Im$ be the ideal generated by the 1-form

$$\zeta = \omega_{12} - c_1\omega_1 - c_2\omega_2 - c_3\omega_{13} - c_4\omega_{23}.$$

Differentiating and using the structure equations we have

$$d\zeta \;=\; d\omega_{12} - c_1 d\omega_1 - c_2 d\omega_2 - c_3 d\omega_{13} - c_4 d\omega_{23} = -K\omega_1\wedge\omega_2 + \omega_{12}\wedge\mu,$$

where $\mu = -c_1\omega_2 + c_2\omega_1 - c_3\omega_{23} + c_4\omega_{13}$. Substituting $\omega_{12} = \zeta + c_1\omega_1 + c_2\omega_2 + c_3\omega_{13} + c_4\omega_{23}$ and using (2.14) we obtain

$$d\zeta = \zeta\wedge\mu - \frac{1}{r^2 b_2^2\sin^2\phi}\left[\sin^2\theta - 2r(\cos\phi + \cos\rho\cos\theta)H + (r^2\sin^2\rho)K\right]\omega_1\wedge\omega_2.$$

By hypothesis, the constants r, θ, ϕ, ρ satisfy (2.1) and M is a hyperbolic linear Weingarten surface such that $1 + 2\beta H + \gamma K = 0$. Thus, $d\zeta = \zeta\wedge\mu$, i.e., $\Im$ is closed under exterior differentiation. By Frobenius theorem, the equation $\zeta = 0$ is integrable. Therefore, there exists an adapted frame $\{e_1, e_2, e_3\}$ such that (2.20) holds in a neighborhood of p_0, with initial condition $e_1(p_0) = \dfrac{v_0^T}{\mid v_0^T \mid}$. Since the angle between v_0 and $N_{p_0} = e_3(p_0)$ is equal to ϕ and the unit vectors $e_3(p_0)$, $e_1(p_0)$ and v_0 are coplanar then $v_0 = \sin\phi e_1(p_0) + \cos\phi e_3(p_0)$. Define, in this neighborhood, the vector function

$$v = \sin\phi e_1 + \cos\phi e_3.$$

By hypothesis, $e_1(p_0)$ is not a principal direction hence we can assume, by continuity, that e_1 is not a principal direction on an open subset V of this neighborhood.

We consider V parametrized by $X : U \subset R^2 \longrightarrow V \subset M \subset \mathbb{R}^3$ and define $X' : U \longrightarrow \mathbb{R}^3$ by

$$X' = X + rv = X + r\sin\phi e_1 + r\cos\phi e_3.$$

Differentiating and using the structure equations, we obtain $dX' = z_1\omega_1 + z_2\omega_{13}$, where

$$z_1 = e_1 - \frac{b_1}{b_2}e_2, \qquad z_2 = -r\cos\phi e_1 + \frac{rb_3\sin\rho}{b_2}e_2 + r\sin\phi e_3.$$

Since e_1 is not a principal direction and $r\sin\phi \neq 0$ we conclude that $M' = X'(U)$ is a regular surface and z_1, z_2 are tangent to M'. Moreover, $e_3' = b_1e_1 + b_2e_2 + \cos\theta e_3$ is a unit vector normal to M'. Consequently, M' is related to $X(U)$ by a hyperbolic linear Weingarten congruence, l with constants r, θ, ϕ, ρ. Using Theorem 2.5, we conclude that M' is a hyperbolic linear Weingarten surface satisfying $1 + 2\beta' H' + \gamma' K' = 0$, where β', γ' are given by (2.2). □

Observe that Theorem 2.8 shows that given a hyperbolic linear Weingarten surface M in $\mathbb{R}^3$ there exists a 3-parameter family of surfaces M' associated to M by a hyperbolic linear Weingarten congruence. The three parameters are determined by the unit vector v_0 and the four constants (r, θ, ϕ, ρ) satisfying two conditions given by (2.1).

2.3. The Geometric Permutability Theorem

In this section, we consider the composition of Bäcklund transformations for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$. We observe that applying a Bäcklund transformation to a surface in $\mathbb{R}^3$ satisfying $1 + 2\beta H + \gamma K = 0$, we obtain new surfaces of the same type but with different constants β and γ. We will now consider a composition of such transformations so that the surface obtained by this composition has the same constants as the surface we started with. This is obtained by imposing certain conditions on the parameters and in this case, the composition is commutative.

Let M, M', M'' be hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$. Suppose that M satisfies $1 + 2\beta H + \gamma K = 0$ and that there are hyperbolic linear Weingarten congruences $l_1 : M \to M'$ and $l_2 : M \to M''$ with constants $(r_1, \theta_1, \phi_1, \rho_1)$ and $(r_2, \theta_2, \phi_2, \rho_2)$ respectively, where $r_i > 0$, $0 < \theta_i < \pi$ and $0 < \rho_i, \phi_i \leq \frac{\pi}{2}$ $(i = 1, 2)$, with $\theta_1 \neq \theta_2$. We want to construct a hyperbolic linear Weingarten surface M^*, with the same constants β, γ, and hyperbolic linear Weingarten congruences $l_2^* : M' \to M^*$ and $l_1^* : M'' \to M^*$ with constants $(r_2, \theta_2, \phi_2, \rho_2)$ and $(r_1, \theta_1, \phi_1, \rho_1)$, respectively, such that

$$l_2^* \circ l_1 = l_1^* \circ l_2.$$

The definition of the hyperbolic linear Weingarten congruences l_1 and l_2 together with the Bäcklund Theorem (Theorem 2.5) allows us to obtain the following equalities from (2.1)

$$\begin{aligned} r_2\sin\rho_2 &= \delta r_1\sin\rho_1, \\ r_2(\cos\phi_2 + \cos\rho_2\cos\theta_2) &= \delta^2 r_1(\cos\phi_1 + \cos\rho_1\cos\theta_1), \end{aligned} \tag{2.21}$$

where δ is the positive constant defined by

$$\delta = \frac{\sin\theta_2}{\sin\theta_1}. \tag{2.22}$$

Moreover, admitting the existence of the hyperbolic linear Weingarten surface M^* and of the hyperbolic linear Weingarten congruences l_1^* and l_2^*, as previously described then, using again Theorem 2.5, we conclude that necessarily the following equalities hold

$$\begin{array}{lcl} r_2\sin\phi_2 & = & \delta r_1\sin\phi_1, \\ r_2(\cos\rho_2+\cos\phi_2\cos\theta_2) & = & \delta^2 r_1(\cos\rho_1+\cos\phi_1\cos\theta_1), \end{array} \tag{2.23}$$

where δ is given by (2.22). Therefore, assuming that (2.23) is satisfied and using (2.21) we obtain

$$r_i\cos\rho_i \quad = \quad \frac{r_i\cos\phi_i(\cos\theta_1\cos\theta_2-1)+r_j\cos\phi_j\sin^2\theta_i}{\cos\theta_i-\cos\theta_j}, \quad 1\le i\ne j\le 2. \tag{2.24}$$

As a direct consequence of this equation we obtain

$$r_1\cos\rho_1\cos\theta_2 - r_2\cos\rho_2\cos\theta_1 + r_2\cos\phi_2 - r_1\cos\phi_1 = 0. \tag{2.25}$$

Let $\{e_1, e_2, e_3\}$ and $\{\overline{e}_1, \overline{e}_2, e_3\}$ be orthonormal frames adapted to M and denote by E the orthogonal 2×2 matrix (with positive determinant) such that

$$\overline{e}_i = \sum_{j=1}^{2} E_{ij}e_j, \qquad i=1,2, \qquad E_{22}=E_{11}, \qquad E_{21}=-E_{12}. \tag{2.26}$$

We consider also orthonormal frames $\{e_1', e_2', e_3'\}$ in M' and $\{e_1'', e_2'', e_3''\}$ in M'' as in Theorem 2.5, i.e., if $v_1 = v_1(p)$ (resp. $v_2 = v_2(p)$) is the direction of the straight line joining the points $p \in M$ and $p_1 = l_1(p) \in M'$ (resp. $p_2 = l_2(p) \in M'$) then the sets $\{e_1, e_3, v_1\}$ and $\{\overline{e}_1, e_3, v_2\}$ are linearly dependent. Then,

$$\left\{\begin{array}{lcl} v_1 & = & \sin\phi_1 e_1+\cos\phi_1 e_3, \\ v_2 & = & \sin\phi_2\overline{e}_1+\cos\phi_2 e_3. \end{array}\right. \tag{2.27}$$

Let a_{ij}' and a_{ij}'' $(1\le i,j\le 3)$ be such that

$$e_i' = \sum_{i=1}^{3} a_{ij}'e_j \qquad \text{and} \qquad e_i'' = \sum_{i=1}^{3} a_{ij}''\overline{e}_j, \qquad \overline{e}_3 = e_3. \tag{2.28}$$

Then using the proof of Theorem 2.5 we have that a_{ij}' (resp. a_{ij}'') are given by (2.4) taking $r=r_1$, $\theta=\theta_1$, $\phi=\phi_1$ and $\rho=\rho_1$ (resp. $r=r_2$, $\theta=\theta_2$, $\phi=\phi_2$ and $\rho=\rho_2$). Let δ be the real number given by (2.22). It follows from (2.21) and (2.23) that

$$a_{13}''=\delta a_{13}', \qquad a_{31}''=\delta a_{31}', \qquad a_{32}''=\delta a_{32}' \qquad \text{and} \qquad a_{23}''=\delta a_{23}'. \tag{2.29}$$

We consider $E = (E_{ij})$ the orthogonal matrix, with positive determinant, defined by (2.26). Since $\theta_1 \neq \theta_2$, we observe that the function

$$\xi = \sin\theta_1 \sin\theta_2 E_{11} + \cos\theta_1 \cos\theta_2 - 1 \tag{2.30}$$

is strictly negative. Furthermore defining

$$F_{11} = \frac{-(\cos\theta_1\cos\theta_2 - 1)E_{11} - \sin\theta_1\sin\theta_2}{\xi}, \quad F_{12} = \frac{(\cos\theta_1 - \cos\theta_2)E_{12}}{\xi}, \tag{2.31}$$

we have that the matrix $F = \begin{pmatrix} F_{11} & F_{12} \\ -F_{12} & F_{11} \end{pmatrix}$ is orthogonal, since $FF^t = I$.

Our next theorem analyzes the composition of Bäcklund transformations for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$ satisfying $1 + 2\beta H + \gamma K = 0$. We show that, imposing conditions (2.23) on the parameters, this composition is commutative and it provides a surface of the same type with the same constants β, γ.

