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Semigroup Ideals and Generalized Semiderivations of Prime Near Rings

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ABSTRACT: Let N be a near ring. An additive mapping $F : N \longrightarrow N$ is said to be a generalized semiderivation on N if there exists a semiderivation $d : N \longrightarrow N$ associated with a function $g : N \longrightarrow N$ such that $F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and $F(g(x)) = g(F(x))$ for all $x, y \in N$. In this paper we prove some theorems in the setting of semigroup ideal of a 3-prime near ring admitting a nonzero generalized semiderivation associated with a nonzero semiderivation, thereby extending some known results on derivations, semiderivations and generalized derivations.

Key Words: 3-prime near-rings, Semiderivations, Generalized Semiderivations.

Contents

1. Introduction

Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative centre Z; and for any pair of elements $x, y \in N$, $[x, y]$ denotes the commutator $xy - yx$ while the symbol (x, y) denotes the additive commutator $x + y - x - y$. An element x of N is said to be distributive if $(y + z)x = yx + zx$, for all $y, z \in N$. A near ring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields that $x0 = 0$). The near ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is called 2-torsion free if $(N, +)$ has no element of order 2. A nonempty subset U of N is called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$); and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. An additive mapping $F : N \longrightarrow N$ is said to be a right (resp. left) generalized derivation with associated derivation D if $F(xy) = F(x)y + xd(y)$ (resp.

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$F(xy) = d(x)y + xF(y)$), for all $x, y \in N$ and F is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation D. Motivated by a definition given by Bergen [8] for rings. The first author in [1] defined an additive mapping $d : N \longrightarrow N$ is said to be a semiderivation on a near ring N if there exists a function $g : N \longrightarrow N$ such that $(i)d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $(ii)d(g(x)) = g(d(x))$, for all $x, y \in N$. In case g is the identity map on N, d is of course just a derivation on N, so the notion of semiderivation generalizes that of derivation. But the generalization is not trivial for example take $N = N_1 \oplus N_2$, where N_1 is a zero symmetric near ring and N_2 is a ring. Then the map $d : N \longrightarrow N$ defined by $d((x, y)) = (0, y)$ is a semiderivation associated with function $g : N \longrightarrow N$ such that $g(x, y) = (x, 0)$. However d is not a derivation on N. An additive mapping $F : N \rightarrow N$ is said to be a generalized semiderivation of N if there exists a semiderivation $d : N \longrightarrow N$ associated with a map $g : N \longrightarrow N$ such that $(i)F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and $(ii)F(g(x)) = g(F(x))$ for all $x, y \in N$. All semiderivations are generalized semiderivations. Moreover, if g is the identity map on N, then all generalized semiderivations are merely generalized derivations, again the notion of generalized semiderivation generalizes that of generalized derivation. Moreover, the generalization is not trivial as the following example shows:

Example 1.1. *Let S be a 2-torsion free left near ring and let*

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}.$$

Define maps $F, d, g : N \rightarrow N$ by

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

and

$$g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be verified that N is a left near ring and F is a generalized semiderivation with associated semiderivation d and a map g associated with d. However F is not a generalized derivation on N.

Very recently Bell et al. [5] proved some results characterizing commutativity of a 3-prime near ring satisfying certain identities involving derivation and generalized derivation.

In this paper we investigate some identities with generalized semiderivations. More precisely we prove that a 3-prime near ring N is commutative ring if one of the

following holds:
(i) $F([u,v]) = \pm[u,v]$; $(ii) F(u \circ v) = \pm(u \circ v)$, (iii) $F([u,v]) = \pm[F(u),v]$ for all $u,v \in U$, a nonzero semigroup ideal of N.
Infact our theorems extend the results of [1], [3], [4], [5], [6], [7] proved for derivations, semiderivations and generalized derivations.

2. Preliminary Results

We begin with several Lemmas, some of them have been proved elsewhere.

Lemma 2.1. *[3, Lemma 1.3 and 1.4]* Let N be 3-prime near ring and U be a nonzero semigroup ideal of N.

(i) If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

(ii) If $x \in N$ and $xU = \{0\}$, or $Ux = \{0\}$, then $x = 0$.

(iii) If $x \in N$ centralizes U, then $x \in Z$.

Lemma 2.2. *[3, Lemma 1.2]* Let N be 3-prime near ring.

(i) If $z \in Z\backslash\{0\}$, then z is not a zero divisor.

(ii) If $Z\backslash\{0\}$ contains an element z for which $z + z \in Z$, then $(N,+)$ is abelian.

(iii) If $z \in Z\backslash\{0\}$ and x is an element of N such that $xz \in Z$, then $x \in Z$.

Lemma 2.3. *[3, Lemma 1.5]* If N is a 3-prime near ring and Z contains a nonzero semigroup ideal, then N is a commutative ring.

Lemma 2.4. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d associated with a map g, then $d \neq 0$ on U.

Proof: Let $d(u) = 0$, for all $u \in U$. Replacing u by xu, we get $d(xu) = 0$, for all $x \in N$ and $u \in U$. Thus $d(x)g(u) + xd(u) = 0$, for all $x \in N$ and $u \in U$, i.e., $d(x)g(u) = 0$. The result follows by Lemma 2.1(ii). □

Lemma 2.5. Let N be a 3-prime near ring admitting a nonzero semiderivation d with a map g such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. Then N satisfies the following partial distributive law:

$$(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

Proof: Let $x, y, z \in N$, by defining d we have

$$d(xyz) = d(xy)z + g(xy)d(z) = (d(x)y + g(x)d(y))z + g(x)g(y)d(z). \quad (2.1)$$

On the other hand,

$$d(xyz) = d(x)yz + g(x)d(yz) = d(x)yz + g(x)(d(y)z + g(y)d(z))$$
$$= d(x)yz + g(x)d(y)z + g(x)g(y)d(z). \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

□

Lemma 2.6. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $a \in N$ and $ad(U) = 0$, (or $d(U)a = 0$), then $a = 0$.

