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Approximation Properties Of (p, q)-Variant Of Stancu-Schurer Operators *

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Abstract: In this article, we introduce (p, q)-variant of Stancu-Schurer operators and discuss the rate of convergence for continuous functions. Further, We discuss recursive estimates, Korovkin-type theorems and direct approximation results using second order modulus of continuity, Peetre's K-functional and Lipschitz class.

Key Words: (p, q)-Integers; (p, q)-Bernstein Operators; (p, q)-Stancu-Schurer.

Contents

1. Introduction

In 1885, Weierstrass gave a very famous result known as Weierstrass approximation theorem which plays an important role in the development of approximation theory. It was considered to be typical until Bernstein gave an elegant proof of it. Bernstein [1] considered polynomials for the continuous functions $f \in C[0, 1]$ defined as follows

$$B_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \ n \in \mathbb{N},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$ and $x \in [0, 1]$. It is a powerful tool for numerical analysis, computer added geometric design (CAGD) and solutions of differential equations. Barbosu introduced two important generalizations of Bernstein i.e. Schurer-Stancu type operators [2] and Kantorovich form of Schurer-Stancu operators [3].

For the last two decades, the application of q-calculus emerged as a new area in the field of approximation theory. Motivated by the applications of q-calculus,

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Lupaş [4] introduced a sequence of Bernstein polynomials based on q-integer. Another form of q-Bernstein operators was given by Philips [5]. Several researchers introduced different type of operators based on q-integers ([6]-[7]). Recently, Mursaleen et al. applied (p,q)-calculus in approximation theory and introduced (p,q)-analogue of Bernstein operators [8], Bernstein-Kantorovich operators [11], Bernstein-Stancu operators [9] (see also [10]). Later on, Szász-Mirakyan operators [12], Szász-Mirakyan-Baskakov operators [13], Bivariate Bernstein Chlodovsky operators [14], Bivariate Schurer Stancu operators [15] (see also references therein) are investigated based on $p;q$-integers. The aim of (p,q)-integers was to generalize several forms of q-oscillator algebras in the earlier physics litrature [16].

Let $0 < q < p \leq 1$. Then, (p,q)-integers for non negative integers n,k are given by

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} \qquad and \qquad [k]_{p,q} = 1 \qquad for \qquad k = 0.$$

(p,q)-binomial coefficient

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

and (p,q)-binomial expansion

$$\begin{aligned} (ax+by)^n_{p,q} &= \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k, \\ (x+y)^n_{p,q} &= (x+y)(px+py)(p^2x+q^2y)...(p^{n-1}x-q^{n-1}y). \end{aligned}$$

Mursaleen et al. [10] defined Bernstein-Schurer operators in the following way

$$B^{p,q}_{n,l}(f;x) = \sum_{\nu=0}^{n+l} b^{\nu}_{n,l}(x;p,q) f\left(\frac{[\nu]_{p,q}}{p^{\nu-(n+l)}[n]_{p,q}}\right), \tag{1.1}$$

for $l = 0,1,2,3,...,n = 1,2,3,...,$ where

$$b^{\nu}_{n,l}(x;p,q) = \frac{1}{p^{\frac{(n+l)(n+l-1)}{2}}} \binom{n+l}{\nu}_{p,q} p^{\frac{\nu(\nu-1)}{2}} x^{\nu} \prod_{j=0}^{n+l-\nu-1} (p^j - q^j x).$$

Motivated by the above developments, we define (p,q)-variant of Stancu-Schurer operators for any
$f \in C[0,l+1], x \in [0,1]$ and $0 \leq \alpha \leq \beta$, as follows

$$S^{\alpha,\beta}_{n,l}(f;x,p,q) = \sum_{\nu=0}^{n+l} b^{\nu}_{n,l}(x;p,q) f\left(\frac{p^{(n+l)-\nu}[\nu]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right), \tag{1.2}$$

for $\nu = 0, 1, 2, 3, ..., n = 1, 2, 3, ...$ One can notice that
(i) for $\alpha = \beta = 0$, (2) reduce to (1).
(ii) for $\alpha = \beta = 0$ and $p = 1$, (2) reduces to q-Stancu-Schurer [17] operators.
In the present paper, we investigate the rate of convergence and Korovkin type theorem the operators defined by (2). Moreover, we discuss direct approximation results using second order modulus of continuity, Peeter's K-functional and Lipschitz class of functions.