Theorem 2.9. (Geometric Permutability Theorem) *Let M, M', M'' be hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$. Suppose that M satisfies $1+2\beta H+\gamma K = 0$ and that there are hyperbolic linear Weingarten congruences $l_1 : M \to M'$ and $l_2 : M \to M''$ as in Theorem 2.8, with constants $(r_1, \theta_1, \phi_1, \rho_1)$ and $(r_2, \theta_2, \phi_2, \rho_2)$ respectively, with $\theta_1 \neq \theta_2$, satisfying (2.1) and (2.23). Given $p \in M$, $p_1 = l_1(p) \in M'$ and $p_2 = l_2(p) \in M''$, we denote by N_p, N'_{p_1} and N''_{p_2} the unit vectors normal to M at p, to M' at p_1 and to M'' at p_2, respectively and $v_1 = v_1(p)$ (resp. $v_2 = v_2(p)$) the unit vector in the direction of the line connecting p to p_1 (resp. p_2). We suppose that $\{N_p, N'_{p_1}, v_1\}$ and $\{N_p, N''_{p_2}, v_2\}$ are sets of linearly independent vectors. Then there exists a regular surface $M^* \subset \mathbb{R}^3$ and hyperbolic linear Weingarten congruences $l_2^* : M' \to M^*$ and $l_1^* : M'' \to M^*$ with constants $(r_2, \theta_2, \rho_2, \phi_2)$ and $(r_1, \theta_1, \rho_1, \phi_1)$ respectively, such that*

$$l_2^* \circ l_1 = l_1^* \circ l_2.$$

Moreover, the Gaussian curvature K^* and the mean curvature H^* of M^* satisfy $1 + 2\beta H^* + \gamma K^* = 0$.

Proof: Let X be a local parametrization of M in a neighborhood of p. Since $l_1 : M \to M'$ and $l_2 : M \to M''$ are hyperbolic linear Weingarten congruences, we have that $X_1 = l_1(X) = X + r_1 v_1$ and $X_2 = l_2(X) = X + r_2 v_2$ are local parametrizations of M' and M'' at p_1 and p_2, respectively. By hypothesis, $r_i > 0$, $0 < \theta_i < \pi$, $0 < \phi_i, \rho_i \leq \frac{\pi}{2}$ $(i = 1, 2)$ are such that (2.1) is satisfied, i.e.,

$$\beta = \frac{-r_i(\cos\phi_i + \cos\rho_i\cos\theta_i)}{\sin^2\theta_i}, \qquad \gamma = \frac{r_i^2 \sin^2\rho_i}{\sin^2\theta_i}, \qquad i = 1, 2. \tag{2.32}$$

Observe that finding hyperbolic linear Weingarten congruences l_1^* and l_2^* as required by the theorem is equivalent to obtaining unit vector fields u_1, u_2 satisfying

$$r_1 v_1 + r_2 u_1 = r_2 v_2 + r_1 u_2. \tag{2.33}$$

We consider new orthonormal frames $\{\overline{e}'_1, \overline{e}'_2, e'_3\}$ adapted to M' and $\{\overline{e}''_1, \overline{e}''_2, e''_3\}$ adapted to M'' given by

$$\overline{e}'_i = F_{ij}e'_j, \quad \overline{e}''_i = F_{ij}e''_j, \quad \text{with} \quad F_{22} = F_{11} \quad \text{and} \quad F_{21} = -F_{12}. \tag{2.34}$$

Define the vector fields

$$\left\{ \begin{array}{lcl} u_1 & = & \sin\rho_2 \overline{e}'_1 + \cos\rho_2 e'_3, \\ u_2 & = & \sin\rho_1 \overline{e}''_1 + \cos\rho_1 e''_3. \end{array} \right. \tag{2.35}$$

The idea is to show that these vectors u_1, u_2 satisfy equation (2.33). Initially, using (2.27), we observe that

$$r_1 v_1 \quad = \quad r_1 \sin\phi_1 e_1 + r_1 \cos\phi_1 e_3.$$

Similarly, using (2.34), (2.35), (2.28), (2.21) and the constant δ given by (2.22), we have

$$\begin{aligned} r_2 u_1 \quad = \quad & [\delta r_1 \sin\rho_1 a'_{11} F_{11} + \delta r_1 \sin\rho_1 a'_{21} F_{12} + r_2 \cos\rho_2 a'_{31}]e_1 + \\ & + [\delta r_1 \sin\rho_1 a'_{12} F_{11} + \delta r_1 \sin\rho_1 a'_{22} F_{12} + r_2 \cos\rho_2 a'_{32}]e_2 + \\ & + [\delta r_1 \sin\rho_1 a'_{13} F_{11} + \delta r_1 \sin\rho_1 a'_{23} F_{12} + r_2 \cos\rho_2 a'_{33}]e_3. \end{aligned}$$

Moreover, it follows from (2.26), (2.27) and (2.22), that

$$r_2 v_2 = \delta r_1 \sin\phi_1 E_{11} e_1 + \delta r_1 \sin\phi_1 E_{12} e_2 + r_2 \cos\phi_2 e_3.$$

Finally, using the relations (2.26), (2.27), (2.34), (2.35) and δ given by (2.22), we obtain

$$\begin{aligned} r_1 u_2 \quad = \quad & [r_1 \sin\rho_1 (a''_{11} E_{11} - a''_{12} E_{12}) F_{11} + r_1 \sin\rho_1 (a''_{21} E_{11} - a''_{22} E_{12}) F_{12} + \\ & + \delta r_1 \cos\rho_1 (a'_{31} E_{11} - a'_{32} E_{12})]e_1 + \\ & + [r_1 \sin\rho_1 (a''_{11} E_{12} + a''_{12} E_{11}) F_{11} + r_1 \sin\rho_1 (a''_{21} E_{12} + a''_{22} E_{11}) F_{12} + \\ & + \delta r_1 \cos\rho_1 (a'_{31} E_{12} + a'_{32} E_{11})]e_2 + \\ & + [\delta r_1 \sin\rho_1 a'_{13} F_{11} + \delta r_1 \sin\rho_1 a'_{23} F_{12} + r_1 \cos\rho_1 a''_{33}]e_3. \end{aligned}$$

Therefore, equation (2.33) is equivalent to the following linear system

$$\left\{ \begin{array}{l} r_1 \sin\rho_1 \left[(\delta a'_{11} - (a''_{11} E_{11} + a''_{12} E_{21})) F_{11} + (\delta a'_{21} - (a''_{21} E_{11} + a''_{22} E_{21})) F_{12} \right] = \\ \qquad \delta r_1 \cos\rho_1 (a'_{31} E_{11} + a'_{32} E_{21}) + \delta r_1 \sin\phi_1 E_{11} - r_1 \sin\phi_1 - r_2 \cos\rho_2 a'_{31}, \\ \\ r_1 \sin\rho_1 \left[(\delta a'_{12} - (a''_{11} E_{12} + a''_{12} E_{22})) F_{11} + (\delta a'_{22} - (a''_{21} E_{12} + a''_{22} E_{22})) F_{12} \right] = \\ \qquad \delta r_1 \cos\rho_1 (a'_{31} E_{12} + a'_{32} E_{22}) + \delta r_1 \sin\phi_1 E_{12} - r_2 \cos\rho_2 a'_{32}, \\ \\ r_1 \cos\rho_1 a''_{33} - r_1 \cos\phi_1 + r_2 \cos\phi_2 - r_2 \cos\rho_2 a'_{33} = 0, \end{array} \right. \tag{2.36}$$

where δ is given by (2.22), E_{11} and E_{12} are given by (2.26) and the real numbers a'_{ij} and a''_{ij} defined by (2.28) are given by (2.4), taking $r = r_k$, $\theta = \theta_k$, $\phi = \phi_k$ and $\rho = \rho_k$, $k = 1, 2$ respectively.

We observe that, as a consequence of (2.4), $a'_{33} = \cos\theta_1$ and $a''_{33} = \cos\theta_2$. Then using (2.25) we conclude that the third equation of the linear system (2.36) is satisfied. Substituting the expressions of F_{11} and F_{12} given by (2.31) and using equations (2.21)-(2.28), (2.4) and (2.30), we conclude that the first and the second equations of this linear system are also satisfied.

We consider the surface M^* parametrized by $X^* = X + r_1 v_1 + r_2 u_1$. It follows from Theorem 2.5 and (2.32) that the surface M^* satisfies $1 + 2\beta H^* + \gamma K^* = 0$. □

Remark 2.10. *If $M \subset \mathbb{R}^3$ is a hyperbolic linear Weingarten surface, such that $1 + 2\beta H + \gamma K = 0$, then Theorem 2.9 shows that the composition of Bäcklund transformations provides a 4-parameter family of surfaces M^* satisfying $1 + 2\beta H^* + \gamma K^* = 0$. The four parameters are determined by the two unit vectors v_i, $i = 1, 2$ and the 8 constants $(r_i, \theta_i, \phi_i, \rho_i)$ satisfying a total of 6 equations, namely (2.32) and (2.23).*

3. Analytic interpretation of Bäcklund transformation

In this section we will present an analytic interpretation of the Geometric Integrability Theorem (Theorem 2.8) and of the Geometric Permutability Theorem (Theorem 2.9) given in the previous section. We start recalling that given a hyperbolic linear Weingarten surface in $\mathbb{R}^3$ satisfying $1 + 2\beta H + \gamma K = 0$, then $D = \gamma - \beta^2 > 0$ and there exists a solution ψ of the sine-Gordon equation

$$\psi_{x_1x_1} - \psi_{x_2x_2} = \sin(\psi + C_{\beta\gamma}), \tag{3.1}$$

where $C_{\beta\gamma}$ is a real constant defined by

$$\sin C_{\beta\gamma} = \frac{2\varepsilon_2\beta\sqrt{D}}{\gamma}, \quad \cos C_{\beta\gamma} = \frac{\gamma - 2\beta^2}{\gamma}, \quad \varepsilon_2^2 = 1. \tag{3.2}$$

Conversely, given a solution ψ of equation (3.1), where $C_{\beta\gamma}$ is a real constant defined by (3.2), there exists a hyperbolic linear Weingarten surface in $\mathbb{R}^3$ satisfying $1 + 2\beta H + \gamma K = 0$, parametrized by lines of curvature, whose first and second fundamental forms are given by $I = g_1^2 dx_1^2 + g_2^2 dx_2^2$ and $II = -\lambda_1 g_1^2 dx_1^2 - \lambda_2 g_2^2 dx_2^2$, where

$$g_1 = \sqrt{\gamma}\cos\frac{\psi}{2}, \qquad g_2 = \sqrt{\gamma}\sin\frac{\psi}{2}, \tag{3.3}$$

$$\lambda_1 = \frac{-1}{g_1}\left[S_1\cos\frac{\psi}{2} + S_2\sin\frac{\psi}{2}\right], \qquad \lambda_2 = \frac{-1}{g_2}\left[-S_2\cos\frac{\psi}{2} + S_1\sin\frac{\psi}{2}\right], \tag{3.4}$$

with

$$S_1 = \frac{-\beta}{\sqrt{\gamma}} \quad \text{and} \quad S_2 = \varepsilon_1\frac{\sqrt{D}}{\sqrt{\gamma}}, \quad \varepsilon_1^2 = 1, \quad \varepsilon_1\varepsilon_2 = -1. \tag{3.5}$$

For more details, see Tenenblat [19].

