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Arithmetic Convergent Sequence Space Defined By Modulus Function *

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ABSTRACT: The aim of this article is to introduce the sequence spaces $AC(f)$ and $AS(f)$ using arithmetic convergence and modulus function, and study algebraic and topological properties of this space, and certain inclusion results.

Key Words: Arithmetic convergence, sequence space, summability, modulus function.

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1. Introduction

Throughout, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the set of natural, real and complex numbers, respectively and $x = (x_k)$ denotes a sequence whose k^{th} term is x_k. Similarly $w, c, \ell_\infty, \ell_1$ denotes the space of *all, convergent, bounded, absolutely summable* sequences of complex terms, respectively.

For a sequence $x = (x_k)$ defined on $\mathbb{N}$ and $n \in \mathbb{N}$, the notation $\sum\limits_{k|n} x_k$ means the finite sum of all the numbers x_k as k ranges over the integers that divide n including 1 and n. In general for integers k and n we write $k|n$ to mean 'k divides n' or 'n is a multiple of k'. We use the symbol $< m, n >$ to denote the greatest common divisor of two integers m and n.

In [17], Ruckle was introduced the notions arithmetic summability and arithmetic convergence as follows:

(i) A sequence $x = (x_k)$ defined on $\mathbb{N}$ is called *arithmetically summable* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left| \sum\limits_{k|m} x_k - \sum\limits_{k|<m,n>} x_k \right| < \varepsilon$.

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(ii) A sequence $y = (y_k)$ is called *arithmetically convergent* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $|y_m - y_{<m,n>}| < \varepsilon$.

We studied arithmetic convergence and arithmetic continuity in [18,19,20].

The notion of a modulus function was introduced by Nakano [12] in the year 1953 (also see [11,13]). We recall that a modulus f is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Because of (ii), $|f(x) - f(y)| \leq f(|x - y|)$ so that in view of (iv), f is continuous everywhere on $[0, \infty)$. A modulus function may be unbounded (for example, $f(x) = x^p, 0 < p \leq 1$) or bounded (for example, $f(x) = \frac{x}{1+x}$).
It is easy to see that $f_1 + f_2$ is a modulus function when f_1 and f_2 are modulus functions, and that the function f_i(i is a positive integer), the composition of a modulus function f with itself i times, is also a modulus function. Ruckle [14] used the idea of a modulus function f to construct a class of FK spaces

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [14] proved that, for any modulus f, $X(f) \subset l_1$.

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f\left(|x_k|\right) < \infty.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in [6]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling [4,5], Köthe [10] and Ruckle [15,16]. After then Kolk [8,9] gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space $X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$. Also we refer to readers [1,2,3] for different types of sequence spaces defined by modulus function.

2. Definitions and Preliminaries

Now we give the following definition for establishing certain results of this article.

Definition 2.1. *A sequence space E said to be sequence algebra if $(x_k) * (y_k) = (x_k y_k) \in E$ whenever $(x_k); (y_k) \in E$.*

Lemma 2.2. *[13, Proposition 1] Let f be a modulus function and let $0 < \delta < 1$. Then for each $x \geq \delta$, we have $f(x) \leq 2f(1)\delta^{-1}x$.*

3. Arithmetic convergent sequence space $AC(f)$

In this section we study certain algebraic and topological properties of arithmetic convergent sequence space $AC(f)$ defined by modulus function f. We define

$$AC(f) = \{(x_m) : \text{for } \varepsilon > 0 \text{ and an integer } n,\ f\left(|x_m - x_{<m,n>}|\right) < \varepsilon,\ \forall m\}.$$

Theorem 3.1. *The sequence space $AC(f)$ is a linear space.*

Proof: Let $(x_m), (y_m)$ be two sequences in $AC(f)$. By definition of $AC(f)$, for $\varepsilon > 0$ and an integer n, we have

$$f\left(|x_m - x_{<m,n>}|\right) < \varepsilon \text{ and } f\left(|y_m - y_{<m,n>}|\right) < \varepsilon \text{ for all } m.$$

