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Coefficient Inequalities For A Class Of Analytic Functions Associated With The Lemniscate Of Bernoulli

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abstract: In this paper, a new subclass of analytic functions ML^*_λ associated with the right half of the lemniscate of Bernoulli is introduced. The sharp upper bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for both real and complex μ are considered. Further, the sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the function f in the class ML^*_λ using Toeplitz determinant is studied. Relevances of the main results are also briefly indicated.

Key Words: Starlike Function, Fekete-Szegö Inequality, Hankel Determinant, Lemniscate of Bernoulli.

Contents

1. Introduction and Motivation

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.
Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α, $(0 \leq \alpha < 1)$, denoted by $\mathcal{S}(\alpha)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}). \tag{1.2}$$

It may be noted that for $\alpha = 0$, the class $\mathcal{S}(\alpha) = \mathcal{S}^*$, the familiar subclass of starlike functions in $\mathbb{U}$. Similarly, a function $f \in \mathcal{A}$ is said to be in the class $\tilde{\mathcal{R}}(\alpha)$, $\alpha > 0$, if it satisfies the inequality

$$|(f'(z))^2 - \alpha| < \alpha \quad (z \in \mathbb{U}). \tag{1.3}$$

The class $\tilde{\mathcal{R}}(1) = \tilde{\mathcal{R}}$ was considered by Sahoo and Patel [28].

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Let f and g be two analytic functions in $\mathbb{U}$. We say f is subordinate to g, written $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if and only if there exists an analytic function w in $\mathbb{U}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, we have the following (see [19]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In 1966, Pommerenke [26] defined the q th Hankel determinant of f for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \cdot & \cdot\ \cdot & & \cdot \\ \cdot & \cdot\ \cdot & & \cdot \\ \cdot & \cdot\ \cdot & & \cdot \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

A good amount of literature is available about the importance of Hankel determinant. It is useful in the study of power series with integral coefficients (see [5]), meromorphic functions (see [32]) and also singularities (see [7]). Noonan and Thomas [22] studied about the second Hankel determinant of a really mean p-valent functions. Noor [23] determined the rate of growth of $H_q(n)$ as $n \longrightarrow \infty$ for the functions in $\mathcal{S}$ with a bounded boundary. Ehrenborg [9] studied the Hankel determinant of exponential polynomials.

For $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

It is well-known that for $f \in \mathcal{S}$ and given by (1.1) (see [8]), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. Fekete-Szegö (see [10]) problem is to estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in \mathcal{S}$. For details, see [6,24,25]. Given family $\mathcal{F}$ of the functions in $\mathcal{A}$, the functional $|H_2(2)|$ is popularly known as the second Hankel determinant. Second Hankel determinant for various subclasses of analytic functions were obtained by different researchers including Janteng et al. [14], Mishra and Gochhayat [20] and Murugusundaramoorthy and Magesh [21]. For some more recent works see [1,3,4,11,12,13,15,31].

Sokół and Stankiewicz [29](also see [2,30]) introduced the class SL^* consisting of normalized analytic functions f in $\mathbb{U}$ satisfying the condition $\left| \left[\frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < 1$, $(z \in \mathbb{U})$. We called such function as Sokół-Stankiewicz starlike function. Recently, Raza and Malik [27] determined the upper bound of third Hankel determinant $H_3(1)$ for the class SL^*. Further, Sahoo and Patel [28] obtained the upper bound to the second Hankel determinant for the class $\tilde{\mathcal{R}} = \{f \in \mathcal{A} : |(f'(z))^2 - 1| < 1,\ z \in \mathbb{U}\}$.