Proof: Let $ad(u) = 0$, for all $u \in U$. Replacing u by uv, we get $ad(uv) = 0$, for all $u, v \in U$ or $a(d(u)g(v)+ud(v)) = 0$, for all $u, v \in U$, we get $ad(u)g(v)+aud(v) = 0$, for all $u, v \in U$ or $aud(v) = 0$, for all $u, v \in U$. Choosing v such that $d(v) \neq 0$ and applying Lemma 2.1(i), we get $a = 0$.
Also, let $d(v)a = 0$, for all $v \in U$. Replacing v by uv, we get $d(uv)a = 0$, for all $u, v \in U$ or $(d(u)v + g(u)d(v))a = 0$, for all $u, v \in U$. Using Lemma 2.5, we get $d(u)va + g(u)d(v)a = 0$, for all $u, v \in U$ or $d(u)va = 0$ or $d(u)va = 0$, for all $u, v \in U$. Choosing u such that $d(u) \neq 0$ and applying Lemma 2.1(i), we get $a = 0$.
□

Lemma 2.7. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If d is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$, then $d^2(U) \neq \{0\}$.

Proof: Suppose $d^2(U) = \{0\}$. Then for $u, v \in U$ exploit the definition of d in different ways to obtain

$$0 = d^2(uv) = d(d(uv)) = d(d(u)v + g(u)d(v)) \text{ for all } u, v \in U,$$
$$= d^2(u)v + g(d(u))d(v) + d(g(u))d(v) + g(u)d^2(v),$$
$$= d(g(u))d(v) + d(g(u))d(v).$$

Note that $g(d(u)) = d(g(u))$ and $g(U) = U$, we get

$$2d(u)d(v) = 0 \text{ for all } u, v \in U.$$

Since N is a 2-torsion free, we get

$$d(u)d(v) = 0 \text{ for all } u, v \in U. \tag{2.3}$$

Replacing v by wv in (2.3), we get

$$d(u)d(wv) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)(d(w)v + g(w)d(v)) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)d(w)v + d(u)g(w)d(v) = 0 \text{ for all } u, v, w \in U.$$

This implies that

$$d(u)g(w)d(v) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)wd(v) = 0 \text{ for all } u, v, w \in U.$$

$$d(U)Ud(U) = \{0\}.$$

Thus we obtain that $d = 0$, a contradiction. □

Lemma 2.8. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose d is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$. If $d(U) \subseteq Z$, then N is a commutative ring.

Proof: We begin by showing that $(N, +)$ is abelian, which by Lemma 2.2(ii) is accomplished by producing $z \in Z \backslash \{0\}$ such that $z + z \in Z$. Let a be an element of U such that $d(a) \neq 0$. Then for all $x \in N$, $ax \in U$ and $ax + ax = a(x + x) \in U$, so that $d(ax) \in Z$ and $d(ax) + d(ax) \in Z$; hence we need only show that there exists $x \in N$ such that $d(ax) \neq 0$. Suppose this is not the case, so that $d((ax)a) = 0 = d(ax)g(a) + axd(a) = axd(a)$ for all $x \in N$. Since $d(a)$ is not zero divisor by Lemma 2.2(i), we get $aN = \{0\}$, so that $a = 0$ - a contradiction. Therefore $(N, +)$ is abelian as required.
We are given that $[d(u), x] = 0$ for all $u \in U$ and $x \in N$. Replacing u by uv, we get $[d(uv), x] = 0$, which yields $[d(u)v + g(u)d(v), x] = 0$ for all $u, v \in U$ and $x \in N$. Since $(N, +)$ is abelian and $d(U) \subseteq Z$, we have

$$d(u)[v, x] + d(v)[x, g(u)] = 0 \text{ for all } u, v \in U \text{ and } x \in N. \tag{2.4}$$

Replacing x by $g(u)$, we obtain $d(u)[v, g(u)] = 0$ for all $u, v \in U$; and choosing $u \in U$ such that $d(u) \neq 0$ and applying Lemma 2.1(iii), we get $g(u) \in Z$. It then follows from (2.4) that $d(u)[v, x] = 0$ for all $v \in U$ and $x \in N$; therefore $U \subseteq Z$ and Lemma 2.3 completes the proof. □

Lemma 2.9. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that d is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $[d(U), d(U)] = \{0\}$, then N is a commutative ring.

Proof: We are given that $[d(U), d(U)] = \{0\}$. Then $d(u)d(vd(w)) = d(vd(w))d(u)$, for all $u, v, w \in U$, i.e., $d(u)(d(v)g(d(w)) + vd^2(w)) = (d(v)g(d(w)) + vd^2(w))d(u)$, for all $u, v, w \in U$. Then by Lemma 2.5, we get

$$d(u)d(v)g(d(w)) + d(u)vd^2(w)) = d(v)g(d(w))d(u) + vd^2(w)d(u);$$
$$d(u)d(v)d(g(w)) + d(u)vd^2(w)) = d(v)d(g(w))d(u) + vd^2(w)d(u);$$
$$d(u)d(v)d(w) + d(u)vd^2(w)) = d(v)d(w)d(u) + vd^2(w)d(u) \text{ for all } u, v, w \in U$$

and since $[d(U), d(U)] = \{0\}$, we obtain

$$d(u)vd^2(w)) = vd^2(w)d(u) \text{ for all } u, v, w \in U. \tag{2.5}$$

Replace v by xv in (2.5), to get

$$d(u)xvd^2(w)) = xvd^2(w)d(u) \text{ for all } u, v, w \in U \text{ and } x \in N.$$

Using (2.5), the above relation yields that $d(u)xvd^2(w)) = xd(u)vd^2(w)$, for all $u, v, w \in U$ and $x \in N$, i.e.,$[d(u), x]vd^2(w) = 0$, for all $u, v, w \in U$ and $x \in N$ by Lemma 2.5. Thus $[d(u), x]Ud^2(w) = 0$, for all $u, w \in U$ and $x \in N$. Since $d^2(U) \neq 0$ by Lemma 2.7, Lemma 2.1(i) gives $d(U) \subseteq Z$, and the result follows by Lemma 2.8. □

Lemma 2.10. Let N be a 3-prime near ring admitting a generalized semiderivation F associated with a semiderivation d. If g is an onto map associated with d such that $g(xy) = g(x)g(y)$ for all $x, y \in N$, then N satisfies the following partial distributive laws:

(i) $(F(x)y + g(x)d(y))z = F(x)yz + g(x)d(y)z$ for all $x, y, z \in N$.