Let ψ be a solution of the sine-Gordon equation (3.1), where $C_{\beta\gamma}$ is a real constant defined by (3.2). We consider the hyperbolic linear Weingarten surface $M \subset \mathbb{R}^3$ satisfying $1+2\beta H+\gamma K=0$. Let $r>0$, $0<\theta<\pi$ and $0<\phi,\rho\leq\frac{\pi}{2}$ be real numbers satisfying (2.1) and (2.12). Using the Geometric Integrability Theorem, we can construct an orthonormal frame $\{e_1,e_2,e_3\}$ tangent to M, locally defined, with dual forms ω_1, ω_2 and connection forms $\omega_{12}, \omega_{13}, \omega_{23}$ associated to this frame satisfying the Bäcklund transformation (2.6), where c_1, c_2, c_3 and c_4 are given by (2.13). Moreover, the correspondence between hyperbolic linear Weingarten surfaces and solutions of the sine-Gordon equation allows us to conclude that the Bäcklund transformation (2.6) is equivalent to the system of partial differential equations

$$\begin{cases} \psi'_{x_1}+\psi_{x_2} &= 2S_3\cos\frac{\psi}{2}\cos\frac{\psi'}{2}-2S_4\cos\frac{\psi}{2}\sin\frac{\psi'}{2}+ \\ &\quad +2S_5\sin\frac{\psi}{2}\cos\frac{\psi'}{2}-2S_6\sin\frac{\psi}{2}\sin\frac{\psi'}{2}, \\ \psi'_{x_2}+\psi_{x_1} &= 2S_3\sin\frac{\psi}{2}\sin\frac{\psi'}{2}+2S_4\sin\frac{\psi}{2}\cos\frac{\psi'}{2}+ \\ &\quad -2S_5\cos\frac{\psi}{2}\sin\frac{\psi'}{2}-2S_6\cos\frac{\psi}{2}\cos\frac{\psi'}{2}, \end{cases} \tag{3.6}$$

where

$$S_3=c_1\sqrt{\gamma}+c_3S_1,\qquad S_4=c_2\sqrt{\gamma}+c_4S_1,\qquad S_5=c_3S_2,\qquad S_6=c_4S_2, \tag{3.7}$$

and c_1,c_2,c_3,c_4 are given by (2.13). Using these real numbers we define the following constants

$$S_7=-S_3^2-S_4^2+S_5^2+S_6^2,\qquad S_8=S_3S_5+S_4S_6, \tag{3.8}$$

$$S_7'=-S_3^2+S_4^2-S_5^2+S_6^2,\qquad S_8'=-S_3S_4-S_5S_6. \tag{3.9}$$

Remark 3.1. (A particular case) *Observe that (3.6) reduces to the classical analytic Bäcklund transformation for the sine-Gordon equation, when* $\phi=\rho=\pi/2$. *In fact, in this case,* $\beta=0$, $\gamma=r^2/\sin^2\theta$, $C_{\beta\gamma}=0$, $b_1=0$, $b_2=-\sin\theta$, $b_3=-\cos\theta$, $c_1=c_4=0$, $c_2=-1/r$, $c_3=\cos\theta/\sin\theta$, $\epsilon_1=1$, $\epsilon_2=-1$, $S_1=S_3=S_6=0$, $S_2=1$, $S_4=-1/\sin\theta$, $S_5=\cos\theta/\sin\theta$. *Moreover,* ψ *is a solution of the sine-Gordon equation*

$$\psi_{x_1x_1}-\psi_{x_2x_2}=\sin\psi \tag{3.10}$$

and (3.6) reduces to the classical result, namely

$$\begin{cases} \psi'_{x_1}+\psi_{x_2} &= 2csc\theta\cos\frac{\psi}{2}\sin\frac{\psi'}{2}+2cot\theta\sin\frac{\psi}{2}\cos\frac{\psi'}{2}, \\ \psi'_{x_2}+\psi_{x_1} &= -2csc\theta\sin\frac{\psi}{2}\cos\frac{\psi'}{2}-2cot\theta\cos\frac{\psi}{2}\sin\frac{\psi'}{2}, \end{cases} \tag{3.11}$$

3.1. Analytic Interpretation of the Integrability Theorem

We will prove that the system of partial differential equations (3.6) is integrable and that each of its solutions ψ' satisfies the sine-Gordon equation

$$\psi'_{x_1x_1} - \psi'_{x_2x_2} = \sin(\psi' + C_{\beta'\gamma'}), \tag{3.12}$$

where β', γ' are given by (2.2) and $C_{\beta'\gamma'}$ is a real constant defined by

$$\sin C_{\beta'\gamma'} = \frac{2\beta'\sqrt{D}}{\gamma'}, \qquad \cos C_{\beta'\gamma'} = \frac{\gamma' - 2(\beta')^2}{\gamma'}, \tag{3.13}$$

with $D := \gamma - \beta^2$. Initially, we will prove the following lemma

Lemma 3.2. *Given real numbers β, γ such that $\gamma - \beta^2 > 0$, we consider the real constant $C_{\beta\gamma}$ defined by (3.2). Let S_1, S_2 be the constants given by (3.5). We choose real numbers $r > 0$, $0 < \theta < \pi$ and $0 < \phi, \rho \leq \frac{\pi}{2}$ satisfying (2.1). Consider the real numbers b_1, b_2, b_3 given by (2.11), c_1, c_2, c_3, c_4 by (2.13), β' and γ' by (2.2) and S_3, S_4, S_5, S_6 by (3.7). Let S_7, S_8 (resp. S_7', S_8') be the constants given by (3.8) (resp. (3.9)). Then*

$$S_7 = -\cos C_{\beta\gamma}, \qquad S_8 = \frac{-\sin C_{\beta\gamma}}{2}, \tag{3.14}$$

$$\cos C_{\beta'\gamma'} = \frac{\sin^2\theta - 2b_1^2}{\sin^2\theta}, \qquad \sin C_{\beta'\gamma'} = \frac{-2b_1b_2}{\sin^2\theta}, \tag{3.15}$$

$$S_7' = \cos C_{\beta'\gamma'}, \qquad S_8' = \frac{\sin C_{\beta'\gamma'}}{2}, \tag{3.16}$$

where $C_{\beta'\gamma'}$ is the constant defined by (3.13).

Proof: Substituting (3.7) into (3.8), we get

$$\begin{aligned} S_7 &= -(c_1^2 + c_2^2)\gamma - (c_3^2 + c_4^2)(S_1^2 - S_2^2) - 2\sqrt{\gamma}S_1(c_1c_3 + c_2c_4), \\ S_8 &= \sqrt{\gamma}S_2(c_1c_3 + c_2c_4) + (c_3^2 + c_4^2)S_1S_2, \end{aligned}$$

where S_1 and S_2 are given by (3.5). Using the identities (2.14), the values of β and γ given by (2.1), the real constants S_1 and S_2 defined in (3.5) and the definition of the $C_{\beta\gamma}$ given by (3.2), we obtain

$$\begin{aligned} S_7 &= \frac{-\sin^2\theta}{r^2\sin^2\phi b_2^2}\left[\gamma + 2\beta\sqrt{\gamma}S_1 + \gamma(S_1^2 - S_2^2)\right] + (S_1^2 - S_2^2) = \frac{2\beta^2 - \gamma}{\gamma} \\ &= -\cos C_{\beta\gamma}, \\ S_8 &= \frac{\sin^2\theta}{r^2\sin^2\phi b_2^2}\left[\beta\sqrt{\gamma}S_2 + \gamma S_1S_2\right] - S_1S_2 = -\varepsilon_2\frac{\beta\sqrt{D}}{\gamma} = \frac{-\sin C_{\beta\gamma}}{2}. \end{aligned}$$

Let b_1 and b_2 be the constants defined by (2.11). Since $D = \gamma - \beta^2$ then using (2.2) and (2.16) we have

$$\sqrt{D} = \frac{-r\sin\phi b_2}{\sin^2\theta} \qquad \text{and} \qquad \beta' = \frac{r\sin\phi b_1}{\sin^2\theta}.$$

Substituting into (3.13) and using the constant γ' defined by (2.2), we obtain (3.15). Finally, substituting (3.7) into (3.9) we get

$$\begin{aligned} S_7' &= (-c_1^2 + c_2^2)\gamma + 2\sqrt{\gamma}S_1(-c_1c_3 + c_2c_4) - c_3^2 + c_4^2, \\ S_8' &= -c_1c_2\gamma - \sqrt{\gamma}S_1(c_1c_4 + c_2c_3) - c_3c_4. \end{aligned}$$

Thus, using (2.15) and (3.5) we obtain

$$\begin{aligned} S_7' &= \frac{(\sin^2\theta - 2b_1^2)(\gamma + 2\beta r\cos\phi + r^2 cos^2\phi) + 2rb_1\sin\phi\cos\theta(\beta + r\cos\phi)}{r^2\sin^2\phi b_2^2} - \frac{cos^2\theta}{b_2^2}, \\ S_8' &= \frac{-b_1(\gamma + 2r\beta\cos\phi + r^2 cos^2\phi) + r\sin\phi\cos\theta(\beta + r\cos\phi)}{r^2\sin^2\phi b_2}. \end{aligned}$$

It follows from (2.17), (2.18), (3.15) and 2.11) that

$$\begin{aligned} S_7' &= \frac{(\sin^2\theta - 2b_1^2)(\sin^2\theta - b_1^2)}{b_2^2\sin^2\theta} = \cos C_{\beta'\gamma'}, \\ S_8' &= \frac{-b_1(\sin^2\theta - b_1^2)}{b_2\sin^2\theta} = \frac{\sin C_{\beta'\gamma'}}{2}. \end{aligned}$$

□

The following theorem provides an analytic interpretation of the Geometric Integrability Theorem (Theorem 2.8).

