Bol. Soc. Paran. Mat. (3s.) **v. 37** 4 (2019): 97–102.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i4.32949

Linear Representation Of a Graph *

E. Montenegro V, E. Cabrera A., J. González C. and R. Manríquez P.

ABSTRACT: In this paper the linear representation of a graph is defined. A linear representation of a graph is a subgroup of $GL(p, \mathbb{R})$, the group of invertible matrices of order p and real coefficients. It will be demonstrated that every graph admits a linear representation. In this paper, simple and finite graphs will be used, framed in the graphs theory's area.

Key Words: Graph, Graphs and Abstract Algebra.

Contents

Introduction

It is customary to define or to describe a graph by means of a diagram in which each vertex is represented by a point and each edge $e = uv$ is represented by a line segment or curve joining the points corresponding to u and v. A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ can also be described by means of matrices. One such matrix is then $n \times n$ adjacency matrix $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G) \\ 0 & \text{if } v_iv_j \notin E(G). \end{cases}$$

Another matrix is the $n \times m$ incidence matrix $B(G) = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are incident} \\ 0 & \text{otherwise.} \end{cases}$$

(See [2]).

* This work was supported and financed by DGI of Universidad de Playa Ancha, through the CNE 01-1718 project.
2010 *Mathematics Subject Classification:* 05C10, 05C25.
Submitted August 01, 2016. Published May 23, 2017

In computer science, a graph is an abstract data type that is meant to implement the undirected graph and directed graph. For the representation of graphs, adjacency matrix, incidence matrix and Adjacency list are used. The latter, is a collection of unordered lists used to represent a finite graph. Each list describes the set of neighbours of a vertex in the graph. (See [3]).

1. Preliminaries

The organization of the following definitions is presented so as this article is self-contained, for this reason we describe some basic concepts for the understanding of our work.

1.1. Simples and finites Graphs

Definition 1.1. *A graph G is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges.*

The graphs to be considered will be simple and finite and with a nonempty set of edges. For a graph G, $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. The cardinality of $V(G)$ is called order of G and the cardinality of $E(G)$ is called size of G. Other concepts used in this work and not defined explicitly can be found in the reference [2], [4], [5].

Definition 1.2. *An automorphism of a graph G is an isomorphism between G and itself. Thus an automorphism de G is a permutation of $V(G)$ that preserves adjacency (and nonadjacency).*

Remark 1.3. *The set of all automorphisms of the graph G form a group under the operation of composition, called the automorphism group or simply the group of G and denoted by $Aut(G)$.*

See more [2], [6], [7], [8]

1.2. Pertmutation matrix

Definition 1.4. *Let $S_n = \{f : \{1, ..., n\} \to \{1, ..., n\} : f \text{ is bijective}\}$ be , the set of permutations.*

Remark 1.5. *S_n form a group under the operation of composition, called symmetric group.*

Definition 1.6. *Let $GL(p, \mathbb{R}) = \{X \in M(p, \mathbb{R}) : \det(X) \neq 0\}$ be, with identity element*

$$I_p = \begin{bmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_p \end{bmatrix},$$

where $e_i = [\delta_{ij}]$ is the i-th row of I_p.

Definition 1.7. *Let $\rho \in S_n$ be, we will say that $M\rho \in GL(n,\mathbb{R})$ defined by*

$$M\rho = [C_i],$$

is the permutation matrix of ρ, if only if, for all $i = 1, ..., n$,

$$C_i = e^t_{\rho(i)},$$

where $C_i = [a_{ij}]$ is the i-th column of $M\rho$.