(ii) $(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z$ for all $x, y, z \in N$.

Proof: (i) Let $x, y, z \in N$, by defining F we have

$$F(xyz) = F(xy)z + g(xy)d(z) = (F(x)y + g(x)d(y))z + g(x)g(y)d(z). \tag{2.6}$$

On the other hand,

$$\begin{aligned} F(xyz) = F(x)yz + g(x)d(yz) &= F(x)yz + g(x)(d(y)z + g(y)d(z)) \\ &= F(x)yz + g(x)d(y)z + g(x)g(y)d(z). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$(F(x)y + g(x)d(y))z = F(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

(ii) Let $x, y, z \in N$, by defining F we have

$$F(xyz) = F(xy)z + g(xy)d(z) = (d(x)g(y) + xF(y))z + g(x)g(y)d(z). \tag{2.8}$$

On the other hand,

$$F(xyz) = d(x)g(yz) + xF(yz) = d(x)g(y)g(z) + x(F(y)z + g(y)d(z))$$

$$= d(x)g(y)z + xF(y)z + g(x)g(y)d(z). \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$(d(x)g(y) + xF(y))z = d(x)g(y)z + xF(y)z \text{ for all } x, y, z \in N.$$

□

Lemma 2.11. Let N be a 3-prime near ring and U a nonzero semigroup ideal of N. If F is a nonzero generalized semiderivation of N with associated semiderivation d and a map g associated with d such that $g(U) = U$, then $F \neq 0$ on U.

Proof: Let $F(u) = 0$ for all $u \in U$. Replacing u by ux, we get $F(ux) = 0$ for all $u \in U$ and $x \in N$. Thus

$$F(u)x + g(u)d(x) = 0 = Ud(x) \text{ for all } x \in N$$

and it follows by Lemma 2.1(ii) that $d = 0$. Therefore, we have

$$F(xu) = F(x)u = 0 \text{ for all } u \in U \text{ for all } x \in N$$

and another appeal to Lemma 2.1(ii) gives $F = 0$, which is a contradiction. □

Lemma 2.12. Let N be a 3-prime near ring and U a nonzero semigroup ideal of N. Suppose that F is a nonzero generalized semiderivation of N with associated semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $a \in N$ and $aF(U) = 0$ (or $F(U)a = 0$) then $a = 0$.

Proof: Suppose that $aF(U) = \{0\}$. Then for $u, v \in U$

$$aF(uv) = aF(u)v + ag(u)d(v) = aud(v) = 0 \text{ for all } u, v \in U \text{ and } a \in N.$$

So by Lemma 2.1(i), $a = 0$ or $d(U) = \{0\}$. If $d(U) = \{0\}$, then

$$ad(u)g(v) + auF(v) = 0 = auF(v) \text{ for all } u, v \in U$$

and since $F(U) \neq \{0\}$ by Lemma 2.11, $a = 0$.

Suppose that $F(U)a = \{0\}$. Then for $u, v \in U$

$$F(uv)a = (F(u)v + g(u)d(v))a = 0.$$

Using Lemma 2.10(i), we get

$$F(u)va + g(u)d(v)a = ud(v)a = 0 \text{ for all } u, v \in U \text{ and } a \in N.$$

Thus by Lemma 2.1(i), $a = 0$ or $d(U) = \{0\}$. If $d(U) = \{0\}$, then

$$d(u)g(v)a + uF(v)a = 0 = uF(v)a \text{ for all } u, v \in U.$$

Since $F(U) \neq \{0\}$, it follows by Lemma 2.11 that $a = 0$.

□

Lemma 2.13. Let N be a 2-torsion free 3-prime near ring and U a nonzero semigroup ideal of N. If N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $F(U) \subseteq U$, then $F^2 \neq \{0\}$.

Proof: Let $F^2(U) = \{0\}$. Then for $u, v \in U$, we have

$$0 = F^2(uv) = F(F(uv)).$$

$$= F(F(u)v + g(u)d(v))$$

$$= F(F(u)v) + F(g(u)d(v))$$

$$= g(F(u))d(v) + F(u)d(v) + g(u)d^2(v)$$

$$= F(g(u))d(v) + F(u)d(v) + ud^2(v)$$

$$= F(u)d(v) + F(u)d(v) + ud^2(v).$$

This implies that

$$2F(u)d(v) + ud^2(v) = 0 \text{ for all } u, v \in U. \tag{2.10}$$

Replacing u by $F(u)$ in (2.10), we get

$$2F(F(u))d(v) + F(u)d^2(v) = 0 \text{ for all } u, v \in U.$$

This implies that

$$2F^2(u)d(v) + F(u)d^2(v) = 0.F(u)d^2(v) = 0.$$

By Lemma 2.12, we obtain that $d^2(v) = 0$ or $F(U) = \{0\}$. If $d^2(v) = 0$, then $d = 0$ by Lemma 2.7, a contradiction. So, we find $F(U) = \{0\}$, again a contradiction by Lemma 2.11. □

3. Main Results

The theorems that we prove in this section extend the results proved [2, Theorem 3.1], [3, Theorem 3.1], [4, Theorem 2.1] and [6, Theorem 3.3].

Theorem 3.1. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F with associated semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $F(U) \subseteq Z$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.