Theorem 3.3. (Analytic Integrability Theorem) *Let ψ be a solution of the sine-Gordon equation (3.1), where β and γ are fixed real numbers such that $\gamma - \beta^2 > 0$ and $C_{\beta\gamma}$ is the constant given by (3.2). We consider $r > 0, 0 < \phi, \rho \leq \frac{\pi}{2}$ and $0 < \theta < \pi$ real numbers satisfying (2.1). Let $C_{\beta'\gamma'}$ be the constant defined by (3.13), where β', γ' are given by (2.2). Consider the numbers b_1, b_2, b_3 given by (2.11), c_1, c_2, c_3, c_4 by (2.13), S_1, S_2 by (3.5) and S_3, S_4, S_5, S_6 by (3.7). Then the system of partial differential equations (3.6) is integrable. Moreover, the function ψ', obtained by integrating this system, provides a 3-parameter family of solutions of the sine-Gordon equation (3.12).*

Proof: Differentiating the first equation of the system (3.6) with respect to x_2 and subtracting from the derivative of the second equation with respect to x_1, we obtain

$$\begin{aligned} &\psi'_{x_1x_2} - \psi'_{x_2x_1} = \sin(\psi + C_{\beta\gamma}) + \\ &+ \left[\psi_{x_1} + \psi'_{x_2}\right]\left[-S_3\cos\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} - S_4\cos\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} - S_5\sin\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} - S_6\sin\tfrac{\psi}{2}\cos\tfrac{\psi'}{2}\right] + \\ &+ \left[\psi_{x_2} + \psi'_{x_1}\right]\left[-S_3\sin\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} + S_4\sin\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} + S_5\cos\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} - S_6\cos\tfrac{\psi}{2}\sin\tfrac{\psi'}{2}\right], \end{aligned}$$

where we used the fact that ψ is a solution of the sine-Gordon equation (3.1). Thus, using (3.6) and the relations given by (3.8) and (3.14), we have

$$\psi'_{x_1x_2} - \psi'_{x_2x_1} = \sin(\psi + C_{\beta\gamma}) + S_7 \sin\psi + 2S_8 \cos\psi = 0,$$

ie, the system (3.6) is integrable.

Similarly, differentiating the first equation of (3.6) with respect to x_1 and subtracting from the derivative of the second equation with respect to x_2, we obtain

$$\begin{aligned}
&\psi'_{x_1x_1} - \psi'_{x_2x_2} = \\
&[\psi_{x_1} + \psi'_{x_2}]\left[-S_3 \sin\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} + S_4 \sin\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} + S_5\cos\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} - S_6\cos\tfrac{\psi}{2}\sin\tfrac{\psi'}{2}\right] + \\
&+ [\psi_{x_2} + \psi'_{x_1}]\left[-S_3\cos\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} - S_4\cos\tfrac{\psi}{2}\cos\tfrac{\psi'}{2} - S_5\sin\tfrac{\psi}{2}\sin\tfrac{\psi'}{2} - S_6\sin\tfrac{\psi}{2}\cos\tfrac{\psi'}{2}\right],
\end{aligned}$$

where we used the fact that ψ is differentiable. Therefore, using (3.6) and the relations given by (3.9) and (3.16), we have

$$\psi'_{x_1x_1} - \psi'_{x_2x_2} = S'_7 \sin\psi' + 2S'_8 \cos\psi' = \sin(\psi' + C_{\beta'\gamma'}),$$

ie, ψ' is a solution of the sine-Gordon equation (3.12).

The functions ψ' obtained by integrating (3.6) depend on 3-parameters, namely the initial condition $\psi'(x_1^0, x_2^0)$, and four constants (r, θ, ϕ, ρ) satisfying two equations given by (2.1). □

Definition 3.4. *Let ψ be a solution of the sine-Gordon equation (3.1). We say that a function ψ' is associated to ψ by* a Bäcklund transformation $BT(r, \theta, \phi, \rho)$ *if ψ' is a solution of the system (3.6).*

3.2. Analytic Interpretation of the Permutability Theorem

Let ψ be a solution of the sine-Gordon equation (3.1), where $C_{\beta\gamma}$ is the constant given by (3.2) and β, γ are constants such that $\gamma - \beta^2 > 0$. The Geometric Permutability Theorem (Theorem 2.9) and the correspondence between hyperbolic linear Weingarten surfaces and solutions of the sine-Gordon equation allows us to construct a new solution ψ^* of the sine-Gordon equation (3.1). The analytic interpretation of the Permutability Theorem (Theorem 3.5) will allow us to obtain ψ^* algebraically. This is the content of our next result. However, the proof of this theorem is highly technical and, therefore, it will be presented in the Appendix.

Theorem 3.5. (Analytic Permutability Theorem) *Let ψ be a solution of the sine-Gordon equation (3.1), where $C_{\beta\gamma}$ is the real constant given by (3.2) and the real numbers β, γ are such that $\gamma - \beta^2 > 0$. We consider real numbers $r_i > 0, 0 < \phi_i, \rho_i \leq \frac{\pi}{2}$ and $0 < \theta_i < \pi$ $(i = 1, 2)$ with $\theta_1 \neq \theta_2$, satisfying (2.1) and (2.23). Let ψ_i , $i = 1, 2$ be solutions of equation (3.12), associated to ψ by the Bäcklund transformations $BT(r_i, \theta_i, \phi_i, \rho_i)$, where $C_{\beta'\gamma'}$ is the constant given by (3.13) and β', γ' are given by (2.2), when $r = r_i$, $\theta = \theta_i$, $\phi = \phi_i$ and $\rho = \rho_i$. Then there* *exists a unique solution* ψ^* *of the sine-Gordon equation* (3.1) *associated to* ψ_i *by* $BT(r_j, \theta_j, \rho_j, \phi_j)$, $1 \leq i \neq j \leq 2$. *Moreover,* ψ^* *is determined algebraically by*

$$\tan\left(\frac{\psi^* - \psi}{4}\right) = \varepsilon_1 \frac{\sin\left(\frac{\theta_2+\theta_1}{2}\right)}{\sin\left(\frac{\theta_2-\theta_1}{2}\right)} \tan\left(\frac{\psi_1 - \psi_2}{4}\right), \qquad \varepsilon_1^2 = 1. \tag{3.17}$$

Observe that in Theorem 3.5 the constants β' and γ' defined by (2.2) are independent of i since (2.23) is satisfied.

4. The composition of Bäcklund transformations and the Ribaucour transformation for hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$

We consider a hyberbolic linear Weingarten surface in $\mathbb{R}^3$ parametrized by orthogonal lines of curvatures $X(x_1, x_2)$ satisfying $1 + 2\beta H + \gamma K = 0$, where β and γ are real constants such that $\beta^2 - \gamma < 0$. There are two methods which provide 4-parameter families of linear Weingarten surfaces, with the same constants β and γ, associated to the surface $X(x_1, x_2)$. Namely, the composition of Bäcklund transformations, as we have seen in the previous sections and the Ribaucour transformation. In general the surfaces obtained by these two methods are not congruent. In fact, by starting with the pseudo-sphere, Goulart-Tenenblat [11] proved, with an explicit example, that a composition of Bäcklund transformations is not a Ribaucour transformation. In this section, we will determine necessary and sufficient conditions for the hyberbolic linear Weingarten surfaces constructed by using these two methods, to be congruent.

4.1. Ribaucour Transformation

We state the main concepts and results of the theory of Ribaucour transformations for surfaces in $\mathbb{R}^3$, in particular for linear Weingarten surfaces, that will be used in the following subsections. More details of the theory can be found in [6] or [8].

Definition 4.1. *Let* M *and* $\tilde{M}$ *be orientable surfaces in* $\mathbb{R}^3$ *and let* N *and* $\tilde{N}$ *be their Gauss maps. We say that* $\tilde{M}$ *is associated to* M *by a Ribaucour transformation if, and only if, there exists a differentiable function* h *defined on* M *and a diffeomorphism* $l : M \longrightarrow \tilde{M}$ *such that* $p + h(p)N(p) = l(p) + h(p)\tilde{N}(l(p))$, $\forall p \in M$, *the subset* $p + h(p)N(p)$, $p \in M$ *is a surface in* $\mathbb{R}^3$ *and the diffeomorphism* l *preserves lines of curvature.*

We say that M and $\tilde{M}$ are *locally associated by a Ribaucour transformation* if for all $p \in M$ there exists a neighborhood of p in M that is associated to a open subset of $\tilde{M}$ by a Ribaucour transformation. Similarly, we define parametrized surfaces associated by such transformations.

The Ribaucour transformation is characterized in terms of a differential equation which must be satisfied by map h of the definition (see [6] or [8]).

Theorem 4.2. *Let* M *be an orientable surface in* $\mathbb{R}^3$, *without umbilic points and let* N *be its Gauss map. We consider* $\{e_i\}$, $i = 1, 2$, *orthonormal principal direction* *vector fields and* $-\lambda_i$ *the corresponding principal curvatures, ie,* $dN(e_i) = \lambda_i e_i$. *A surface* $\tilde{M}$ *is locally associated to* M *by a Ribaucour transformation if, and only if, there exist parametrizations* $X : U \subset R^2 \to M$ *and* $\tilde{X} : U \subset R^2 \to \tilde{M}$ *and a differentiable function* $h : U \to R$ *such that* $1 + h\lambda_i \neq 0$,

$$\tilde{X} = X + h(N - \tilde{N})$$

and $\tilde{N}$ *is a Gauss map of* $\tilde{M}$ *given by*

$$\tilde{N} = \frac{1}{1 + Z_1^2 + Z_2^2}\left[2\sum_{i=1}^{2} Z_i e_i + (Z_1^2 + Z_2^2 - 1)N\right],$$

where

$$Z_i = \frac{dh(e_i)}{1 + h\lambda_i} \tag{4.1}$$

and h *satisfies the differential equation*

$$dZ_j(e_i) + Z_i\omega_{ij}(e_i) - Z_i Z_j \lambda_i = 0, \qquad 1 \leq i \neq j \leq 2, \tag{4.2}$$

where ω_{ij} *are the connection forms associated to* $\{e_i\}$.

We observe that the differential equation (4.2) is of second order and highly non linear. The proposition below shows how the problem of obtaining the function h can be linearized.