2. Basic estimates for Schurer-Stancu operators $S_{n,l}^{\alpha,\beta}(f; x, p, q)$

Lemma 2.1. *[10] Let $B_{n,l}^{p,q}(f; x)$ be given by (1). Then for any $x \in [0, 1]$ and $0 < q < p \leq 1$, we have the following identities*

$$\begin{aligned} B_{n,l}^{p,q}(1; x) &= 1, \\ B_{n,l}^{p,q}(t; x) &= \frac{[n + l]_{p,q} x}{[n]_{p,q}}, \\ B_{n,l}^{p,q}(t^2; x) &= \frac{[n + l]_{p,q} p^{n+l-1} x}{[n]_{p,q}^2} + \frac{q[n + l]_{p,q}[n + l - 1]_{p,q} x^2}{[n]_{p,q}^2}. \end{aligned}$$

Lemma 2.2. *Let $x \in [0, 1]$ and $0 < q < p \leq 1$. For the operators $S_{n,l}^{\alpha,\beta}(f; x, p, q)$, we have*

$$S_{n,l}^{\alpha,\beta}(t^m; x, p, q) = \frac{[n]_{p,q}^m}{([n]_{p,q} + \beta)^m} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{[n]_{p,q}}\right)^{m-i} B_{n,l}^{p,q}(t^i; x).$$

Proof: From (2), we get

$$\begin{aligned} S_{n,l}^{\alpha,\beta}(t^m; x, p, q) &= \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \left(\frac{p^{(n+l)-\nu}[\nu]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right)^m \\ &= \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \frac{[n]_{p,q}^m}{([n]_{p,q} + \beta)^m} \left(\frac{p^{(n+l)-\nu}[\nu]_{p,q} + \alpha}{[n]_{p,q}}\right)^m \\ &= \frac{[n]_{p,q}^m}{([n]_{p,q} + \beta)^m} \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{[n]_{p,q}}\right)^{m-i} \left(\frac{p^{(n+l)-\nu}[\nu]_{p,q}}{[n]_{p,q}}\right)^i \\ &= \frac{[n]_{p,q}^m}{([n]_{p,q} + \beta)^m} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{[n]_{p,q}}\right)^{m-i} \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \left(\frac{p^{(n+l)-\nu}[\nu]_{p,q}}{[n]_{p,q}}\right)^i \\ &= \frac{[n]_{p,q}^m}{([n]_{p,q} + \beta)^m} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{[n]_{p,q}}\right)^{m-i} B_{n,l}^{p,q}(t^i; x). \end{aligned}$$

$\square$

Lemma 2.3. *For $S_{n,l}^{\alpha,\beta}$, we have*

$$\begin{aligned}
S_{n,l}^{\alpha,\beta}(1;x,p,q) =& 1,\\
S_{n,l}^{\alpha,\beta}(t;x,p,q) =& \frac{[n+l]_{p,q}x+\alpha}{[n]_{p,q}+\beta},\\
S_{n,l}^{\alpha,\beta}(t^2;x,p,q) =& \frac{[n+l]_{p,q}p^{n+l-1}+2\alpha)x}{([n]_{p,q}+\beta)^2}+\frac{q[n+l]_{p,q}[n+l-1]_{p,q}x^2}{([n]_{p,q}+\beta)^2}\\
&+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}.
\end{aligned}$$