Motivated by the above mentioned works obtained by earlier researchers, we introduce the following subclass of analytic function as below:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class ML^*_λ, $0 \le \lambda \le 1$, if it satisfies the condition*

$$\left| \left[\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} \right]^2 - 1 \right| < 1 \quad (z \in \mathbb{U}). \tag{1.4}$$

Note that for $\lambda = 0$, the class ML^*_0 reduces to the class SL^*, studied by Raza and Malik [27] and while for $\lambda = 1$, the class ML^*_1 reduces to $\tilde{\mathcal{R}}$ studied by Sahoo and Patel [28]. In term of subordination, relation (1.4) can be written as.

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} \prec q(z) = \sqrt{1+z} \quad (z \in \mathbb{U}), \tag{1.5}$$

where $q(0) = 1$. To state the geometrical significance of the class ML^*_λ, consider

$$w = q(e^{i\theta}) = \sqrt{1 + e^{i\theta}} \quad (0 \le \theta \le 2\pi). \tag{1.6}$$

It follows from (1.6) that $w^2 - 1 = e^{i\theta}$, which implies $|w^2 - 1| = 1$. Taking $w = u + iv$ and simplifying we get

$$(u^2 + v^2)^2 = 2(u^2 - v^2).$$

Therefore, $q(\mathbb{U})$ is the region bounded by the right half of the lemniscate of Bernoulli given by $(u^2 + v^2)^2 - 2(u^2 - v^2) = 0$.

In this paper, following the techniques devised by Libera and Złotkiewicz [16, 17], we solve the Fekete-Szegö problem and also determine the upper bounds of the Hankel determinant $|H_2(1)|$ for a subclass ML^*_λ.

2. Preliminaries

Let $\mathcal{P}$ be the class of analytic functions p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

such that

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Each of the following results will be required in our present investigation.

Lemma 2.1. *[18] Let $p \in \mathcal{P}$ and of the form (2.1). Then*

$$|p_2 - \nu p_1^2| \le \begin{cases} -4\nu + 2, & \nu < 0 \\ 2, & 0 \le \nu \le 1, \\ 4\nu - 2, & \nu > 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if $p(z) = \left(\frac{1}{2} + \frac{\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\frac{1-z}{1+z}$, $(0 \le \eta \le 1)$, or one of its rotations. If $\nu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Although the above upper bound is sharp, when $0 < \nu < 1$, it can be improved as follows:

$$|p_2 - \nu p_1^2| + \nu |p_1|^2 \le 2 \quad \left(0 < \nu \le \frac{1}{2}\right),$$

and

$$|p_2 - \nu p_1^2| + (1-\nu)|p_1|^2 \le 2 \quad \left(\frac{1}{2} < \nu \le 1\right).$$

Lemma 2.2. *[18] Let $p \in \mathcal{P}$ be of the form (2.1), then for any complex number ν,*

$$|p_2 - \nu p_1^2| \le 2\max(1, |2\nu - 1|). \tag{2.2}$$

This result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.3. *([16], [17, p. 254]) Let the function $p \in \mathcal{P}$ be given by the power series (2.1). Then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{2.3}$$

and

$$4p_3 = p_1^3 + 2(4-p_1^2)p_1 x - (4-p_1^2)p_1 x^2 + 2(4-p_1^2)(1-|x|^2)z \tag{2.4}$$

for some complex numbers x, z satisfying $|x| \le 1$ and $|z| \le 1$.

3. Main Results

The first two theorems give the results related to Fekete-Szegö functional, which is a special case of the Hankel determinant.

Theorem 3.1. *Let the function f given by (1.1) be in the class ML_λ^*. Then for real μ, we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1-3\lambda^2-2\lambda-2\lambda\mu-4\mu}{8(2+\lambda)(1+\lambda)^2}, & \mu < \delta_1, \\ \frac{1}{2(2+\lambda)}, & \delta_1 \le \mu \le \delta_2, \\ -\left[\frac{1-3\lambda^2-2\lambda-2\lambda\mu-4\mu}{8(2+\lambda)(1+\lambda)^2}\right], & \mu > \delta_2. \end{cases} \tag{3.1}$$