Proof: We begin by showing that $(N, +)$ is abelian, which by Lemma 2.2(ii) is accomplished by producing $z \in Z\backslash\{0\}$ such that $z + z \in Z$. Let a be an element of U such that $F(a) \neq 0$. Then for all $x \in N$, $ax \in U$ and $ax + ax = a(x + x) \in U$, so that $F(ax) \in Z$ and $F(ax) + F(ax) \in Z$; hence we need only to show that there exists $x \in N$ such that $F(ax) \neq 0$. Suppose that this is not the case, so that $F((ax)a) = 0 = F(ax)a + g(ax)d(a) = g(a)g(x)d(a) = axd(a)$ for all $x \in N$. By Lemma 2.1(i) either $a = 0$ or $d(a) = 0$.

If $d(a) = 0$, then $F(xa) = F(x)a + g(x)d(a)$; that is, $F(xa) = F(x)a \in Z$, for all $x \in N$. Thus, $[F(u)a, y] = 0$ for all $y \in N$ and $u \in U$. This implies that $F(u)[a, y] = 0$ for all $u \in U$ and $y \in N$ and Lemma 2.2(i) gives $a \in Z$. Thus, $0 = F(ax) = F(xa) = F(x)a$ for all $x \in N$. Replacing x by $u \in U$, we have $F(U)a = 0$, and by Lemma 2.2(i) and Lemma 2.11, we have a contradiction.

To complete the proof, we show that if N is 2-torsion free, then N is commutative.
Consider first case $d = 0$. This implies that $F(ux) = F(u)x \in Z$ for all $u \in U$ and $x \in N$. By Lemma 2.11, we have $u \in U$ such that $F(u) \in Z\backslash\{0\}$, so N is commutative by Lemma 2.2(iii).

Now consider the case $d \neq 0$. Let $c \in Z\backslash\{0\}$. This implies that $x \in U$, $F(xc) = F(x)c + g(x)d(c) = F(x)c + xd(c) \in Z$. Thus $(F(x)c + xd(c))y = y(F(x)c + xd(c))$ for all $x, y \in U$ and $c \in Z$. Therefore, by Lemma 2.10(i), $F(x)cy + xd(c)y = yF(x)c + yxd(c)$ for all $x, y \in U$ and $c \in Z$. Since $d(c) \in Z$ and $F(x) \in Z$, we obtain $d(c)[x, y] = 0$ for all $x, y \in U$ and $c \in Z$. Let $d(Z) \neq \{0\}$. Choosing c such that $d(c) \neq 0$ and noting that $d(c)$ is not a zero divisor, we have $[x, y] = 0$ for all $x, y \in U$. By Lemma 2.1(iii), $U \subseteq Z$; hence N is commutative by Lemma 2.3.
The remaining case is $d \neq 0$ and $d(Z) = \{0\}$. Suppose we can show that $U \cap Z \neq \{0\}$. Taking $z \in (U \cap Z)\backslash\{0\}$ and $x \in N$, we have $F(xz) = F(x)z \in Z$; therefore $F(N) \subseteq Z$ by Lemma 2.2(iii). Let $F(x) \in Z$ for all $x \in N$.
Since $d(Z) = 0$, for all $x, y \in N$. We have

$$0 = d(F(xy))$$

$$0 = d(F(x)y + g(x)d(y)),$$

$$0 = F(x)d(y) + g(x)d^2(y) + d(g(x))g(d(y)) \text{ for all } x, y \in Z.$$

Hence $F(xd(y)) = -d(g(x))g(d(y)) \in Z$ for all $x, y \in N$. By hypothesis, we have $d(x)d(y) \in Z$ for all $x, y \in N$. This implies that

$$d(x)(d(x)d(y) - d(y)d(x)) = 0 \text{ for all } x, y \in N.$$

Left multiplying by $d(y)$, we arrive at

$$d(y)d(x)N(d(x)d(y) - d(y)d(x)) = \{0\} \text{ for all } x, y \in N.$$

Since N is a 3-prime near ring, we get

$$[d(x), d(y)] = 0 \text{ for all } x, y \in N.$$

Using Lemma 2.9, N is a commutative ring.

Assume that $U \cap Z = \{0\}$. For each $u \in U$, $F(u^2) = F(u)u + g(u)d(u) = F(u)u + ud(u) = u(F(u) + d(u)) \in U \cap Z$. So $F(u^2) = 0$, thus for all $u \in U$ and $x \in N$, $F(u^2x) = F(u^2)x + g(u^2)d(x) = u^2d(x) \in U \cap Z$. So $u^2d(x) = 0$ and Lemma 2.6, $u^2 = 0$. Since $F(xu) = F(x)u + g(x)d(u) = F(x)u + xd(u) \in Z$ for all $u \in U$ and $x \in N$. We have $(F(x)u + xd(u))u = u(F(x)u + xd(u))$ and right multiplying by u gives $uxd(u)u = 0$. Consequently, $d(u)uNd(u)u = \{0\}$. So that $d(u)u = 0$ for all $u \in U$,so $F(u)u = 0$ for all $u \in U$. But by Lemma 2.11, there exist $u_0 \in U$ for which $F(u_0) \neq 0$; and $F(u_0) \in Z$, we get $u_0 = 0$, contradiction. Therefore, $U \cap Z \neq \{0\}$ as required. □

Theorem 3.2. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F associated with a semiderivation d and a map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $[F(U), F(U)] = \{0\}$, then $(N, +)$ is abelian.