Proposition 4.3. *If* h *is a nonvanishing function, defined on a simply connected domain, which satisfies equation (4.2) then* $h = \dfrac{\Omega}{W}$, *where* Ω *and* W *are nonvanishing functions satisfying*

$$\begin{aligned} d\Omega_i(e_j) &= \Omega_j\omega_{ij}(e_j), i \neq j, \\ d\Omega &= \textstyle\sum_{i=1}^{2} \Omega_i\omega_i, \\ dW &= -\textstyle\sum_{i=1}^{2} \Omega_i\lambda_i\omega_i. \end{aligned} \tag{4.3}$$

Conversely, if Ω *and* W *satisfy (4.3) and* $W(W + \Omega\lambda_i) \neq 0$ *then* $h = \dfrac{\Omega}{W}$ *is a solution of (4.2).*

Observe that Ω_i $i = 1, 2$ are the covariant derivatives of Ω. Moreover, considering Z_i defined by (4.1), one can show that $Z_i = \Omega_i/W$ (see [8]).

Next theorem shows that, by imposing an additional condition, the Ribaucour transformation of a linear Weingarten surface, satisfying $\alpha + 2\beta H + \gamma K = 0$ provides a family of surface this same type, with the same constants α, β, γ.

Theorem 4.4. (Corro-Ferreira-Tenenblat [8]) *Let M be a surface of $\mathbb{R}^3$, without umbilic points and let $\tilde{M}$ be associated to M by a Ribaucour transformation, such that the normal lines at corresponding points intersect at a distance h. Assume that $h = \dfrac{\Omega}{W}$ is not constant along the lines of curvature and suppose that the functions Ω and W satisfy the additional condition*

$$| \nabla\Omega |^2 + W^2 = 2C_R(\alpha\Omega^2 + 2\beta\Omega W + \gamma W^2), \tag{4.4}$$

where α, β, γ, $C_R \neq 0$ are real constants. Then $\tilde{M}$ is a linear Weingarten surface satisfying $\alpha + 2\beta\tilde{H} + \gamma\tilde{K} = 0$ if, and only if, M satisfies $\alpha + 2\beta H + \gamma K = 0$, where H and K (resp. $\tilde{H}$ and $\tilde{K}$) are, respectively, the mean and Gaussian curvatures of M (resp. $\tilde{M}$).

Observe that we are denoting by C_R the constant of the Ribaucour transformation.

Theorem 4.5. (Corro-Ferreira-Tenenblat [8]) *Let $M \subset R^3$ be a linear Weingarten surface satisfying $\alpha + 2\beta H + \gamma K = 0$, with no umbilic points. Let e_i, $i = 1, 2$ be orthonormal principal direction vector fields. Let ω_i, ω_{ij} and ω_{i3} be the dual and the connection forms. Then the system*

$$\begin{aligned} d\Omega &= \textstyle\sum_{i=1}^{2} \Omega_i\omega_i, \\ dW &= \textstyle\sum_{i=1}^{2} \Omega_i\omega_{i3}, \\ d\Omega_i &= \Omega_j\omega_{ij} + C_R(2\alpha\Omega + 2\beta W)\omega_i + \\ & \quad -[(1 - 2\gamma C_R)W - 2C_R\beta\Omega]\omega_{i3}, \quad i \neq j. \end{aligned} \tag{4.5}$$

is integrable, for any constant $C_R \neq 0$. On a simply connected domain, any solution, whose initial conditions satisfy (4.4), satisfies (4.4) identically. If M is locally parametrized by $X : U \subset R^2 \to M$ and Ω, W is a non trivial solution of (4.5) satisfying (4.4), then each surface of the family

$$\tilde{X} = X - \frac{2\Omega}{|\nabla\Omega|^2 + W^2}(\nabla\Omega - WN) \tag{4.6}$$

is a linear Weingarten surface, locally associated to X by a Ribaucour transformation, satisfying $\alpha + 2\beta\tilde{H} + \gamma\tilde{K} = 0$, where $\tilde{H}$ and $\tilde{K}$ are the mean and Gaussian curvatures of $\tilde{X}$.

Remark 4.6. *Considering Z_i given by (4.1), since $Z_i = \Omega_i/W$, we can rewrite condition (4.4) as*

$$Z_1^2 + Z_2^2 + 1 = 2C_R(\alpha h^2 + 2\beta h + \gamma). \tag{4.7}$$

Remark 4.7. Let $M \subset \mathbb{R}^3$ be a linear Weingarten surface satisfying $\alpha + 2\beta H + \gamma K = 0$. If M is parametrized by orthogonal lines of curvature $X(x_1, x_2)$, then the system of differential equations (4.5) can be written as follows

$$\begin{aligned}
\frac{\partial \Omega}{\partial x_i} &= g_i \Omega_i, \\
\frac{\partial \Omega_i}{\partial x_j} &= \frac{1}{g_i}\frac{\partial g_j}{\partial x_i}\Omega_j, \qquad i \neq j, \\
\frac{\partial W}{\partial x_i} &= -\lambda_i g_i \Omega_i, \\
\frac{\partial \Omega_i}{\partial x_i} &= -\frac{1}{g_j}\frac{\partial g_i}{\partial x_j}\Omega_j + 2C_R(\alpha - \beta\lambda_i) g_i \Omega + \\
&\quad + [2C_R\beta + (1 - 2C_R\gamma)\lambda_i] g_i W, \quad i \neq j,
\end{aligned} \tag{4.8}$$

where $i, j = 1, 2$, $g_i = |X_{x_i}|$, $-\lambda_i$ are the principal curvatures of M and $C_R \neq 0$ is a real constant.

Proposition 4.8. (Lemes-Roitman-Tenenblat-Tribuzi [13]) Let $M \subset \mathbb{R}^3$ be a linear Weingarten surface satisfying $\alpha + 2\beta H + \gamma K = 0$. If $\tilde{M}$ is associated to M by a Ribaucour transformation as in Theorem 4.5, then the first fundamental form of $\tilde{M}$ is given by $\tilde{I} = \tilde{\omega}_1^2 + \tilde{\omega}_2^2$, where

$$\tilde{\omega}_i = \pm \frac{(\gamma - \alpha h^2) + (2\beta h^2 + 2\gamma h)\lambda_i}{\alpha h^2 + 2\beta h + \gamma}\omega_i, \quad i = 1, 2, \tag{4.9}$$

and $h = \dfrac{\Omega}{W}$.

4.2. Necessary and sufficient conditions

Given a hyberbolic linear Weingarten surface M in $\mathbb{R}^3$, satisfying $1 + 2\beta H + \gamma K = 0$, one can consider the surfaces $\tilde{M}$ associated to M by Ribaucour transformations as in Theorem 4.5 and the surfaces M^* associated to M by composition of Bäcklund transformations as in Theorem 2.9. We will determine necessary and sufficient conditions for $\tilde{M}$ and M^* to be congruent.

Let $X(x_1, x_2)$ be a parametrization by orthogonal lines of curvature of a surface M satisfying $1 + 2\beta H + \gamma K = 0$. Let $C_{\beta\gamma}$ be the real constant defined by (3.2). We consider ψ a solution of the sine-Gordon equation (3.1) such that the first and second fundamental forms of X are given by $I = g_1^2 dx_1^2 + g_2^2 dx_2^2$ and $II = -\lambda_1 g_1^2 dx_1^2 - \lambda_2 g_2^2 dx_2^2$, where

$$g_1 = \sqrt{\gamma}\cos\frac{\psi}{2}, \qquad g_2 = \sqrt{\gamma}\sin\frac{\psi}{2}, \tag{4.10}$$

$$\lambda_1 = -\frac{1}{\gamma}\left[-\beta + \varepsilon_1 \frac{g_2}{g_1}\sqrt{D}\right], \qquad \lambda_2 = -\frac{1}{\gamma}\left[-\beta + \varepsilon_2 \frac{g_1}{g_2}\sqrt{D}\right], \tag{4.11}$$

and $D = \gamma - \beta^2$ (see (3.2)-(3.5)).

Remark 4.9. For later use, let us establish the following notation

1. $\tilde{X}(x_1, x_2)$: surface associated to $X(x_1, x_2)$ by a Ribaucour transformation satisfying $1 + 2\beta\tilde{H} + \gamma\tilde{K} = 0$ whose normal lines, at corresponding points, intersect at a distance $h(x_1, x_2)$,

2. $X^*(x_1, x_2)$: surface satisfying $1+2\beta H^*+\gamma K^* = 0$, obtained from X by a composition of Bäcklund transformation $BT(r_1, \theta_1, \phi_1, \rho_1)$ and $BT(r_2, \theta_2, \phi_2, \rho_2)$, where $r_i > 0$, $0 < \theta_i < \pi$, $0 < \phi_i, \rho_i < \frac{\pi}{2}$ $(i = 1, 2)$ are real constants, with $\theta_1 \neq \theta_2$ such that (2.1) and (2.23) are satisfied.

3. $\psi_i(x_1, x_2)$ $(i = 1, 2)$: solutions of the sine-Gordon equation (3.12), associated to ψ by Bäcklund transformation $BT(r_i, \theta_i, \phi_i, \rho_i)$, where $C_{\beta'\gamma'}$ is the real constant given by (3.13) and β', γ' are given by (2.2).

4. $\psi^*(x_1, x_2)$: solution of the sine-Gordon equation (3.1) given by (3.17).