Proof: Using Lemma 2.1 and Lemma 2.2, we have

$$S_{n,l}^{\alpha,\beta}(t^m;x,p,q)=\frac{[n]_{p,q}^m}{([n]_{p,q}+\beta)^m}\sum_{i=0}^{m}\binom{m}{i}\left(\frac{\alpha}{[n]_{p,q}}\right)^{m-i}B_{n,l}^{p,q}(t^i;x).$$

For $m=0$

$$S_{n,l}^{\alpha,\beta}(1;x,p,q)=B_{n,l}^{p,q}(1;x)=1.$$

For $m=1$

$$\begin{aligned}
S_{n,l}^{\alpha,\beta}(t^1;x,p,q) &= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)}\sum_{i=0}^{1}\binom{1}{i}\left(\frac{\alpha}{[n]_{p,q}}\right)^{1-i}B_{n,l}^{p,q}(t^i;x)\\
&= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)}\left(\frac{\alpha}{[n]_{p,q}}B_{n,l}^{p,q}(1;x)+B_{n,l}^{p,q}(t^1;x)\right)\\
&= \frac{[n+l]_{p,q}x+\alpha}{[n]_{p,q}+\beta}.
\end{aligned}$$

And for $m=2$

$$\begin{aligned}
S_{n,l}^{\alpha,\beta}(t^2;x,p,q) &= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2}\sum_{i=0}^{2}\binom{2}{i}\left(\frac{\alpha}{[n]_{p,q}}\right)^{2-i}B_{n,l}^{p,q}(t^i;x)\\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2}\left(\frac{\alpha^2}{[n]_{p,q}^2}B_{n,l}^{p,q}(t^2;x)+2\frac{\alpha}{[n]_{p,q}}B_{n,l}^{p,q}(t^1;x)+1B_{n,l}^{p,q}(1;x)\right)\\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2}\left(\frac{\alpha^2}{[n]_{p,q}^2}\left(\frac{[n+l]_{p,q}p^{n+l-1}x}{[n]_{p,q}^2}+\frac{q[n+l]_{p,q}[n+l-1]_{p,q}x^2}{[n]_{p,q}^2}\right)\right)\\
&\quad+\frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2}\left(2\frac{\alpha}{[n]_{p,q}}\left(\frac{[n+l]_{p,q}x}{[n]_{p,q}}\right)+1\right)\\
&= \frac{[n+l]_{p,q}p^{n+l-1}+2\alpha)x}{([n]_{p,q}+\beta)^2}+\frac{q[n+l]_{p,q}[n+l-1]_{p,q}x^2}{([n]_{p,q}+\beta)^2}+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}.
\end{aligned}$$

□

Lemma 2.4. *Let $\psi_x^i(t)=(t-x)^i$ and $S_{n,l}^{\alpha,\beta}$ be the operators defined by (2). Then, we get*

$$\begin{aligned}
S_{n,l}^{\alpha,\beta}(\psi_x^0(t);x,p,q) &= 1,\\
S_{n,l}^{\alpha,\beta}(\psi_x^1(t);x,p,q) &= \left(\frac{[n+l]_{p,q}}{[n]_{p,q}+\beta}-1\right)x+\frac{\alpha}{[n]_{p,q}+\beta},\\
S_{n,l}^{\alpha,\beta}(\psi_x^2(t);x,p,q) &= \frac{[n+l]_{p,q}[n+l-1]_{p,q}q-2[n+l]_{p,q}([n]_{p,q}+\beta)}{([n]_{p,q}+\beta)^2}x^2\\
&\quad+\frac{([n]_{p,q}+\beta)^2}{([n]_{p,q}+\beta)^2}x^2+\frac{[n+l]_{p,q}(p^{n+l-1}+2\alpha)}{([n]_{p,q}+\beta)^2}x\\
&\quad-\frac{2\alpha([n]_{p,q}+\beta)}{([n]_{p,q}+\beta)^2}x+\frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Proof: Lemma 2.4 follows directly from Lemma 2.3. □