Proof: Assume that $x \in U$ such that

$$[x, F(U)] = [x + x, F(U)] = 0 \text{ and } u, v \in U \text{ such that } u + v \in U.$$

Then

$$[x + x, F(u + v)] = 0$$

$$(x + x)F(u + v) = F(u + v)(x + x),$$

$$(x + x)(F(u) + F(v)) = F(u + v)x + F(u + v)x,$$

$$(x + x)F(u) + (x + x)F(v) = xF(u + v) + xF(u + v),$$

$$F(u)(x + x) + F(v)(x + x) = x(F(u) + F(v)) + x(F(u) + F(v)),$$

$$F(u)(x + x) + F(v)(x + x) = xF(u) + xF(v) + xF(u) + xF(v),$$

$$xF(u) + xF(u) + F(v) + xF(v) = xF(u) + xF(v) + xF(u) + xF(v),$$

$$xF(u) + xF(v) = xF(v) + xF(u),$$

i.e.,

$$xF(u) + xF(v) - xF(v) - xF(u) = 0,$$

$$x(F(u) + F(v) - F(v) - F(u)) = 0,$$

$$xF(u + v - u - v) = 0 \text{ for all } u, v, x \in U.$$

This gives that $xF(c) = 0$, where $c = u + v - u - v$. Let $a, b \in U$. Then $ab \in U$ and $ab + ab = a(b + b) \in U$. Since $[F(U), F(U)] = \{0\}$, we take $x = F(ab)$ so that

$$[F(ab) + F(ab), F(u + v)] = 0$$

which gives $F(c)F(ab) = 0$, i.e., $F(c)F(U^2) = 0$.

Since U^2 is also a semigroup ideal. By Lemma 2.12, $F(c) = 0$, i.e.,

$$F(u + v - u - v) = 0 \text{ for all } u, v \in U \text{ such that } u + v \in U. \tag{3.1}$$

Replacing u by ry and v by rz for $y, z \in N$ and $u, r \in U$, we have $u, v \in U$ and $u + v = ry + rz = r(y + z) \in U$.

Then (3.1) implies that

$$F(ry + rz - ry - rz) = 0 \text{ for all } r \in U \text{ and } y, z \in N. \tag{3.2}$$

Now replace r by wr, to get

$$F(wry + wrz - wry - wrz) = 0 \text{ for all } r, w \in U \text{ and } y, z \in N,$$

$$F(w(ry + rz - ry - rz)) = 0 \text{ for all } r, w \in U \text{ and } y, z \in N.$$

This implies that

$$d(w)g(ry + rz - ry - rz) + wF(ry + rz - ry - rz) = 0.$$

Using (3.2), we obtain

$$d(w)g(ry + rz - ry - rz) = 0 \text{ for all } r, w \in U \text{ and } y, z \in N,$$

$$d(U)g(ry + rz - ry - rz) = \{0\} \text{ for all } r \in U \text{ and } y, z \in N.$$

Application of Lemma 2.6 gives that

$$(ry + rz - ry - rz) = 0 \text{ for all } r \in U \text{ and } y, z \in N,$$

$$r(y + z - y - z) = 0 \text{ for all } r \in U \text{ and } y, z \in N,$$

$$U(y + z - y - z) = \{0\} \text{ and } y, z \in N.$$

Now using Lemma 2.1(ii), we get $y + z - y - z = 0$ for all $y, z \in N$. Hence $(N, +)$ is abelian. □

Theorem 3.3. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F with associated nonzero semiderivation d and a map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $[F(U), F(U)] = \{0\}$, then N is a commutative ring.

Proof: We are given that $[F(U), F(U)] = \{0\}$. Then

$$F(u)F(F(w)v) = F(F(w)v)F(u) \text{ for all } u, v, w \in U,$$

$$F(u)(d(F(w))g(v) + F(w)F(v)) = (d(F(w))g(v) + F(w)F(v))F(u).$$

Using Lemma 2.10(ii), we get

$$F(u)d(F(w))g(v) + F(u)F(w)F(v) = d(F(w))g(v)F(u) + F(w)F(v)F(u).$$

Then we have

$$F(u)d(F(w))g(v) = d(F(w))g(v)F(u) \text{ for all } u, v, w \in U. \tag{3.3}$$

Thus

$$F(u)d(F(w))v = d(F(w))vF(u). \tag{3.4}$$

Replacing v by vt for all $t \in N$ and using (3.4), we get

$$d(F(w))vF(u)t = d(F(w))vtF(u),$$

i.e.

$$d(F(w))v[F(u), t] = \{0\} \text{ for all } u, v, w \in U \text{ and } t \in N,$$

$$d(F(w))U[F(u), t] = \{0\} \text{ for all } u, w \in U \text{ and } t \in N.$$

By Lemma 2.1(i), we get either $[F(u), t] = 0$ or $d(F(w)) = 0$. In the first case $F(U) \subseteq Z$ and Theorem 3.1 completes the proof. Let us assume that $d(F(U)) = \{0\}$. Then $0 = d(F(uv)) = d(F(u)v + g(u)d(v)) = d(F(u)v) + d(ud(v)) = F(u)d(v) + d(u)g(d(v)) + ud^2(v) = F(u)d(v) + d(u)d(v) + ud^2(v)$ for all $u, v \in U$, we have

$$F(u)d(v) + d(u)d(v) + ud^2(v) = 0. \tag{3.5}$$

Now replacing u by uw, we get

$$F(uw)d(v) + d(uw)d(v) + uwd^2(v) = 0,$$

$$(d(u)g(w) + uF(w))d(v) + (d(u)g(w) + ud(w))d(v) + uwd^2(v) = 0.$$

Using Lemma 2.5 and 2.10(ii), we have

$$d(u)g(w)d(v) + uF(w)d(v) + d(u)g(w)d(v) + ud(w)d(v) + uwd^2(v) = 0,$$

$$2d(u)wd(v) + u\{F(w)d(v) + d(w)d(v) + wd^2(v)\} = 0.$$

Using (3.5), we obtain

$$2d(u)wd(v) = 0.$$

This implies that

$$2d(u)Ud(v) = 0 \text{ for all } u, v \in U.$$

Since N is 2-torsion free, we get

$$d(u)Ud(v) = \{0\}.$$

Thus, we obtain that $d(U) = \{0\}$, a contradiction by Lemma 2.4. $\square$

Theorem 3.4. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and an additive map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $F([u, v]) = \pm[u, v]$, for all $u, v \in U$, then N is a commutative ring.