Since f is a modulus function, for $\varepsilon > 0$ and an integer n and scalars α and β,

$$\begin{aligned} f\left(|(\alpha x_m + \beta y_m) - (\alpha x_{<m,n>} + \beta y_{<m,n>})|\right) &\leq f\left(|\alpha|\,|x_m - x_{<m,n>}|\right) \\ &\quad + f\left(|\beta|\,|y_m - y_{<m,n>}|\right) \\ &\leq f\left(|x_m - x_{<m,n>}|\right) \\ &\quad + f\left(|y_m - y_{<m,n>}|\right) \\ &< \varepsilon + \varepsilon = 2\varepsilon \text{ for all } m. \end{aligned}$$

Thus $AC(f)$ is a linear space. □

Theorem 3.2. *The sequence space $AC(f)$ is a sequence algebra.*

Proof: Let $(x_m), (y_m)$ be sequences $AC(f)$. Then by definition, for $\varepsilon > 0$ and an integer n, we have

$$f\left(|x_m - x_{<m,n>}|\right) < \varepsilon \text{ and } f\left(|y_m - y_{<m,n>}|\right) < \varepsilon \text{ for all } m.$$

Then we have

$$f\left(|x_m y_m - x_{<m,n>} y_{<m,n>}|\right) < \varepsilon \text{ for all } m. \text{ (see [18])}$$

Hence the sequence $(x_m y_m) \in AC(f)$. This completes the proof. □

Theorem 3.3. *Let f_1 and f_2 be two modulus functions, then $AC(f_1) \subset AC(f_1 f_2)$.*

Proof: Let $(x_m) \in AC(f_2)$. Then for $\varepsilon > 0$ and an integer n, we have
$f_2\left(|x_m - x_{<m,n>}|\right) < \varepsilon$.
We choose δ with $0 < \delta < 1$ such that $f_1(x) < \varepsilon$ for $0 \leq x \leq \delta$. Let us denote $y_m = f_2\left(|x_m - x_{<m,n>}|\right)$ and consider

$$\lim_m f_1(y_m) = \lim_{y_m \leq \delta} f_1(y_m) + \lim_{y_m > \delta} f_1(y_m),\ m \in \mathbb{N}$$

since f_1 is a modulus function,

$$\lim_{y_m \leq \delta} f_1(y_m) < \varepsilon.$$

For $y_m > \delta$ we use the fact that

$$y_m < \frac{y_m}{\delta} < 1 + \left[\frac{y_m}{\delta}\right],$$

where $[x]$ denotes the integral part of x.
Since f_1 is a modulus function, by definition of modulus function and Lemma 2.2, we have for $y_m > \delta$

$$f_1(y_m) < f_1(1)\left(1 + \left[\frac{y_m}{\delta}\right]\right) \leq 2f_1(1)\frac{y_m}{\delta}.$$

Thus

$$\lim_{y_m > \delta} f_1(y_m) \leq 2f_1(1)\delta^{-1} \lim_{y_m > \delta} y_m < \varepsilon.$$

Hence $(x_m) \in AC(f_1 f_2)$. □

Corollary 3.4. *The sequence space of all arithmetic convergent sequences, AC is subset of $AC(f)$ .i.e. $AC \subset AC(f)$.*

Theorem 3.5. *Let f_1 and f_2 be two modulus functions. Then we have*

$$AC(f_1) \cap AC(f_2) \subset AC(f_1 + f_2).$$

Proof: Let $(x_m) \in AC(f_1) \cap AC(f_2)$. Then
$(x_m) \in AC(f_1)$ and $(x_m) \in AC(f_2)$
$\Rightarrow$ for $\varepsilon > 0$ $\exists$ positive integer n such that

$$f_1\left(|x_m - x_{<m,n>}|\right) < \varepsilon \text{ and } f_2\left(|x_m - x_{<m,n>}|\right) < \varepsilon \text{ for all } m.$$