Substituting (4.10) into (4.11) and using (4.9), we obtain that the first fundamental form of $\tilde{X}$ is given by $\tilde{I} = \tilde{g}_1^2 dx_1^2 + \tilde{g}_2^2 dx_1^2$, where

$$\begin{aligned} \tilde{g}_1 &= \frac{2\varepsilon_2\sqrt{D}(\beta h^2 + \gamma h)\sin\frac{\psi}{2} + (-(\gamma - 2\beta^2)h^2 + 2\beta\gamma h + \gamma^2)\cos\frac{\psi}{2}}{\sqrt{\gamma}(h^2 + 2\beta h + \gamma)}, \\ \tilde{g}_2 &= \frac{(-(\gamma - 2\beta^2)h^2 + 2\beta\gamma h + \gamma^2)\sin\frac{\psi}{2} + 2\varepsilon_1\sqrt{D}(\beta h^2 + \gamma h)\cos\frac{\psi}{2}}{\sqrt{\gamma}(h^2 + 2\beta h + \gamma)}. \end{aligned} \tag{4.12}$$

We obseve that the first fundamental form of X^* is given by $I^* = (g_1^*)^2 dx_1^2 + (g_2^*)^2 dx_2^2$, where $g_1^* = -\sqrt{\gamma}\cos\frac{\psi^*}{2}$ and $g_2^* = -\sqrt{\gamma}\sin\frac{\psi^*}{2}$. Introducing the notation

$$\eta = \frac{\sin\left(\dfrac{\theta_2 + \theta_1}{2}\right)}{\sin\left(\dfrac{\theta_2 - \theta_1}{2}\right)}, \tag{4.13}$$

we define the functions

$$\varphi = \varepsilon_1 \eta \tan\frac{\psi_1 - \psi_2}{4}, \qquad \Lambda = \frac{\varphi\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \varphi\sin\frac{\psi}{2}}. \tag{4.14}$$

Using the Analytic Permutability Theorem (Theorem 3.5), we observe that $\varphi = \tan\frac{\psi^* - \psi}{4}$. Therefore,

$$\begin{aligned} g_1^* &= \sqrt{\gamma}\left[\frac{2\varphi}{1+\varphi^2}\sin\frac{\psi}{2} + \frac{-1+\varphi^2}{1+\varphi^2}\cos\frac{\psi}{2}\right], \\ g_2^* &= \sqrt{\gamma}\left[\frac{-1+\varphi^2}{1+\varphi^2}\sin\frac{\psi}{2} - \frac{2\varphi}{1+\varphi^2}\cos\frac{\psi}{2}\right]. \end{aligned} \tag{4.15}$$

Considering a hyperbolic linear Weingarten surface M immersed in $\mathbb{R}^3$, our next theorem establishes the necessary and sufficient conditions for a composition of Bäcklund transformations and a Ribaucour transformation of M to be congruent.

Theorem 4.10. *Let $M \subset \mathbb{R}^3$ be a linear hyperbolic Weingarten surface satisfying $1 + 2\beta H + \gamma K = 0$, parametrized by lines of curvature $X(x_1, x_2)$. Let $X^*(x_1, x_2)$ be a surface associated to X by a composition of Bäcklund transformations as in Theorem 2.9. Let $\tilde{X}(x_1, x_2)$ be a hyperbolic linear Weingarten surface associated to X by a Ribaucour transformation as in Theorem 4.5, such that the normal lines at corresponding points intersect at a distance $h(x_1, x_2)$. Then, with the notation of Remark 4.9, $\tilde{X}$ and X^* are congruent if, and only if, h is one of the following functions*

$$\frac{-\gamma}{\beta + \varepsilon_1\sqrt{D}\varphi}, \qquad \frac{-\gamma}{\beta - \varepsilon_1\sqrt{D}\varphi^{-1}}, \qquad \frac{-\gamma}{\beta - \varepsilon_1\sqrt{D}\Lambda}, \qquad \frac{-\gamma}{\beta + \varepsilon_1\sqrt{D}\Lambda^{-1}}, \tag{4.16}$$

where φ and Λ are given by (4.14) and $D = \gamma - \beta^2$.

Proof: We observe that the first fundamental form of a linear Weingarten surface determines its second fundamental form. Considering the notation established in Remark 4.9, let $\tilde{g}_1$, $\tilde{g}_2$ and g_1^*, g_2^* given by (4.12) and (4.15), respectively. Since the fundamental forms of X^* are determined by the solution ψ^* of the sine-Gordon equation (3.1) given by (3.17), then $\tilde{X}$ and X^* are congruent if, and only if, $\tilde{g}_1 = \pm g_1^*$ and $\tilde{g}_2 = \pm g_2^*$. Observe that the equality $\tilde{g}_i = \pm g_i^*$ $(i = 1, 2)$ is a quadratic equation for h in terms of φ.

A straighforward computation shows that $\tilde{g}_1 = g_1^*$ and $\tilde{g}_2 = g_2^*$ (*resp.* $\tilde{g}_1 = -g_1^*$ and $\tilde{g}_2 = -g_2^*$) if, and only if, $h = \dfrac{-\gamma}{\beta + \varepsilon_1\sqrt{D}\varphi}$ $\left(\textit{resp. } h = \dfrac{-\gamma}{\beta - \varepsilon_1\sqrt{D}\varphi^{-1}}\right)$. Similarly, $\tilde{g}_1 = g_1^*$ and $\tilde{g}_2 = -g_2^*$ (*resp.* $\tilde{g}_1 = -g_1^*$ and $\tilde{g}_2 = g_2^*$) if, and only if, $h = \dfrac{-\gamma}{\beta - \varepsilon_1\sqrt{D}\Lambda}$ $\left(\textit{resp. } h = \dfrac{-\gamma}{\beta + \varepsilon_1\sqrt{D}\Lambda^{-1}}\right)$. Therefore, h should be one of the functions in (4.16). □

As we have seen in the previous sections, (see Remark 3.1), if two surfaces are associated by a Bäcklund transformation $BT(r, \theta, \frac{\pi}{2}, \frac{\pi}{2})$, i.e., $\phi = \rho = \dfrac{\pi}{2}$, then Theorems 2.5, 2.8 and 2.9 reduce to the classical theorems of Bäcklund transformations for surfaces in $\mathbb{R}^3$ with constant Gaussian curvature $K = -\dfrac{\sin\theta}{r^2} < 0$. In particular, r is determined by θ, since K is a fixed negative number. For this reason, in this case, we denote the Bäcklund transformation just by $BT(\theta)$. Without loss of generality one may consider $K = -1$, i.e. $\gamma = 1$. Moreover, since $\beta = 0$, $\gamma = 1$ and $\epsilon_1 = 1$, as an immediate consequence of the previous theorem, we get the following corollary.

Corollary 4.11. *Under the same conditions as in Theorem 4.10, if $\beta = 0$ and $\gamma = 1$, i.e. if the surfaces X, X^* and $\tilde{X}$ have Gaussian curvature equal to -1, then $\tilde{X}$ and X^* are congruent if, and only if, h is one of the following functions*

$$-\frac{1}{\varphi}, \qquad \varphi, \qquad -\Lambda \qquad \textit{or} \qquad \frac{1}{\Lambda},$$

where φ and Λ are given by (4.14).

5. Appendix

We now prove the analytic version of the permutability theorem (Theorem 3.5), for the Bäcklund transformations $BT(r_i, \theta_i, \phi_i, \rho_i)$, $i = 1, 2$.

Remark 5.1. *Given a solution ψ of the sine-Gordon equation (3.1), where $C_{\beta\gamma}$ is the constant defined by (3.2) and β, γ are real numbers such that $\gamma - \beta^2 > 0$, we choose real numbers $r_i > 0$, $0 < \theta_i < \pi$ and $0 < \rho_i, \phi_i \leq \frac{\pi}{2}$ $(i = 1, 2)$ and $\theta_1 \neq \theta_2$ satisfying (2.1), (2.12), (2.21) and (2.23). For each $i = 1, 2$, taking $r = r_i$, $\theta = \theta_i$, $\phi = \phi_i$ and $\rho = \rho_i$ we define b_{ji}, $1 \leq j \leq 3$ given by (2.11), the constants c_{ki}, $1 \leq k \leq 4$ defined by (2.13) and the constants S_{ni}, $4 \leq n \leq 8$ given by (3.7) and (3.8). Moreover, we define the constants β', γ' defined by (2.2) which are independent of i since (2.23) is satisfied.*

In order to achieve our goal, we need to prove some lemmas. We define the real numbers L_ℓ $(1 \leq \ell \leq 6)$ below,

$$
\begin{aligned}
L_1 &= S_{31}S_{42} + S_{32}S_{41} + S_{51}S_{62} + S_{52}S_{61},\\
L_2 &= S_{31}S_{32} - S_{41}S_{42} + S_{51}S_{52} - S_{61}S_{62},\\
L_3 &= S_{31}S_{41} + S_{32}S_{42} + S_{51}S_{61} + S_{52}S_{62},\\
L_4 &= S_{32}^2 - S_{62}^2 - S_{41}^2 + S_{51}^2 = S_{31}^2 - S_{61}^2 - S_{42}^2 + S_{52}^2,\\
L_5 &= -S_{31}S_{62} + S_{32}S_{61} - S_{41}S_{52} + S_{42}S_{51},\\
L_6 &= -S_{31}S_{52} + S_{32}S_{51} + S_{41}S_{62} - S_{42}S_{61}.
\end{aligned}
\tag{5.1}
$$

and the functions m_s $(1 \leq s \leq 8)$ as

$$
\begin{aligned}
m_1 &= S_{52}\cos\tfrac{\psi_1}{2} - S_{62}\sin\tfrac{\psi_1}{2} - S_{51}\cos\tfrac{\psi_2}{2} + S_{61}\sin\tfrac{\psi_2}{2},\\
m_2 &= S_{32}\cos\tfrac{\psi_1}{2} - S_{42}\sin\tfrac{\psi_1}{2} - S_{31}\cos\tfrac{\psi_2}{2} + S_{41}\sin\tfrac{\psi_2}{2},\\
m_3 &= S_{42}\cos\tfrac{\psi_1}{2} + S_{32}\sin\tfrac{\psi_1}{2} - S_{41}\cos\tfrac{\psi_2}{2} - S_{31}\sin\tfrac{\psi_2}{2},\\
m_4 &= -S_{62}\cos\tfrac{\psi_1}{2} - S_{52}\sin\tfrac{\psi_1}{2} + S_{61}\cos\tfrac{\psi_2}{2} + S_{51}\sin\tfrac{\psi_2}{2},\\
m_5 &= S_{51}\cos\tfrac{\psi_1}{2} - S_{61}\sin\tfrac{\psi_1}{2} - S_{52}\cos\tfrac{\psi_2}{2} + S_{62}\sin\tfrac{\psi_2}{2},\\
m_6 &= S_{31}\cos\tfrac{\psi_1}{2} - S_{41}\sin\tfrac{\psi_1}{2} - S_{32}\cos\tfrac{\psi_2}{2} + S_{42}\sin\tfrac{\psi_2}{2},\\
m_7 &= S_{41}\cos\tfrac{\psi_1}{2} + S_{31}\sin\tfrac{\psi_1}{2} - S_{42}\cos\tfrac{\psi_2}{2} - S_{32}\sin\tfrac{\psi_2}{2},\\
m_8 &= -S_{61}\cos\tfrac{\psi_1}{2} - S_{51}\sin\tfrac{\psi_1}{2} + S_{62}\cos\tfrac{\psi_2}{2} + S_{52}\sin\tfrac{\psi_2}{2}.
\end{aligned}
\tag{5.2}
$$