3. Convergence properties of $S_{n,l}^{\alpha,\beta}(f;x,p_n,q_n)$

Theorem 3.1. *Let $(q_n)_n,(p_n)_n$ be two real sequences such that $0<q_n<p_n\leq 1$, $\lim_{n\to\infty}q_n=1$, $\lim_{n\to\infty}p_n=1$ and $\lim_{n\to\infty}q_n^n=a$, $\lim_{n\to\infty}p_n^n=b$ with $0<a,b<1$. Then, for each $f\in C[0,l+1]$, we have*

$$\parallel S_{n,l}^{\alpha,\beta}(f;x,p_n,q_n)-f(x)\parallel_{C[0,l+1]}=0.$$

Proof: It is sufficient to show that for $i=0,1,2$

$$\parallel S_{n,l}^{\alpha,\beta}(f;x,p_n,q_n)(t^i;x)-x^i\parallel\to 0 \qquad as \qquad n\to\infty.$$

For $i=0$, it is obvious.
For $i=1$, we have

$$\begin{aligned}
|S_{n,l}^{\alpha,\beta}(t;x,p_n,q_n)-x| &= \left|\frac{[n+l]_{p_n,q_n}x+\alpha}{[n]_{p_n,q_n}+\beta}-x\right|\\
&\leq \left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\\
\parallel S_{n,l}^{\alpha,\beta}(t;x,p_n,q_n)-x\parallel &\to 0 \qquad as \quad n\to\infty.
\end{aligned}$$

For $i = 2$,

$$
\begin{aligned}
& |S_{n,l}^{\alpha,\beta}(t^2; x, p_n, q_n) - x^2| = \\
= \quad & \left| \frac{[n+l]_{p_n,q_n}(p_n^{n+l-1} + 2\alpha)x}{([n]_{p_n,q_n} + \beta)^2} + \frac{q_n [n+l]_{p_n,q_n} [n+l-1]_{p_n,q_n} x^2}{([n]_{p_n,q_n} + \beta)^2} \right. \\
& \left. + \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} - x^2 \right| \\
\leq \quad & \left| \frac{[n+l]_{p_n,q_n}(p_n^{n+l-1} + 2\alpha)x}{([n]_{p_n,q_n} + \beta)^2} \right| + \left| \frac{q_n [n+l]_{p_n,q_n} [n+l-1]_{p_n,q_n}}{([n]_{p_n,q_n} + \beta)^2} - 1 \right| x^2 \\
& + \left| \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} \right| \\
\to 0 \quad & \text{as } \; n \to \infty
\end{aligned}
$$

and this proves the theorem. □

Theorem 3.2. *Let $0 < q_n < p_n \leq 1$ be two real sequences such that $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n^n = a$, $\lim_{n\to\infty} p_n^n = b$ with $0 < a, b < 1$. Then*

$$|S_{n,l}^{\alpha,\beta}(f; x, p_n, q_n) - f(x)| \leq 2\omega(f; \sqrt{\delta_{n,l}^{\alpha,\beta}(x)}),$$

for all $f \in C[0, l+1]$ and $\delta_{n,l}^{\alpha,\beta}(x) = \sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t); x, p_n, q_n)}$.

Proof: Calculating the difference, we find

$$
\begin{aligned}
& |S_{n,l}^{\alpha,\beta}(f; x, p_n, q_n) - f(x)| = \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \left| f\left(\frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} \right) - f(x) \right| \\
\leq \quad & \left\{ 1 + \frac{1}{\delta_n^{\alpha,\beta}} \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \left| \frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} - x \right| \right\} \omega(f; \delta_n^{\alpha,\beta}(x)) \\
\leq \quad & \left\{ 1 + \frac{1}{\delta_n^{\alpha,\beta}} \sqrt{ \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x; p, q) \left(\left| \frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} - x \right| \right)^2 } \right\} \omega(f; \delta_n^{\alpha,\beta}(x)) \\
= \quad & \left\{ 1 + \frac{1}{\delta_n^{\alpha,\beta}} \sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t); x, p_n, q_n)} \right\} \omega(f; \delta_n^{\alpha,\beta}(x))
\end{aligned}
$$

choosing $\delta_{n,l}^{\alpha,\beta} = \sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t); x, p_n, q_n)}$, we get the desired result.