Proof: By hypothesis

$$F([u, v]) = \pm[u, v] \text{ for all } u, v \in U. \tag{3.6}$$

Replacing v by vu and using $[u, vu] = [u, v]u$, we get

$$F([u, v]u) = ([u, v]u) \text{ for all } u, v \in U.$$

Thus

$$F([u, v])u + g([u, v])d(u) = \pm([u, v])u \text{ for all } u, v \in U.$$

Using (3.6), we get

$$\pm([u, v])u + g([u, v])d(u) = \pm([u, v])u.$$

This implies that

$$g([u, v])d(u) = 0 \text{ for all } u, v \in U.$$

Thus

$$g([u, v])d(U) = \{0\} \text{ for all } u, v \in U$$

and Lemma 2.6 yields that

$$g([u, v]) = 0 \text{ for all } u, v \in U.$$

Therefore, we get

$$[g(u), g(v)] = 0 \text{ for all } u, v \in U.$$

Since $g(U) = U$, we get

$$[u, v] = 0 \text{ for all } u, v \in U. \tag{3.7}$$

Now replacing u by ur, for all $r \in N$ in (3.7), we find

$$U[r, v] = \{0\}.$$

Using Lemma 2.1(ii) we have $[r, v] = 0$ for all $v \in U$ and $r \in N$. Hence $U \subseteq Z$ and N is commutative ring by Lemma 2.3. □

Theorem 3.5. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and an additive map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $F(u \circ v) = 0$, for all $u, v \in U$, then N is a commutative ring.

Proof: Assume that
$$F(u \circ v) = 0 \text{ for all } u, v \in U. \tag{3.8}$$
Replacing v by vu in (3.8), we get
$$F(u \circ vu) = 0 \text{ for all } u, v \in U.$$
$$F(u(vu) + (vu)u) = 0,$$
$$F((uv + vu)u) = 0,$$
$$F((u \circ v)u) = 0,$$
$$g(u \circ v)d(u) = 0.$$
This implies that
$$g(u \circ v)d(U) = \{0\} \text{ for all } u, v \in U.$$
Using Lemma 2.6, we get
$$g(uv + vu) = 0 \text{ for all } u, v \in U.$$
Since g is additive, we have
$$g(uv) + g(vu) = 0 \text{ for all } u, v \in U.$$
$$g(u)g(v) + g(v)g(u) = 0 \text{ for all } u, v \in U.$$
-i.e.,
$$g(u)g(v) = -g(v)g(u) \text{ for all } u, v \in U. \tag{3.9}$$
Replacing v by vw in (3.9), we get
$$g(u)g(v)g(w) = -g(v)g(w)g(u) \text{ for all } u, v, w \in U.$$
Using (3.9), we obtain
$$g(v)g(u)g(w) = g(v)g(w)g(u) \text{ for all } u, v, w \in U.$$
$$g(v)[g(u), g(w)] = 0 \text{ for all } u, v, w \in U,$$
-i.e.,
$$U[g(u), g(w)] = \{0\} \text{ for all } u, w \in U.$$
By Lemma 2.1(ii), $[g(u), g(w)] = 0$ for all $u, w \in U$. Since $g(U) = U$, it follows that $[u, w] = 0$ for all $u, w \in U$. Arguing in the similar manner as in the proof of Theorem 3.4, we get the result. □

Theorem 3.6. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and an additive map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $F(u \circ v) = \pm(u \circ v)$, for all $u, v \in U$, then N is a commutative ring.

Proof: By hypothesis, we have

$$F(u \circ v) = \pm(u \circ v), \text{ for all } u, v \in U. \tag{3.10}$$

Substituting vu for v in (3.10), we get

$$F(u \circ vu) = \pm(u \circ vu),$$

$$F(u(vu) + (vu)u) = \pm(u(vu) + vu(u)),$$
$$F((uv + vu)u) = \pm((uv + vu)u),$$
$$F((u \circ v)u) = \pm((u \circ v)u),$$
$$F(u \circ v)u + g(u \circ v)d(u) = \pm((u \circ v)u).$$

Using (3.10), we get

$$\pm((u \circ v)u) + g(u \circ v)d(u) = \pm((u \circ v)u).$$

This implies that

$$g(u \circ v)d(u) = 0.$$

Arguing in the similar manner as in the proof of Theorem 3.4 and Theorem 3.5, we get the required result. □

Theorem 3.7. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and an additive map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$, then the following assertions are equivalent:

(i) $F([u, v]) = [F(u), v]$ for all $u, v \in U$.

(ii) $F([u, v]) = -[F(u), v]$ for all $u, v \in U$.

(iii) N is commutative ring.

Proof: It is obvious that (iii) implies both (i) and (ii). Now we prove that (i) implies (iii). By hypothesis, we have

$$F([u, v]) = [F(u), v] \text{ for all } u, v \in U. \tag{3.11}$$

Taking uv instead of v in (3.11) and noting that $[u, uv] = u[u, v]$, we get

$$F([u, uv]) = [F(u), uv],$$

$$F(u[u, v]) = [F(u), uv],$$

$$uF([u, v]) + d(u)g([u, v]) = F(u)uv - uvF(u) \text{ for all } u, v \in U.$$

Using (3.11) and recall that $F(u)u = uF(u)$, we have

$$u[F(u), v] + d(u)g([u, v]) = F(u)uv - uvF(u),$$

$$uF(u)v - uvF(u) + d(u)g([u, v]) = F(u)uv - uvF(u),$$
$$F(u)uv - uvF(u) + d(u)g([u, v]) = F(u)uv - uvF(u),$$
$$d(u)g([u, v]) = 0 \text{ for all } u, v \in U,$$
$$d(u)g(u)g(v) = d(u)g(v)g(u) \text{ for all } u, v \in U. \tag{3.12}$$