Lemma 5.2. *Let ψ be a solution of the sine-Gordon equation (3.1) and let r_i, θ_i, ϕ_i, ρ_i, β', γ', $i = 1, 2$ be real numbers as described in Remark 5.1. Let $C_{\beta'\gamma'}$ and L_ℓ $(1 \leq \ell \leq 6)$ be the real constants given by (3.13) and (5.1), respectively. Then*

$$\begin{aligned} L_1 &= \left[\frac{\cos\theta_1\cos\theta_2 - 1}{\sin\theta_1\sin\theta_2}\right]\sin C_{\beta'\gamma'}, & L_2 &= \left[\frac{\cos\theta_1\cos\theta_2 - 1}{\sin\theta_1\sin\theta_2}\right]\cos C_{\beta'\gamma'}, \\ L_3 &= -\sin C_{\beta'\gamma'}, & L_4 &= -\cos C_{\beta'\gamma'}, \\ L_5 &= -\varepsilon_1\left[\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1\sin\theta_2}\right]\cos C_{\beta'\gamma'}, & L_6 &= \varepsilon_1\left[\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1\sin\theta_2}\right]\sin C_{\beta'\gamma'}. \end{aligned}$$

Proof: Substituting the constants S_{ni}, $3 \leq n \leq 6$, $i = 1, 2$ given by (3.7) and Remark 5.1 in (5.1) and using (2.21), (2.23) and (3.5), we obtain

$$\begin{aligned} L_1 &= \frac{2}{\delta}c_{11}c_{21}\gamma - \beta\left(c_{11}c_{42} + c_{21}c_{32} + \frac{1}{\delta}(c_{11}c_{41} + c_{21}c_{31})\right) + c_{31}c_{42} + c_{32}c_{41}, \\ L_2 &= \frac{1}{\delta}(c_{11}^2 - c_{21}^2)\gamma + \beta\left[-c_{11}\left(c_{32} + \frac{1}{\delta}c_{31}\right) + c_{21}\left(c_{42} + \frac{1}{\delta}c_{41}\right)\right] + c_{31}c_{32} - c_{41}c_{42}, \end{aligned}$$

where δ is given by (2.22). It follows from the expressions given by (2.13) for the constants c_{ki}, $1 \leq k \leq 4$, $i = 1, 2$, defined in Remark 5.1, that

$$\begin{aligned} \delta r_1^2\sin^2\phi_1 b_{21}^2 L_1 &= b_{11}b_{21}(\gamma + 2\beta r_2\cos\phi_2 + r_2^2 cos^2\phi_2) + \\ &\quad + b_{11}b_{21}(\gamma + 2\beta r_1\cos\phi_1 + r_1^2 cos^2\phi_1) + \\ &\quad - b_{11}b_{21}(r_1\cos\phi_1 - r_2\cos\phi_2)^2 + \\ &\quad - r_1\sin\phi_1 b_{21}\left[\cos\theta_1(\beta + r_2\cos\phi_2) + \cos\theta_2(\beta + r_1\cos\phi_1)\right]. \\ \delta r_1^2\sin^2\phi_1 b_{21}^2 L_2 &= \tfrac{1}{2}(2b_{11}^2 - \sin^2\theta_1)[\gamma + 2\beta r_1\cos\phi_1 + r_1^2 cos^2\phi_1] + \\ &\quad + \tfrac{1}{2}(2b_{11}^2 - \sin^2\theta_1)[\gamma + 2\beta r_2\cos\phi_2 + r_2^2 cos^2\phi_2] + \\ &\quad - \tfrac{1}{2}(2b_{11}^2 - \sin^2\theta_1)(r_1\cos\phi_1 - r_2\cos\phi_2)^2 + \\ &\quad - r_1\sin\phi_1 b_{11}[\cos\theta_1(\beta + r_2\cos\phi_2) + \cos\theta_2(\beta + r_1\cos\phi_1)] + \\ &\quad + r_1^2\sin^2\phi_1\cos\theta_1\cos\theta_2. \end{aligned}$$

We observe from (2.24) that

$$r_1\sin\phi_1 b_{11}(\cos\theta_1 - \cos\theta_2) = (r_1\cos\phi_1 - r_2\cos\phi_2)\sin^2\theta_1, \tag{5.3}$$

where b_{jj}, $1 \leq j \leq 3$ are given by (2.11) and Remark 5.1. Hence, using the identities (2.17), (2.18), (2.22), (2.23) we can write

$$L_1 = \frac{2(1 - \cos\theta_1\cos\theta_2)b_{11}b_{21}}{\sin^3\theta_1\sin\theta_2}, \qquad L_2 = \frac{(\sin^2\theta_1 - 2b_{11}^2)(\cos\theta_1\cos\theta_2 - 1)}{\sin^3\theta_1\sin\theta_2}.$$

As a direct consequence of (3.15) we obtain

$$L_1 = \left[\frac{\cos\theta_1\cos\theta_2 - 1}{\sin\theta_1\sin\theta_2}\right]\sin C_{\beta'\gamma'}, \qquad L_2 = \left[\frac{\cos\theta_1\cos\theta_2 - 1}{\sin\theta_1\sin\theta_2}\right]\cos C_{\beta'\gamma'}.$$

The identities

$$L_3 = -\sin C_{\beta'\gamma'}, \qquad L_4 = -\cos C_{\beta'\gamma'}$$

follow directly from Lemma 3.2, Remark 5.1 and the real constants S_{ni}, S'_{ni}, $n = 7, 8$, $i = 1, 2$ defined in (3.7) and (3.8). Finally, in order, to prove the last two equalities, we use the relations (2.22), (3.5), (3.7) and Remark 5.1 to obtain from (5.1) that

$$\begin{aligned} L_5 &= \varepsilon_1\sqrt{D}\left[c_{11}\left(-c_{42} + \frac{1}{\delta}c_{41}\right) + c_{21}\left(-c_{32} + \frac{1}{\delta}c_{31}\right)\right], \\ L_6 &= \varepsilon_1\sqrt{D}\left[c_{11}\left(-c_{32} + \frac{1}{\delta}c_{31}\right) - c_{21}\left(-c_{42} + \frac{1}{\delta}c_{41}\right)\right], \end{aligned}$$

where δ is given by (2.22) and $D = \gamma - \beta^2$. Using the expressions given by (2.13) for the constants c_{ki}, $1 \le k \le 4$, $i = 1, 2$ defined in Remark 5.1, we can write

$$\begin{aligned} \delta r_1^2 \sin^2\phi_1 b_{21}^2 L_5 &= \varepsilon_1\sqrt{D}[-2b_{11}b_{21}(r_1\cos\phi_1 - r_2\cos\phi_2) + \\ &\quad + r_1\sin\phi_1 b_{21}(\cos\theta_1 - \cos\theta_2)], \\ \delta r_1^2 \sin^2\phi_1 b_{21}^2 L_6 &= \varepsilon_1\sqrt{D}[(r_1\cos\phi_1 - r_2\cos\phi_2)(\sin^2\theta_1 - 2b_{11}^2) + \\ &\quad + r_1\sin\phi_1 b_{11}(\cos\theta_1 - \cos\theta_2)]. \end{aligned}$$

It follows from the expressions of β and γ given by (2.1) and of b_{21}, given in (2.11) and Remark 5.1 that $\sqrt{D}\sin^2\theta_1 = -r_1\sin\phi_1 b_{21}$. It follows from (5.3) that

$$L_5 = \frac{-\varepsilon_1(\cos\theta_1 - \cos\theta_2)(\sin^2\theta_1 - 2b_{11}^2)}{\delta\sin^4\theta_1}, \qquad L_6 = \frac{-2\varepsilon_1 b_{11}b_{21}(\cos\theta_1 - \cos\theta_2)}{\delta\sin^4\theta_1}.$$

Therefore, using (2.22) and (3.15) we obtain

$$L_5 = -\varepsilon_1\left[\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1\sin\theta_2}\right]\cos C_{\beta'\gamma'}, \qquad L_6 = \varepsilon_1\left[\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1\sin\theta_2}\right]\sin C_{\beta'\gamma'}.$$

□

Lemma 5.3. *Let ψ, ψ_1, ψ_2 be the function described in Remark 4.9. We consider $L_k, 1 \le k \le 3$ the real constants defined in (5.1) and m_s, $1 \le s \le 8$ the functions given by (5.2). If*

$$\begin{aligned} \Gamma &= m_1m_4 - m_2m_3, \\ \triangle_1 &= m_4m_5 - m_2m_7, \qquad \triangle_2 = m_4m_6 - m_2m_8, \\ \triangle_3 &= m_1m_7 - m_3m_5, \qquad \triangle_4 = m_1m_8 - m_3m_5, \end{aligned} \tag{5.4}$$

then

$$
\begin{aligned}
\Gamma &= \left[L_1 - L_3\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)\right]\cos\left(\tfrac{\psi_1+\psi_2}{2}\right) + \\
&\quad \left[L_2 - L_4\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)\right]\sin\left(\tfrac{\psi_1+\psi_2}{2}\right), \\
\triangle_1 &= \left[L_3 - L_1\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)\right]\cos\left(\tfrac{\psi_1+\psi_2}{2}\right) + \\
&\quad \left[L_4 - L_2\cos\left(\tfrac{\psi_1-\psi_2}{2}\right)\right]\sin\left(\tfrac{\psi_1+\psi_2}{2}\right), \\
\triangle_2 &= -L_5\sin\left(\tfrac{\psi_1-\psi_2}{2}\right)\sin\left(\tfrac{\psi_1+\psi_2}{2}\right) + L_6\sin\left(\tfrac{\psi_1-\psi_2}{2}\right)\cos\left(\tfrac{\psi_1+\psi_2}{2}\right).
\end{aligned}
\tag{5.5}
$$

Moreover, $\triangle_3 = -\triangle_2$ *and* $\triangle_4 = \triangle_1$.

Proof: Using (5.1), (5.2) and (5.4) a straightforward computation shows that

$$
\begin{aligned}
\Gamma &= -\frac{L_3}{2}(\cos\psi_1 + \cos\psi_2) - \frac{L_4}{2}(\sin\psi_1 + \sin\psi_2) + \\
&\quad +L_1\cos\left(\tfrac{\psi_1+\psi_2}{2}\right) + L_2\sin\left(\tfrac{\psi_1+\psi_2}{2}\right), \\
\triangle_1 &= -\frac{L_1}{2}(\cos\psi_1 + \cos\psi_2) - \frac{L_2}{2}(\sin\psi_1 + \sin\psi_2) + \\
&\quad +L_3\cos\left(\tfrac{\psi_1+\psi_2}{2}\right) + L_4\sin\left(\tfrac{\psi_1+\psi_2}{2}\right), \\
\triangle_2 &= \frac{L_5}{2}(\cos\psi_1 - \cos\psi_2) + \frac{L_6}{2}(\sin\psi_1 - \sin\psi_2).
\end{aligned}
$$

Applying some trigonometric identities, we obtain (5.5). Analogously, we prove that $\triangle_3 = -\triangle_2$ and $\triangle_4 = \triangle_1$. □

With the aid of the lemmas above, we will obtain the analytic interpretation of the permutability theorem for linear Weingarten hyperbolic surfaces in $\mathbb{R}^3$.