□

Theorem 3.3. *For $0 \leq \alpha \leq \beta$, $x \in [0,1]$ and $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} p_n = 1$ and $\lim\limits_{n\to\infty} q_n^n = a$, $\lim\limits_{n\to\infty} p_n^n = b$ with $0 < a, b < 1$, we have*

$$\begin{aligned}|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| \quad &\leq \quad \omega_1(([n]_{p,q}+\beta)^{-1}) \\ &\times\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t);x)}\left\{1+\sqrt{([n]_{p,q}+\beta)}\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t);x)}\right\}\end{aligned}$$

where $f'(x)$ has continuous derivative over $[0, l+1]$ and $\omega_1(f;\delta_{n,\beta})$ is the modulus of continuity of $f'(x)$.

Proof: For $t_1, t_2 \in [0,b]$ and $t_1 < \eta < t_2$, it is known that

$$\begin{aligned} f(t_1) - f(t_2) &= (t_1-t_2)f'(\eta), \\ &= (t_1-t_2)f'(t_1) + (t_1-t_2)[f'(\eta) - f'(t_1)], \end{aligned} \tag{3.1}$$

and from Lorentz[[18], p. 21, Theorem 1.6.2], we have

$$|(t_1-t_2)[f'(\eta)-f'(t_1)]| \leq |t_1-t_2|(\lambda+1)\omega_1(\delta), \tag{3.2}$$

where $\lambda = \lambda(x_1, x_2; \delta)$ is the integer $[|t_1 - t_2|\delta^{-1}]$.
Now, we have

$$|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| = \left|\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p,q) f\left(\frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}\right) - f(x)\right|. \tag{3.3}$$

Using (3) and (4), we get

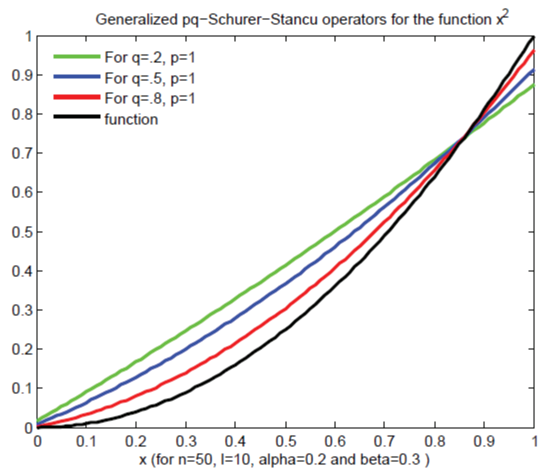
$$\begin{aligned}
&|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| \\
&\leq \left|\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p,q)\left(x - \frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}\right)f'(x)\right| \\
&\quad + \omega_1(\delta_{n,l}^{\alpha,\beta})\sum_{\nu=0}^{n+l}\left|\frac{p_n^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta} - x\right|(\lambda+1)b_{n,l}^{\nu}(x;p,q), \\
&\leq \omega_1(\delta_{n,l}^{\alpha,\beta})\left\{\sum_{\nu=0}^{n+l}\left|\frac{p^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta} - x\right| b_{n,l}^{\nu}(x;p,q)\right. \\
&\quad \left. + \sum_{\nu=0}^{n+l}\left|\frac{p^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta} - x\right|\left(\lambda\Big(x, \frac{[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}; \delta_{n,l}^{\alpha,\beta}\Big)+1\right)b_{n,l}^{\nu}(x;p,q)\right\}, \\
&\leq \omega_1(\delta_{n,l}^{\alpha,\beta})\left\{\sum_{\nu=0}^{n+l}\left|\frac{p^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta} - x\right| b_{n,l}^{\nu}(x;p,q)\right. \\
&\quad \left. + \frac{1}{\delta_n^{\beta}}\sum_{\nu=0}^{n+l}\left(\frac{p^{(n+l)-\nu}[\nu]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta} - x\right)^2 b_{n,l}^{\nu}(x;p,q)\right. \\
&\leq \omega_1(\delta_n^{\beta})\left(\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2;x)} + \frac{S_{n,l}^{\alpha,\beta}(\psi_x^2;x)}{\delta_n^{\beta}}\right)
\end{aligned}$$