Furthermore, for $\delta_1 < \mu \le \delta_1 + \beta$,

$$|a_3 - \mu a_2^2| + (\mu - \delta_1)|a_2|^2 \le \frac{1}{2(2+\lambda)}, \tag{3.2}$$

and for $\delta_1 + \beta < \mu < \delta_1 + 2\beta$,

$$|a_3 - \mu a_2^2| + (\delta_1 + 2\beta - \mu)|a_2|^2 \leq \frac{1}{2(2+\lambda)}, \tag{3.3}$$

where

$$\delta_1 = -\left[\frac{3 + 10\lambda + 7\lambda^2}{2(2+\lambda)}\right], \tag{3.4}$$

$$\delta_2 = \frac{5 + 6\lambda + \lambda^2}{2(2+\lambda)} \tag{3.5}$$

and

$$\beta = \frac{2(1+\lambda)^2}{\lambda + 2}. \tag{3.6}$$

These results are sharp.

Proof. Let $f \in ML_\lambda^*$. In view of Definition 1.1, there exists an analytic function $w(z)$ satisfying the condition of Schwarz lemma such that

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \sqrt{1 + w(z)} \quad (z \in \mathbb{U}). \tag{3.7}$$

Define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{3.8}$$

Clearly $p \in \mathcal{P}$. From (3.8), we get

$$w(z) = \frac{p(z) - 1}{p(z) + 1} \quad (z \in \mathbb{U}). \tag{3.9}$$

From (3.7) and (3.9), we have

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \sqrt{\frac{p(z)-1}{p(z)+1} + 1} = \sqrt{\frac{2p(z)}{1 + p(z)}}. \tag{3.10}$$

Now, by substituting the series expansion of $p(z)$ from (3.8) in (3.10), it follows that

$$\sqrt{\frac{2p(z)}{1+p(z)}} = 1 + \frac{1}{4}p_1 z + \left(\frac{p_2}{4} - \frac{5}{32}p_1^2\right) z^2 + \left(\frac{p_3}{4} - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3 + \cdots. \tag{3.11}$$

Using series expansions for $f(z)$ and $f'(z)$ from (1.1) give

$$\begin{aligned} \frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} &= 1 + (1+\lambda)a_2 z + \{(2+\lambda)a_3 - (1-\lambda^2)a_2^2\}z^2 + \{(3+\lambda)a_4 \\ &\quad -(1-\lambda)(2\lambda+3)a_2 a_3 + (1+\lambda)(1-\lambda)^2 a_2^3\}z^3 + \cdots. \end{aligned} \tag{3.12}$$

Making use of (3.11) and (3.12) in (3.10) and equating the coefficients of z, z^2 and z^3 in the resulting equation, we deduce that

$$a_2 = \frac{p_1}{4(1+\lambda)}, \tag{3.13}$$

$$a_3 = \frac{1}{4(2+\lambda)}\left[p_2 - \frac{7\lambda+3}{8(1+\lambda)}p_1^2\right], \tag{3.14}$$

and

$$a_4 = \frac{1}{4(3+\lambda)}\left[p_3 - \frac{7\lambda^2+16\lambda+7}{4(1+\lambda)(2+\lambda)}p_1p_2 + \frac{13+40\lambda+25\lambda^2}{32(1+\lambda)(2+\lambda)}p_1^3\right]. \tag{3.15}$$