Replacing v by vt in (3.12), we get

$$d(u)g(u)g(vt) = d(u)g(vt)g(u),$$

i.e.,

$$d(u)g(u)g(v)g(t) = d(u)g(v)g(t)g(u) \text{ for all } u, v, t \in U.$$

Using (3.12), we find

$$d(u)g(v)g(u)g(t) = d(u)g(v)g(t)g(u) \text{ for all } u, v, t \in U.$$

$$d(u)v(g(u)t - tg(u)) = 0,$$
$$d(u)v[g(u), t] = 0 \text{ for all } u, v, t \in U,$$
$$d(u)U[u, t] = \{0\} \text{ for all } u, t \in U.$$

By Lemma 2.1(i) either $d(U) = \{0\}$ or $[u, t] = 0$. If $d(U) = \{0\}$, we arrive at a contradiction by Lemma 2.4. On the other hand if $[u, t] = 0$, then replacing u by ur, we obtain

$$u[r, t] = 0 \text{ for all } u, t \in U \text{ and } r \in N.$$

Thus, $U[r, t] = \{0\}$ and by Lemma 2.1(ii), we get

$$[r, t] = 0 \text{ for all } t \in U \text{ and } r \in N.$$

Hence $U \subseteq Z$ and N is a commutative ring by Lemma 2.3.

Similarly, we can prove that (ii) implies (iii). □

Theorem 3.8. Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero generalized semiderivation F associated with a nonzero semiderivation d and an additive map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$, then the following assertions are equivalent:

(i) $F([u, v]) = [u, F(v)]$ for all $u, v \in U$.

(ii) $F([u, v]) = -[u, F(v)]$ for all $u, v \in U$.

(iii) N is commutative ring.

Proof: It is obvious that (iii) implies both (i) and (ii). Now we prove that (i) implies (iii). By hypothesis, we have

$$F([u,v]) = [u, F(v)] \text{ for all } u, v \in U. \tag{3.13}$$

Replacing u by vu in (3.13), we get

$$F([vu,v]) = [vu, F(v)] \text{ for all } u, v \in U.$$

Using the fact that $[vu,v] = v[u,v]$, we arrive at

$$F(v[u,v]) = [vu, F(v)],$$

i.e.

$$vF([u,v]) + d(v)g([u,v]) = vuF(v) - F(v)vu \text{ for all } u, v \in U. \tag{3.14}$$

Using (3.13) and noting that $vF(v) = F(v)v$, we have

$$v[u, F(v)] + d(v)g([u,v]) = vuF(v) - F(v)vu,$$

$$vuF(v) - vF(v)u + d(v)g([u,v]) = vuF(v) - vF(v)u,$$

$$d(v)g([u,v]) = 0 \text{ for all } u, v \in U.$$

Arguing in the similar manner as in the proof of Theorem 3.7, we get the result.

Similarly, we can prove that (ii) implies (iii). □

4. Generalized semiderivations acting as a homomorphism or an antihomomorphism

In [7], Bell and Kappe proved that if R is a semiprime ring and d is a derivation on R which is either an endomorphism or an antiendomorphism on R, then $d = 0$. Of course, derivations which are not endomorphisms or antiendomorphisms on R may behave as such on certain subsets of R; for example, any derivation d behaves as the zero endomorphism on the subring C consisting of all constants (i.e., the elements x for which $d(x) = 0$). In fact in a semiprime ring R, d may behave as an endomorphism on a proper ideal of R. However as noted in [7], the behaviour of d is somewhat restricted in the case of a prime ring. The first author in [2] considered (θ, ϕ)-derivation d acting as a homomorphism or an antihomomorphism on a nonzero Lie ideal of a prime ring and concluded that $d = 0$. In this section we establish similar results in the setting of a semigroup ideal of a 3-prime near ring admitting a generalized semiderivation.

Theorem 4.1. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Let F be a nonzero generalized semiderivation of N associated with a semiderivation d and a map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If F acts as a homomorphism on U, then F is identity map on N and $d = 0$.

Proof: By the hypothesis,

$$F(xy) = d(x)g(y) + xF(y) = F(x)F(y) \text{ for all } x, y \in U.$$

Replacing y by yz in the above relation, we obtain

$$F(x)F(yz) = d(x)g(yz) + xF(yz)$$

$$F(x)F(y)F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)),$$
$$F(xy)F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)),$$
$$(d(x)g(y) + xF(y))F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)).$$

Using Lemma 2.10(ii), we get

$$d(x)g(y)F(z) + xF(y)F(z) = d(x)g(yz) + x(d(y)g(z) + yF(z)) \text{ for all } x, y, z \in U$$

$$d(x)g(y)F(z) + xF(yz) = d(x)g(yz) + x(d(y)g(z) + yF(z)) \text{ for all } x, y, z \in U,$$

$$d(x)g(y)F(z)+x(d(y)g(z)+yF(z)) = d(x)g(yz)+xd(y)g(z)+xyF(z)) \text{ for all } x, y, z \in U.$$

$$d(x)g(y)F(z) + xd(y)g(z) + xyF(z) = d(x)g(yz) + xd(y)g(z) + xyF(z).$$

$$d(x)g(y)F(z) = d(x)g(y)g(z).$$

$$d(x)yF(z) = d(x)yz \text{ for all } x, y, z \in U.$$

$$d(x)y(F(z) - z) = 0 \text{ for all } x, y, z \in U.$$

$$d(x)U(F(z) - z) = \{0\} \text{ for all } x, y, z \in U.$$

It follows by Lemma 2.1(i) either $d(U) = \{0\}$ or $F(z) = z$ for all $z \in U$. In fact, as we now show both of these conditions hold. Suppose that $F(u) = u$ for all $u \in U$. Then for all $u \in U$ and $x \in N$, $F(xu) = xu = d(x)g(u) + xF(u) = d(x)u + xu$, hence $d(x)U = \{0\}$ for all $x \in N$ and $d = 0$.
On the other hand, suppose that $d(U) = \{0\}$, so that $d = 0$. Then for all $x, y \in U$, $F(xy) = F(x)y = F(x)F(y)$, so that $F(x)(y - F(y)) = 0$. Replacing y by zy, $z \in N$ and noting that $F(zy) = zF(y)$, we see that $F(x)N(y - F(y)) = \{0\}$ for all $x, y \in U$. Therefore, $F(U) = \{0\}$ or F is the identity map on U. But $F(U) = \{0\}$ contradicts Lemma 2.11, so F is the identity map on U.
Since F is the identity map on U and $F(xy) = xF(y)$ for all $x, y \in N$, $F(ux) = ux = uF(x)$ for all $u \in U$ and $x \in N$. Thus $U(x - F(x)) = \{0\}$ for all $x \in N$. Hence F is the identity map on N. □