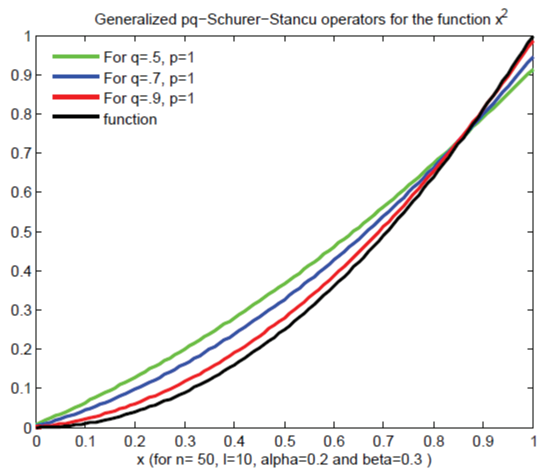
$$=\omega_1(\delta_n^\beta)\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2;x)}\left\{1+\frac{\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2;x)}}{\delta_n^\beta}\right\}.$$

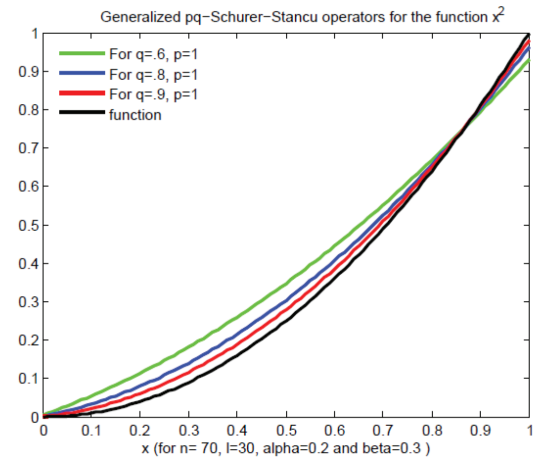
Taking $\delta_n^\beta=([n]_{p,q}+\beta)^{-1}$, we get

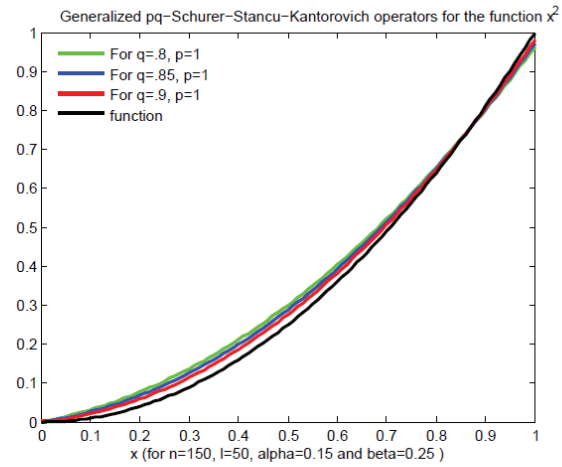
$$|S_{n,l}^{\alpha,\beta}(f;x)-f(x)|\leq\omega_1(([n]_{p,q}+\beta)^{-1})\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t);x)}\left\{1+\sqrt{([n]_{p,q}+\beta)}\sqrt{S_{n,l}^{\alpha,\beta}(\psi_x^2(t);x)}\right\}.$$