Remark 1.8. *From Definition 1.7: $M_\rho = \left[\delta_{j\rho(i)}\right]$.*

Example 1.1. *If $\rho = (123) \in S_3$, then $M_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Indeed,*

$$e_{\rho(1)} = e_2 = [\delta_{2j}] = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Rightarrow C_1 = e_2^t = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [\delta_{j2}]$$

$$e_{\rho(2)} = e_3 = [\delta_{3j}] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow C_2 = e_3^t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\delta_{j3}]$$

$$e_{\rho(1)} = e_1 = [\delta_{1j}] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Rightarrow C_3 = e_1^t = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [\delta_{j1}]$$

Lemma 1.9. *If $\rho, \sigma \in S_p$, then $M_\rho M_\sigma = M_{\sigma\circ\rho}$.*

Proof: Let $\rho, \sigma \in S_p$ be, such that $M_\rho = \left[\delta_{j\rho(i)}\right]$ and $M_\sigma = \left[\delta_{j\sigma(i)}\right]$. Then $M_\rho M_\sigma = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{p} \delta_{k\rho(i)}\delta_{j\sigma(k)} = \begin{cases} 1 & \text{if } \sigma(\rho(i)) = j \\ 0 & \text{if } \sigma(\rho(i)) \neq j. \end{cases} \tag{1.1}$$

On the other hand, $M_{\sigma\circ\rho} = \left[\delta_{j\sigma(\rho(i))}\right]$, where

$$\delta_{j\sigma(\rho(i))} = \begin{cases} 1 & \text{if } \sigma(\rho(i)) = j \\ 0 & \text{if } \sigma(\rho(i)) \neq j. \end{cases} \tag{1.2}$$

Therefore, From (1.1) and (1.2) $M_\rho M_\sigma = M_{\sigma\circ\rho}$. □

2. Linear Group

In this section, we introduce a fundamental definition for our research.

Definition 2.1. *If $H \leq S_p$, then $M(H) = \{M_\rho / \rho \in H\}$ it will be called linear group of H.*

Theorem 2.2. *If $H \leq S_p$, then $M(H) \leq GL(p,\mathbb{R})$.*

Proof:

(i) As $H \leq S_p$, we have to $1_{S_p} \in H$, then $M_{(1)} \in M(H)$. Therefore $M(H) \neq \emptyset$.

(ii) $\forall \rho, \sigma \in H : M_\rho M_{\sigma^{-1}} = M_{\sigma^{-1} \circ \rho} = M_\nu \in M(H)$

From (i), and (ii), then $M(H) \leq GL(p, \mathbb{R})$. □

Theorem 2.3. *If $H \leq S_p$, then $M(H) \cong H$.*

Proof: Let $f : H \to M(H)$ be defined by $f(\rho) = M_{\rho^{-1}}$.
If $\rho, \sigma \in H$ then, $f(\rho \circ \sigma) = M_{(\rho\circ\sigma)^{-1}} = M_{\sigma^{-1}\circ\rho^{-1}} = M_{\rho^{-1}} M_{\sigma^{-1}} = f(\rho)f(\sigma)$.
On the other hand, if $f(\rho) = f(\sigma) \Rightarrow M_{\rho^{-1}} = M_{\sigma^{-1}} \Rightarrow M_\rho = M_\sigma \Rightarrow \rho = \sigma$. Moreover, for each $M \in M(H)$, there exists $\rho \in H$ such that $f(\rho) = M$.
Therefore f is an isomorphism. □

3. Linear representation of a Graph

Definition 3.1. *We will say that $M(H)$ is a linear representation of a graph G, of order p, if only if, $M(H) \cong Aut(G)$.*

Remark 3.2. *The linear representation of a graph G, is denoted $M(G)$. Then,*

$$M(G) = \{M_\rho / \rho \in Aut(G)\}.$$

Example 3.1 (Linear representation of a complete Graph)**.** *Let K_p be a complete Graph (See [2]). We have $Aut(K_p) \cong S_p$ (See [4]) and $M(S_p) = \{M_\rho / \rho \in S_p\}$. Thus, $M(S_p) \cong S_p$.*