For real μ, it follows from (3.13) and (3.14) that

$$|a_3 - \mu a_2^2| = \frac{1}{4(2+\lambda)}|p_2 - \nu p_1^2|, \tag{3.16}$$

where

$$\nu = \frac{3+10\lambda+7\lambda^2+4\mu+2\lambda\mu}{8(1+\lambda)^2}.$$

In view of (3.16) and by an application of Lemma 2.1, we obtain the desired assertion.
The results are sharp for the functions $\psi_i(z)$, $i = 1, 2, 3, 4$ such that

$$\frac{z\psi_1'(z)}{(1-\lambda)\psi_1(z)+\lambda z} = \sqrt{1+z} \quad (\mu < \delta_1 \text{ or } \mu > \delta_2),$$

$$\frac{z\psi_2'(z)}{(1-\lambda)\psi_2(z)+\lambda z} = \sqrt{1+z^2} \quad (\delta_1 < \mu < \delta_2),$$

$$\frac{z\psi_3'(z)}{(1-\lambda)\psi_3(z)+\lambda z} = \sqrt{1+\phi(z)} \quad (\mu = \delta_1),$$

and

$$\frac{z\psi_4'(z)}{(1-\lambda)\psi_4(z)+\lambda z} = \sqrt{1-\phi(z)} \quad (\mu = \delta_2),$$

where

$$\phi(z) = \frac{z(z+\eta)}{1+\eta z} \quad (0 \le \eta \le 1).$$

Thus, the proof of Theorem 3.1 is completed. □

Remark 3.2. *Putting $\lambda = 1$ in Theorem 3.1, we get the result due to Sahoo and Patel (see [28, Corollary 2.2]).*

Remark 3.3. *Putting $\lambda = 0$ in Theorem 3.1, we get the Fekete-Szegö functional for the class SL^* due to Raza and Malik (see [27, Theorem 2.1]).*

Theorem 3.4. *Let the function f given by (1.1) be in the class ML^*_λ. Then, for a complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2(2+\lambda)} \max\left\{1, \left|\frac{3\lambda^2 + 2\lambda + 2\lambda\mu + 4\mu - 1}{4(1+\lambda)^2}\right|\right\}. \tag{3.17}$$

The estimate in (3.17) is sharp.

Proof. From (3.16), we have

$$|a_3 - \mu a_2^2| = \frac{1}{4(2+\lambda)}|p_2 - \nu p_1^2|.$$

Therefore, by virtue of Lemma 2.2, we obtain the desired assertion.
The result is sharp for the function

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \sqrt{1+z},$$

or

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \sqrt{1+z^2}.$$

□

Remark 3.5. *Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 3.4, we get the result of Raza and Malik (see [27, Theorem 2.2]) and Sahoo and Patel (see [28, Thoerem 2.1]) respectively.*

Taking $\lambda = 0$ and $\mu = 1$ in Theorem 3.4, we get the result for $|H_2(1)|$ as follows.

Corollary 3.6. *[27] If the function f, given by (1.1) belongs to the class SL^*, then*

$$|a_3 - a_2^2| \leq \frac{1}{4}.$$

Further, putting $\lambda = \mu = 1$ and $\lambda = 1, \mu = 0$ in Theorem 3.4, we have the following results due to Sahoo and Patel [28].

Corollary 3.7. *[28, Corollary 2.1] If the function f, given by (1.1) belongs to the class $\bar{\mathcal{R}}$, then*

$$|a_3 - a_2^2| \leq \frac{1}{6} \text{ and } |a_3| \leq \frac{1}{6}. \tag{3.18}$$

The estimates are sharp.

Now, we determine the sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the class ML^*_λ.

Theorem 3.8. *Let $f \in \mathcal{A}$ given by (1.1) be in the class ML^*_λ. Assume that its coefficients a_2, a_3 and a_4 are given by (3.13), (3.14) and (3.15), with $p_1 > 0$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{4(2+\lambda)^2}. \tag{3.19}$$

The estimate in (3.19) is sharp.