□









4. Direct results for $S_{n,l}^{\alpha,\beta}(f;x)$

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{0\leq x<\infty} |f(x)|.$$

Then, for any $\delta > 0$, Peeter's K-functional is defined as

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. Now, we know that there exits an absolute constant $C > 0$ Devore and Lorentz[[19], p.177, Theorem 2.4] such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}),$$

where $\omega_2(f;\delta)$ is the second order modulus of continuity given by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 4.1. *Let $f \in C_B^2[0,l+1]$ and $(q_n)_n, (p_n)_n$ be two sequences such that $0 < q_n < p_n \leq 1$, $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n^n = a$ and $\lim\limits_{n\to\infty} p_n^n = b$ with $0 < a, b < 1$. Then for all $x \in [0,1]$ there exists a constant $K > 0$ such that*

$$\begin{aligned} | S_{n,l}^{\alpha,\beta}(f;x) - f(x) | \quad &\leq \quad K\omega_2(f;\sqrt{\Theta_{n,a}^{\alpha,\beta}(x)}) \\ &+\omega\left(f;\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right) \end{aligned}$$

where

$$\Theta_{n,a}^{\alpha,\beta}(x) \quad = \quad S_{n,l}^{\alpha,\beta}((t-x)^2;x) + \left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right)^2.$$

Proof: First, we consider the auxiliary operators $\widehat{S}_{n,l}^{\alpha,\beta}$

$$\widehat{S}_{n,l}^{\alpha,\beta}(f;x) = S_{n,l}^{\alpha,\beta}(f;x) + f(x) - f\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right). \quad (4.1)$$

We find that

$$\widehat{S}_{n,l}^{\alpha,\beta}(1;x) = 1,$$

$$\widehat{S}_{n,l}^{\alpha,\beta}(t-x;x) = 0,$$

$$|\widehat{S}_{n,l}^{\alpha,\beta}(f;x)| \leq 3\|f\|. \quad (4.2)$$

Let $g \in C_B^2[0,\infty)$. By the Taylor's theorem

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv. \quad (4.3)$$

Now

$$\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)=g'(x)\widehat{S}_{n,a}^{\alpha,\beta}(t-x;x)+\widehat{S}_{n,l}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)$$

$$\begin{aligned}
&= \widehat{S}_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)\\
&= S_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)\\
&\quad -\int_x^{\zeta_{\alpha,\beta}}\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}-v\right)g''(v)dv.
\end{aligned}$$

Therefore

$$|\widehat{S}_{n}^{\alpha,\beta}(g;x)-g(x)|\leq\left|S_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)\right|$$

$$+\left|\int_x^{\zeta_{\alpha,\beta}}\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}-v\right)g''(v)dv\right|.$$

Since

$$\left|\int_x^t(t-v)g''(v)dv\right|\leq(t-x)^2\parallel g''\parallel, \tag{4.4}$$

then we have

$$\begin{aligned}
&\left|\int_x^{\zeta_{\alpha,\beta}}\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}-v\right)g''(v)dv\right|\\
&\leq\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right)^2\parallel g''\parallel,
\end{aligned}\tag{4.5}$$

where

$$\zeta_{\alpha,\beta}=\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}.$$

From (6), (7) and (8), we have

$$\begin{aligned}
|\widehat{S}_{n,l}^{\alpha,\beta}(g;x)-g(x)| &\leq S_{n,l}^{\alpha,\beta}((t-x)^2;x)\|g''\|\\
&\quad+\left\{\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right)^2\right\}\|g''\|\\
&= \Theta_{n,l}^{\alpha,\beta}(x)\|g''\|.
\end{aligned}\tag{4.6}$$

Next, we have

$$
\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x)-f(x)| &\leq |\widehat{S}_{n,l}^{\alpha,\beta}(f-g;x)|+|(f-g)(x)|+|\widehat{S}_{n,l}^{\alpha,\beta}(g;x)-g(x)| \\
&\quad +\left|f\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right)\right)-f(x)\right|
\end{aligned}
$$