Theorem 4.2. Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Let F be a nonzero generalized semiderivation of N associated with a semiderivation d and a map g associated with d such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If F acts as an antihomomorphism on U, then $d = 0$, F is the identity map on N and N is a commutative ring.

Proof: First we show that $d = 0$ if and only if F is the identity map on N. Clearly if F is the identity map on N, $xd(y) = 0$ for all $x, y \in N$, and hence $d = 0$. Conversely, assume that $d = 0$, in which case $F(xy) = F(x)y = xF(y)$ for all $x, y \in N$. It follows that for any $x, y, z \in U$,

$$F(yxz) = F(z)F(yx) = F(z)yF(x) = F(zy)F(x) = zF(y)F(x) = zF(xy). \quad (4.1)$$

On the other hand,

$$\begin{aligned} F(yxz) &= F(xz)F(y) = F(x)zF(y) = F(x)F(zy) \\ &= F(x)F(y)F(z) = F(yx)F(z) = F(y)xF(z) \\ &= F(y)F(xz) = F(y)F(x)z = F(xy)z. \end{aligned} \quad (4.2)$$

Comparing (4.1) and (4.2) shows that $F(U^2)$ centralizes U, so that $F(U^2) \subseteq Z$ by Lemma 2.1(iii). Since U^2 is a nonzero semigroup ideal; hence $F(U^2) \neq \{0\}$ by Lemma 2.11. Suppose that $x, y \in U$ such that $F(xy) \neq 0$, we see that for any $z \in U$, $F(xy)z = F(xyz) = F(yz)F(x) = F(y)F(zx) = F(y)F(x)F(z) = F(xy)F(z)$, and hence $F(xy)(z - F(z)) = 0$. Since $F(xy) \in Z\backslash\{0\}$, we get $F(z) = z$ for all $z \in U$. Hence F is the identity map on N. We note now that if the identity map of N acts as an antihomomorphism on U, then U is commutative, so that by Lemma 2.1(iii) and Lemma 2.3, N is a commutative ring. To complete the proof of our theorem, we need only to argue that $d = 0$. By our antihomomorphism hypothesis

$$F(xy) = d(x)g(y) + xF(y) = F(y)F(x) \text{ for all } x, y \in U. \quad (4.3)$$

Substituting xy for y in (4.3), we obtain

$$F(xy)F(x) = F(xxy) = d(x)g(xy) + xF(xy) \text{ for all } x, y \in U.$$

$$F(xy)F(x) = d(x)g(xy) + xF(y)F(x),$$

$$(d(x)g(y) + xF(y))F(x) = d(x)g(xy) + xF(y)F(x).$$

Using Lemma 2.10(ii), we have

$$d(x)g(y)F(x) + xF(y)F(x) = d(x)g(x)g(y) + xF(y)F(x),$$

$$d(x)g(y)F(x) = d(x)g(x)g(y).$$

$$d(x)yF(x) = d(x)xy \text{ for all } x, y \in U. \quad (4.4)$$

Replacing y by yr in (4.4), we get

$$d(x)yrF(x) = d(x)xyr \text{ for all } x, y \in U \text{ and } r \in N.$$

Using (4.2), we have

$$d(x)yrF(x) = d(x)yF(x)r$$

and so,

$$d(x)y[r, F(x)] = 0 \text{ for all } x, y \in U, r \in N.$$

Application of Lemma 2.1(i) yields that either $d(x) = 0$ or $[r, F(x)] = 0$ i.e., $d(x) = 0$ or $F(x) \in Z$. Suppose that there exists $w \in U$ such that $F(w) \in Z \backslash \{0\}$. Then for all $v \in U$ such that $d(v) = 0$, $F(wv) = F(w)v + g(w)d(v) = F(w)v$;

$$F(wv) = F(w)v = F(v)F(w) = F(w)F(v)$$

and hence $F(w)(v - F(v)) = 0 = v - F(v)$.
Now consider arbitrary $x, y \in U$. If one of $F(x), F(y)$ is in Z, then $F(xy) = F(x)F(y)$. If $d(x) = 0 = d(y)$, then $d(xy) = d(x)y + xd(y) = 0$, so $F(xy) = xy = F(x)F(y)$. Therefore $F(xy) = F(x)F(y)$ for all $x, y \in U$, and by Theorem 4.1, F is the identity map on N, and therefore $d = 0$.
The remaining possibility is that for each $x \in U$, either $d(x) = 0$ or $F(x) = 0$. Let $u \in U \backslash \{0\}$ and let $U_1 = uN$. Then U_1 is a nonzero semigroup right ideal contained in U and U_1 is an additive subgroup of N. The sets $\{x \in U_1 | d(x) = 0\}$ and $\{x \in U_1 | F(x) = 0\}$ are additive subgroups of U_1 with union equal to U_1, so $d(U_1) = \{0\}$ or $F(U_1) = \{0\}$. If $d(U_1) = \{0\}$, then $d = 0$ by Lemma 2.4. Suppose, then, that $F(U_1) = \{0\}$. Then for arbitrary $x, y \in N$, $F(uxy) = F(ux)y + g(ux)d(y) = 0 = uxd(y)$, so $uNd(y) = \{0\}$, and again $d = 0$. This completes the proof. $\square$

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