Proof. From (3.13), (3.14) and (3.15), we have

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{p_1}{16(1+\lambda)(3+\lambda)}\left(p_3 - \frac{7\lambda^2+16\lambda+7}{4(1+\lambda)(2+\lambda)}p_1p_2 + \frac{13+40\lambda+25\lambda^2}{32(1+\lambda)(2+\lambda)}p_1^3\right) \\ &\quad - \left[\frac{1}{4(2+\lambda)}\left(p_2 - \frac{3+7\lambda}{8(1+\lambda)}p_1^2\right)\right]^2 \\ &= \frac{1}{16}\left[\frac{p_1p_3}{(1+\lambda)(3+\lambda)} - \frac{p_2^2}{(2+\lambda)^2}\right. \\ &\quad + \frac{-5-6\lambda+\lambda^2}{4(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p_1^2p_2 \\ &\quad \left. + \frac{25+51\lambda-9\lambda^2+\lambda^3}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p_1^4\right]. \end{aligned} \tag{3.20}$$

For convenience of notation, we write $p_1 = p$ $(0 \le p \le 2)$. Putting the values of p_2 and p_3 from Lemma 2.3 in (3.20), we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{1}{16}\left|\frac{p_1\{p^3 + 2(4-p^2)px - (4-p^2)px^2 + 2(4-p^2)(1-|x|^2)z\}}{4(1+\lambda)(3+\lambda)}\right. \\ &\quad - \frac{(5+6\lambda-\lambda^2)p^2\{p^2+(4-p^2)x\}}{8(1+\lambda)^2(2+\lambda)^2(3+\lambda)} \\ &\quad - \frac{\{p^2+(4-p^2x)\}^2}{4(2+\lambda)^2} \\ &\quad \left. + \frac{25+51\lambda-9\lambda^2+\lambda^3}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p^4\right| \\ &= \frac{1}{16}\left|\frac{\lambda^3-\lambda^2+19\lambda+1}{64(1+\lambda)^4(2+\lambda)^2(3+\lambda)}p^4\right. \\ &\quad - \frac{1+2\lambda-\lambda^2}{8(1+\lambda)^2(2+\lambda)^2(3+\lambda)}(4-p^2)p^2x \\ &\quad - \frac{p^2+4\lambda^2+16\lambda+12}{4(1+\lambda)(2+\lambda)^2(3+\lambda)}(4-p^2)x^2 \\ &\quad \left. + \frac{p(4-p^2)(1-|x|^2)z}{2(1+\lambda)(3+\lambda)}\right|, \end{aligned} \tag{3.21}$$

for some x $(|x| \le 1)$ and for some z $(|z| \le 1)$. An application of triangle inequality in (3.21) and replacing $|x|$ by y in the resulting equation with assumption that

$(p^2+4\lambda^2+16\lambda+12-8p-8\lambda p-2\lambda^2 p)>0$, we get

$$\begin{aligned}|a_2a_4-a_3^2| &\le \frac{1}{16}\left[\frac{\lambda^3-\lambda^2+19\lambda+1}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p^4\right.\\ &\quad+\frac{(1+2\lambda-\lambda^2)(4-p^2)p^2y}{8(1+\lambda)^2(2+\lambda)^2(3+\lambda)}\\ &\quad+\frac{p^2+4\lambda^2+16\lambda+12-8p-8\lambda p-2\lambda^2 p}{4(1+\lambda)(2+\lambda)^2(3+\lambda)}(4-p^2)y^2\\ &\quad\left.+\frac{(4-p^2)p}{2(1+\lambda)(3+\lambda)}\right]\\ &= F(p,y;\lambda)\quad (0\le p\le 2,\ 0\le y\le 1)(say). \qquad (3.22)\end{aligned}$$

Differentiating on both sides of (3.22) with respect to y, we have

$$\begin{aligned}\frac{\partial F(p,y;\lambda)}{\partial y} &= \frac{1}{16}\left[\frac{(1+2\lambda-\lambda^2)(4-p^2)p^2}{8(1+\lambda)^2(2+\lambda)^2(3+\lambda)}\right.\\ &\quad\left.+\frac{(p^2+4\lambda^2+16\lambda+12-8p-8\lambda p-2\lambda^2 p)}{2(1+\lambda)(2+\lambda)^2(3+\lambda)}(4-p^2)y\right] \qquad (3.23)\end{aligned}$$