Using (9), we have

$$
\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x)-f(x)| &\leq 4\|f-g\|+\widehat{S}_{n,l}^{\alpha,\beta}(g;x)-g(x)| \\
&+ \left|f\left(\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right)-f(x)\right| \\
&\leq 4\|f-g\|+\Theta_{n,l}^{\alpha,\beta}(x)\|g''\| \\
&\quad +\omega\left(f;\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right).
\end{aligned}
$$

By the definition of Peetre's K-functional, we have

$$
\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x)-f(x)| &\leq C\omega_2\big(f;\sqrt{\Theta_{n,l}^{\alpha,\beta}(x)}\big) \\
&\quad +\omega(f;\left(\frac{[n+l]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right).
\end{aligned}
$$

□

Now, we discuss the rate of convergence of the operators $S_{n,l}^{\alpha,\beta}$ in Lipschitz class $Lip_M(\gamma)$, given by

$$Lip_M(\gamma)=\{f\in C[0,\infty):|f(t)-f(x)|\leq M|t-x|^{\gamma}:x,t\in[0,1]\}$$

where M is a constant and $0<\gamma\leq 1$.

Theorem 4.2. *Let $f\in Lip_M(\gamma)$. Then, we have*

$$|S_{n,l}^{\alpha,\beta}(f;x)-f(x)|\leq M\delta_{n,l}^{\alpha,\beta}(x),$$

where $\delta_{n,l}^{\alpha,\beta}(x)=S_{n,l}^{\alpha,\beta}((t-x)^2;x)$.

Proof: Let $\gamma=1$ and $0<q_n<p_n\leq 1$, $\lim\limits_{n\to\infty}q_n=1$, $\lim\limits_{n\to\infty}p_n=1$, $\lim\limits_{n\to\infty}q_n^n=a$

and $\lim\limits_{n\to\infty} p_n^n = b$ with $0 < a, b < 1$. Then for $f \in Lip_M(1)$, we have

$$\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| &\leq \sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\left| f\left(\frac{p_n{}^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}\right) - f(x)\right| \\
&\leq M\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\left|\frac{p_n{}^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} - x\right| \\
&= M S_{n,l}^{\alpha,\beta}(|t-x|;x) \\
&\leq M(S_{n,l}^{\alpha,\beta}((t-x)^2;x))^{\frac{1}{2}} \\
&= M(\delta_{n,l}^{\alpha,\beta}(x))^{\frac{1}{2}}.
\end{aligned}$$

Thus, the assertion holds for $\gamma = 1$. Now, we will prove for $\gamma \in (0,1)$. From the Hölder inequality with $t_1 = \frac{1}{\gamma}$,
$t_2 = \frac{1}{1-\gamma}$, we have

$$\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| &= \left(\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\left| f\left(\frac{p_n{}^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}\right) - f(x)\right|^{\gamma}\right)^{\frac{1}{\gamma}} \\
&\quad\times\left(\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\right)^{\frac{1}{1-\gamma}} \\
&\leq \left(\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\left| f\left(\frac{p_n{}^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}\right) - f(x)\right|^{\gamma}\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Since $f \in Lip_M(\gamma)$, we obtain

$$\begin{aligned}
|S_{n,l}^{\alpha,\beta}(f;x) - f(x)| &\leq M\left(\sum_{\nu=0}^{n+l} b_{n,l}^{\nu}(x;p_n,q_n)\left|\frac{p_n{}^{(n+l)-\nu}[\nu]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} - x\right|\right)^{\gamma} \\
&\leq M(S_{n,l}^{\alpha,\beta}(|t-x|;x))^{\gamma} \\
&\leq M(S_{n,l}^{\alpha,\beta}((t-x)^2;x))^{\frac{\gamma}{2}} \\
&= M(\delta_{n,l}^{\alpha,\beta}(x))^{\frac{\gamma}{2}}.
\end{aligned}$$

□

Remark 4.3. *Approximation results obtained for Bernstein-Schurer by Mursaleen et al. (see [10]) are the particular case of our results for* $\alpha = \beta = 0$.

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