From the above equations we can easily see that for $\varepsilon > 0$ and integer n

$$(f_1 + f_2)\left(|x_m - x_{<m,n>}|\right) < \varepsilon \text{ for all } m.$$

Thus $(x_m) \in AC(f_1 + f_2)$. Hence the result. □

4. Arithmetic summable sequence space $AS(f)$

In this section we study certain algebraic and topological properties of arithmetic summable sequence space $AS(f)$ defined by modulus function f. We define

$$AS(f)=\left\{(x_m) : \text{for } \varepsilon > 0 \text{ and an integer } n, f\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon, \forall m\right\}.$$

Theorem 4.1. *The sequence space $AS(f)$ is a linear space.*

Proof: Let $(x_m), (y_m) \in AS(f)$. By definition of $AS(f)$, for $\varepsilon > 0$ and an integer n,

$$f\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon \text{ and } f\left(\left|\sum_{k|m} y_k - \sum_{k|<m,n>} y_k\right|\right) < \varepsilon \text{ for all } m.$$

Since f is a modulus function, for $\varepsilon > 0$ and an integer n and scalars α and β,

$$\begin{aligned} & f\left(\left|\sum_{k|m}(\alpha x_k + \beta y_k) - \sum_{k|<m,n>} (\alpha x_k + \beta y_k)\right|\right) \\ & \leq f\left(|\alpha|\left|\sum_{k|m} x_m - \sum_{k|<m,n>} x_k\right|\right) + f\left(|\beta|\left|\sum_{k|m} y_k - \sum_{k|<m,n>} y_k\right|\right) \\ & \leq f\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) + f\left(\left|\sum_{k|m} y_k - \sum_{k|<m,n>} y_k\right|\right) \\ & < \varepsilon + \varepsilon = 2\varepsilon \text{ for all } m. \end{aligned}$$

Thus $AS(f)$ is a linear space. □

Theorem 4.2. *Let f_1 and f_2 be two modulus functions, then $AS(f_1) \subset AS(f_1 f_2)$.*

Proof: The proof of the theorem is analogous to the proof of the Theorem 3.3 □

Corollary 4.3. *The sequence space of all arithmetic summable sequences, AS is subset of $AS(f)$ i.e. $AS \subset AS(f)$.*

Theorem 4.4. *Let f_1 and f_2 be two modulus functions. Then $AS(f_1) \cap AS(f_2) \subset AS(f_1 + f_2)$.*

Proof: Let $(x_m) \in AS(f_1) \cap AS(f_2)$. Then
$(x_m) \in AS(f_1)$ and $(x_m) \in AS(f_2)$
$\Rightarrow$ for $\varepsilon > 0$ $\exists$ positive integer n such that

$$f_1\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon \text{ and } f_2\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon \text{ for all } m.$$

From the above equations we can easily see that for $\varepsilon > 0$ and integer n

$$(f_1 + f_2)\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon \text{ for all } m.$$

Thus $(x_m) \in AS(f_1 + f_2)$. Hence the result. □

The next result gives an important inclusion property between l_1 and $AS(f)$.

Theorem 4.5. $l_1 \subset AS(f)$.

Proof: The result can be easily obtained from the inclusion $l_1 \subset AS$ given by Proposition 16 in [17] and by using the Corollary 4.3. □

The following theorem established a relation between $AS(f)$ and $AC(f)$.

Theorem 4.6. *If the sequence* $(x_m) \in AS(f)$ *then the sequence* (y_m) *defined by* $y_m = \sum_{k|m} x_k$ *is in* $AC(f)$.

Proof: Let $(x_m) \in AS(f)$. Then by definition we have

$$f\left(\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right|\right) < \varepsilon$$
$$\Rightarrow f\left(|y_m - y_{<m,n>}|\right) < \varepsilon$$
$$\Rightarrow y_m \in AC(f).$$

□

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