It is observed that $\frac{\partial F(p,y;\lambda)}{\partial y}>0$ for $0<p<2,\ 0<y<1$. Thus $F(p,y;\lambda)$ is an increasing function of y which implies $F(p,y;\lambda)$ cannot have maximum in the interior on the closed rectangle $[0,2]\times[0,1]$. Therefore, for fixed $p\in[0,2]$,

$$\max_{0\le y\le 1} F(p,y;\lambda)=F(p,1,\lambda)=H(p;\lambda)(say), \qquad (3.24)$$

where

$$\begin{aligned}H(p;\lambda)&=\frac{1}{16}\left[\frac{\lambda^3-\lambda^2+19\lambda+1}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p^4+\frac{(1+2\lambda-\lambda^2)(4-p^2)p^2}{8(1+\lambda)^2(2+\lambda)^2(3+\lambda)}\right.\\ &\quad\left.+\frac{(4-p^2)p}{2(1+\lambda)(3+\lambda)}+\frac{p^2+4\lambda^2+16\lambda+12-8p-8\lambda p-2\lambda^2 p}{4(1+\lambda)(2+\lambda)^2(3+\lambda)}(4-p^2)\right]. \qquad (3.25)\end{aligned}$$

Therefore

$$\begin{aligned}H'(p;\lambda)&=\frac{1}{16}\left[\frac{\lambda^3-\lambda^2+19\lambda+1}{16(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p^3+\frac{p(1+2\lambda-\lambda^2)(2-p^2)}{2(1+\lambda)^2(2+\lambda)^2(3+\lambda)}\right.\\ &\quad\left.-\frac{3p^2}{2(1+\lambda)(3+\lambda)}-\frac{p(4p^2+8\lambda^2+32\lambda+16+16p+6\lambda^2 p+24\lambda p-2p^2)}{4(1+\lambda)(2+\lambda)^2(3+\lambda)}\right],\end{aligned}$$

and

$$\begin{aligned}H''(p;\lambda)&=\frac{1}{16}\left[\frac{3(\lambda^3-\lambda^2+19\lambda+1)}{16(1+\lambda)^2(2+\lambda)^2(3+\lambda)}p^2+\frac{(1+2\lambda-\lambda^2)(2-3p^2)}{2(1+\lambda)^2(2+\lambda)^2(3+\lambda)}\right.\\ &\quad\left.-\frac{3p}{(1+\lambda)(3+\lambda)}-\frac{12p^2+8\lambda^2+32\lambda+16+32p+12\lambda^2 p+48\lambda p-6p^2+16p}{4(1+\lambda)(2+\lambda)^2(3+\lambda)}\right].\end{aligned}$$

By elementary calculus we have $H''(p;\lambda) < 0$ for $0 \le p \le 2$ and $H(p;\lambda)$ has real critical point at $p = 0$. This shows that the maximum of $H(p;\lambda)$ occurs at $p = 0$. Thus, the upper bound in (3.22) corresponds to $p = 0$ and $y = 1$ from which we get the required estimate (3.19).
The result is sharp for the functions

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z} = \sqrt{1+z},$$

or

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z} = \sqrt{1+z^2}.$$

The proof of Theorem 3.8 is thus completed. □

Remark 3.9. *Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 3.8, we get the result of Raza and Malik (see [27]) and Sahoo and Patel (see [28]).*

The sharp upper bound for the fourth coefficient of the function $f \in ML^*_\lambda$ is given by the following theorem.