In particular, $Aut(K_3) \cong S_3$ and $M(K_3) = \{M_\rho / \rho \in Aut(K_3)\}$. Then, $M(K_3)$ has six matrices. These are:

$$M_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; M_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; M_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$
$$M_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; M_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; M_{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Accordingly, the linear representation of K_3 is:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

Example 3.2 (Linear representation of a cycle). *Let C_p be a cycle of order p (See [2]). We have $Aut(C_p) \cong D_{2p}$ (See [4]) and $M(D_{2p}) = \{M_\rho / \rho \in D_{2p}\}$. Thus, $M(D_{2p}) \cong D_{2p}$.*

In particular, $Aut(C_4) \cong D_8$ and $M(C_4) = \{M_\rho / \rho \in Aut(C_4)\}$. Thus, $M(C_4)$ has six matrices. But $D_8 = \langle (1234), (14)(23) \rangle$, therefore:

$$M(C_4) = \left\langle \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

is the linear representation of C_4.

Remark 3.3. *Note that for the generator $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, the characteristic polynomial is: $x^4 - 1$, and its characteristic values: $\pm 1, \pm i$. These can be considered as the vertices of a square inscribed in the unit circle in the complex plane.*

For the Generator $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, the characteristic polynomial is: $x^4 - 2x^2 + 1$, and its characteristic values: ± 1, each of them of multiplicity two. These can be considered as the generators orthogonal planes in space $\mathbb{R}^4$.

Finally, we have the main result of this work.

Theorem 3.4. *Every graph admits a linear representation.*

Proof: Let G be a graph and $Aut(G)$ the group of G. By Cayley Theorem (See [1]), exist $H \leq S_p$ such that $H \cong Aut(G)$ and by Theorem 2.3, $H \cong M(H)$. Therefore, $Aut(G) \cong M(H)$. □

Acknowledgments

The authors wish to thank the Department of Mathematics and Statistics, Universidad de Playa Ancha, for their support in carrying out this paper.

References

1. Cayley A. The theory of groups, graphical representation. *American Journal of Mathematics* 1876, 1, 174-176.
2. Chartrand G.; Lesniak. *Graph and Digraph*; Wadsworth and Brooks/Cole Advanced Books and Software Pacific Grove, C.A, 1996.
3. Cormen T., et. *Introduction to Algorithms*, Second Edition. MIT Press and McGraw-Hill, 2001.

4. Godsil, Graphs, groups and polytopes, in the book *Combinatorial Mathematics*, Proceedings of the International Conference on Combinatorial Theory Canberra, August 16–27, 1977; pp. 157-164.

5. Harary F.; A characterization of block graphs. *Canad. Math. Bull.* 1963, 6, 1-6.

6. Montenegro E., A result on the order and size of graphs that represent a finite groups. *Extracta Mathematicae*,1987, 2, 1, 14-16.

7. Montenegro E., Power D., Ruiz S., Salazar R. Spectra of related graphs and self reproducing Polyhedra. *Proyecciones*, 1992, 11, 1, 1-9.

8. Montenegro E. Graph with given Automorphism group and given Nuclear number. *Proyecciones*, 1992, 11, 1, 21-28

Eduardo Monetengro V.,
Department of Mathematics and Statistics,
Univerisdad de Playa Ancha,
Laboratorio de investigación Lab[e]saM
Chile.
E-mail address: emontene@upla.cl

and

Eduardo Cabrera A.,
Department of Mathematics and Statistics,
Univerisdad de Playa Ancha,
Laboratorio de investigación Lab[e]saM
Chile.
E-mail address: ecabrera@upla.cl

and

José González C.,
Department of Mathematics and Statistics,
Univerisdad de Playa Ancha,
Laboratorio de investigación Lab[e]saM
Chile.
E-mail address: jgonzalez@upla.cl

and

Ronald Manríquez P.,
Department of Mathematics and Statistics,
Univerisdad de Playa Ancha,
Laboratorio de investigación Lab[e]saM
Chile.
E-mail address: ronald.manriquez@upla.cl