Proof of Theorem 3.5: By hypothesis, ψ_i, $i = 1, 2$, are associated to ψ by Bäcklund transformations $BT(r_i, \theta_i, \phi_i, \rho_i)$. Then its follows from (3.6) that

$$
\left\{
\begin{aligned}
\psi_{1,x_1} + \psi_{x_2} &= 2S_{31}\cos\tfrac{\psi}{2}\cos\tfrac{\psi_1}{2} - 2S_{41}\cos\tfrac{\psi}{2}\sin\tfrac{\psi_1}{2} + \\
&\quad +2S_{51}\sin\tfrac{\psi}{2}\cos\tfrac{\psi_1}{2} - 2S_{61}\sin\tfrac{\psi}{2}\sin\tfrac{\psi_1}{2}, \\
\psi_{1,x_2} + \psi_{x_1} &= 2S_{31}\sin\tfrac{\psi}{2}\sin\tfrac{\psi_1}{2} + 2S_{41}\sin\tfrac{\psi}{2}\cos\tfrac{\psi_1}{2} + \\
&\quad -2S_{51}\cos\tfrac{\psi}{2}\sin\tfrac{\psi_1}{2} - 2S_{61}\cos\tfrac{\psi}{2}\cos\tfrac{\psi_1}{2},
\end{aligned}
\right.
\tag{5.6}
$$

$$\left\{\begin{array}{lcl}\psi_{2,x_1}+\psi_{x_2} & = & 2S_{32}\cos\frac{\psi}{2}\cos\frac{\psi_2}{2}-2S_{42}\cos\frac{\psi}{2}\sin\frac{\psi_2}{2}+\\ & & +2S_{52}\sin\frac{\psi}{2}\cos\frac{\psi_2}{2}-2S_{62}\sin\frac{\psi}{2}\sin\frac{\psi_2}{2},\\ \psi_{2,x_2}+\psi_{x_1} & = & 2S_{32}\sin\frac{\psi}{2}\sin\frac{\psi_2}{2}+2S_{42}\sin\frac{\psi}{2}\cos\frac{\psi_2}{2}+\\ & & -2S_{52}\cos\frac{\psi}{2}\sin\frac{\psi_2}{2}-2S_{62}\cos\frac{\psi}{2}\cos\frac{\psi_2}{2},\end{array}\right. \tag{5.7}$$

where $S_{ni},\ 3\leq n\leq 6,\ i=1,2$ are the constants defined by (3.7) taking $r=r_i,\theta=\theta_i,\phi=\phi_i,\rho=\rho_i$.

We want to determine a solution ψ^* of the sine-Gordon equation (3.1) such that ψ_i $(i=1,2)$ is associated to ψ^* by a Bäcklund transformation $BT(r_j,\theta_j,\phi_j,\rho_j)$, $1\leq i\neq j\leq 2$. Without loss of generality, we can change ψ^* by $\psi^*+2\pi$. So, we want to determine a function ψ^* that satisfies the following differential equations

$$\left\{\begin{array}{lcl}\psi_{1,x_1}+\psi^*_{x_2} & = & -2S_{32}\cos\frac{\psi^*}{2}\cos\frac{\psi_1}{2}+2S_{42}\cos\frac{\psi^*}{2}\sin\frac{\psi_1}{2}+\\ & & -2S_{52}\sin\frac{\psi^*}{2}\cos\frac{\psi_1}{2}+2S_{62}\sin\frac{\psi^*}{2}\sin\frac{\psi_1}{2},\\ \psi_{1,x_2}+\psi^*_{x_1} & = & -2S_{32}\sin\frac{\psi*}{2}\sin\frac{\psi_1}{2}-2S_{42}\sin\frac{\psi*}{2}\cos\frac{\psi_1}{2}+\\ & & +2S_{52}\cos\frac{\psi*}{2}\sin\frac{\psi_1}{2}+2S_{62}\cos\frac{\psi*}{2}\cos\frac{\psi_1}{2},\end{array}\right. \tag{5.8}$$

$$\left\{\begin{array}{lcl}\psi_{2,x_1}+\psi^*_{x_2} & = & -2S_{31}\cos\frac{\psi^*}{2}\cos\frac{\psi_2}{2}+2S_{41}\cos\frac{\psi^*}{2}\sin\frac{\psi_2}{2}+\\ & & -2S_{51}\sin\frac{\psi^*}{2}\cos\frac{\psi_2}{2}+2S_{61}\sin\frac{\psi^*}{2}\sin\frac{\psi_2}{2},\\ \psi_{2,x_2}+\psi^*_{x_1} & = & -2S_{31}\sin\frac{\psi^*}{2}\sin\frac{\psi_2}{2}-2S_{41}\sin\frac{\psi^*}{2}\cos\frac{\psi_2}{2}+\\ & & +2S_{51}\cos\frac{\psi^*}{2}\sin\frac{\psi_2}{2}+2S_{61}\cos\frac{\psi^*}{2}\cos\frac{\psi_2}{2}.\end{array}\right. \tag{5.9}$$

We suppose that ψ^* exits. Subtracting the first equation of (5.8) (resp. 5.9) from the first equation of (5.6) (resp. (5.7)) we obtain two expressions for $\psi^*_{x_2}-\psi_{x_2}$. Similarly, subtracting the second equation of (5.8) (resp. 5.9) from the second equation of (5.6) (resp. (5.7)) we obtain two expressions for $\psi^*_{x_1}-\psi_{x_1}$. Equating the expressions obtained in each case, we conclude that ψ^* must satisfy the matrix equation

$$\begin{bmatrix} m_1 & m_2\\ m_3 & m_4\end{bmatrix}\begin{bmatrix}\sin\frac{\psi*}{2}\\ \cos\frac{\psi*}{2}\end{bmatrix}=-\begin{bmatrix} m_5 & m_6\\ m_7 & m_8\end{bmatrix}\begin{bmatrix}\sin\frac{\psi}{2}\\ \cos\frac{\psi}{2}\end{bmatrix},$$

where $m_s,\ 1\leq s\leq 8$ are given by (5.2). Using (5.4), its follows that

$$\begin{bmatrix}\sin\frac{\psi*}{2}\\ \cos\frac{\psi*}{2}\end{bmatrix}=\frac{-1}{\Gamma}\begin{bmatrix}\triangle_1 & \triangle_2\\ -\triangle_2 & \triangle_1\end{bmatrix}\begin{bmatrix}\sin\frac{\psi}{2}\\ \cos\frac{\psi}{2}\end{bmatrix}, \tag{5.10}$$

where the functions Γ and $\triangle_k$, $1 \le k \le 4$, are given by (5.4). Now, using Lemmas 5.2 and 5.3 we have

$$\Gamma = \frac{\left[\cos(\theta_1-\theta_2) - 1 + (\cos(\theta_1+\theta_2)-1)\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)\right]\sin\left(\frac{\psi_1+\psi_2}{2}+C_{\beta'\gamma'}\right)}{\left[1+\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)\right]\sin\theta_1\sin\theta_2},$$

$$\triangle_1 = \frac{\left[-(\cos(\theta_1-\theta_2) - 1) + (\cos(\theta_1+\theta_2)-1)\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)\right]\sin\left(\frac{\psi_1+\psi_2}{2}+C_{\beta'\gamma'}\right)}{\left[1+\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)\right]\sin\theta_1\sin\theta_2},$$

$$\triangle_2 = \frac{2\varepsilon_1(\cos\theta_1-\cos\theta_2)\tan\left(\frac{\psi_1-\psi_2}{4}\right)\sin\left(\frac{\psi_1+\psi_2}{2}+C_{\beta'\gamma'}\right)}{\left[1+\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)\right]\sin\theta_1\sin\theta_2}.$$

We observe that $\dfrac{\cos(\theta_1+\theta_2)-1}{\cos(\theta_1-\theta_2)-1} = \eta^2$ and $\dfrac{\cos\theta_1-\cos\theta_2}{\cos(\theta_1-\theta_2)-1} = -\eta,$ where η is given by (4.13). Therefore,

$$\frac{\triangle_1}{\Gamma} = -\left[\frac{1-\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)}\right] \quad \text{and} \quad \frac{\triangle_2}{\Gamma} = \frac{-2\varepsilon_1\eta\tan\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)}.$$

Substituting these expressions into (5.10), we obtain

$$\begin{bmatrix} \sin\frac{\psi*}{2} \\ \cos\frac{\psi*}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{1-\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)} & \dfrac{2\varepsilon_1\eta\tan\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)} \\ \dfrac{-2\varepsilon_1\eta\tan\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)} & \dfrac{1-\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)}{1+\eta^2\tan^2\left(\frac{\psi_1-\psi_2}{4}\right)} \end{bmatrix} \begin{bmatrix} \sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{bmatrix}.$$

On the other hand, writing $\dfrac{\psi^*}{2} = \dfrac{\psi^*-\psi}{2} + \dfrac{\psi}{2}$, we have

$$\begin{bmatrix} \sin\frac{\psi*}{2} \\ \cos\frac{\psi*}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{1-\tan^2\left(\frac{\psi^*-\psi}{4}\right)}{1+\tan^2\left(\frac{\psi^*-\psi}{4}\right)} & \dfrac{2\tan\left(\frac{\psi^*-\psi}{4}\right)}{1+\tan^2\left(\frac{\psi^*-\psi}{4}\right)} \\ \dfrac{-2\tan\left(\frac{\psi^*-\psi}{4}\right)}{1+\tan^2\left(\frac{\psi^*-\psi}{4}\right)} & \dfrac{1-\tan^2\left(\frac{\psi^*-\psi}{4}\right)}{1+\tan^2\left(\frac{\psi^*-\psi}{4}\right)} \end{bmatrix} \begin{bmatrix} \sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{bmatrix}.$$

Equating the right hand side of the last last equalities, we conclude that ψ^* is determined by the algebraic relation

$$\tan\left(\frac{\psi^*-\psi}{4}\right) = \varepsilon_1\eta\tan\left(\frac{\psi_1-\psi_2}{4}\right),$$

where η is given by (4.13).

Conversely, we can show that the function ψ^* defined by this relation satisfies equations (5.8) and (5.9).

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