Theorem 3.10. *Let the function f given by (1.1) be in the class ML^*_L. Then*

$$|a_4| \le \frac{1}{2(3+\lambda)} \quad (0 \le \lambda \le 1). \tag{3.26}$$

Proof. Proceeding similarly as in the proof of Theorem 3.8 and making use of Lemma 2.2 in (3.15) assuming that $(1-4\lambda-3\lambda^2) > 0$, it follows that

$$\begin{aligned} |a_4| \le \frac{1}{16(3+\lambda)}\Bigg[&\frac{1+5\lambda^2}{8(1+\lambda)(2+\lambda)}p^3 + \frac{(1-4\lambda-3\lambda^2)}{2(1+\lambda)(2+\lambda)}(4-p^2)py \\ &+(4-p^2)py^2 + 2(4-p^2)(1-y^2)\Bigg] \\ &= T(p,y;\lambda) \text{ (say)} . \end{aligned} \tag{3.27}$$

Now we maximize the function $T(p,y;\lambda)$ on the closed rectangle $[0,2]\times[0,1]$. Suppose that the maximum of T occurs at the interior point of $[0,2]\times[0,1]$. Differentiating (3.27) with respect to y, we obtain

$$\frac{\partial T}{\partial y} = \frac{1}{16(3+\lambda)}\left[\frac{(1-4\lambda-3\lambda^2)}{2(1+\lambda)(2+\lambda)}p + 2(p-2)y\right](4-p^2). \tag{3.28}$$

It is clear that $\frac{\partial T}{\partial y} < 0$ for $0 < p < 2$ and $0 < y < 1$. Thus, $T(p,y,\lambda)$ is an decreasing function of y, contradicting our assumption. Therefore,

$$\begin{aligned}\max_{0\le y\le 1} T(p,y;\lambda) = T(p,0,\lambda) = \frac{1}{16(3+\lambda)}\left[\frac{1+5\lambda^2}{8(1+\lambda)(2+\lambda)}p^3 + 2(4-p^2)\right] \\ = S(p) \text{ (say)} . \end{aligned} \tag{3.29}$$

From (3.29), we have

$$S'(p) = \frac{1}{16(3+\lambda)} \left[\frac{3(1+5\lambda^2)p^2}{8(1+\lambda)(2+\lambda)} - 4p \right], \tag{3.30}$$

and

$$S''(p) = \frac{1}{16(3+\lambda)} \left[\frac{3(1+5\lambda^2)p}{4(1+\lambda)(2+\lambda)} - 4 \right]. \tag{3.31}$$

For extreme values of $S(p)$, consider $S'(p) = 0$. From (3.30), we have

$$\frac{3(1+5\lambda^2)p^2}{8(1+\lambda)(2+\lambda)} - 4p = 0$$

$$\Longrightarrow p \left[\frac{3(1+5\lambda^2)p}{8(1+\lambda)(2+\lambda)} - 4 \right] = 0 \tag{3.32}$$

We now discuss the following cases.

Cases 1: If $p = 0$, then

$$S''(p) = -\frac{1}{4(3+\lambda)} < 0.$$

By the second derivative test, $S(p)$ has maximum value at $p = 0$.
Cases 2: If $p \neq 0$, then (3.33) gives

$$p = \frac{32(1+\lambda)(2+\lambda)}{3(1+5\lambda^2)}. \tag{3.33}$$

Using the value of p given in (3.33) in (3.31), we get

$$S''(p) = \frac{1}{4(3+\lambda)} > 0 \quad (0 \leq \lambda \leq 1).$$

Hence by second derivative test, $S(p)$ has minimum value at p, where p is given by (3.33).
From the above discussion, it is clear that $S(p)$ attains its maximum at $p = 0$. Thus, the upper bound in (3.27) corresponds to $p = 0$ and $y = 0$ from which we get the required estimate (3.26).
The estimate in (3.26) is sharp for the function $f \in \mathcal{A}$ given by

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \sqrt{1+z^3} \quad (z \in \mathbb{U}).$$

This complete the prove of Theorem 3.10. □

Remark 3.11. *Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 3.10, we get the upper bounds for $|a_4|$ for the class of SL^* and $\tilde{\mathcal{R}}$ respectively studied by Raza and Malik [27] and Sahoo and Patel [28].*

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