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On the Paranormed Space $\mathcal{M}_u(t)$ of Double Sequences *

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ABSTRACT: In this paper, we introduce the paranormed sequence space $\mathcal{M}_u(t)$ corresponding to the normed space $\mathcal{M}_u$ of all bounded double sequences. We examine general topological properties of this space and determine its alpha-, beta- and gamma-duals. Furthermore, we characterize some classes of four-dimensional matrix transformations concerning this space and its dual spaces.

Key Words: Paranormed double sequence space, Dual spaces, Matrix transformations.

Contents

1. Introduction and Notations

The present section is devoted to recall some basic facts concerning the spaces of double sequences and establish some properties of those spaces.

Let us point out that the theory of (single) sequence spaces is exhaustively discussed in the recent books [1,2].

Ω, the set of all complex valued double sequences, forms a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as *a double sequence space*.

A double sequence $x = (x_{kl})$ in Ω is said to be bounded if

$$\|x\|_\infty = \sup_{k,l\in\mathbb{N}} |x_{kl}| < \infty$$

and the set of all bounded double sequences is denoted by $\mathcal{M}_u$ which is a Banach space with the norm $\|\cdot\|_\infty$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The well-known convergence rule for double sequences was introduced by Pringsheim [3]. A double sequence $x = (x_{kl}) \in \Omega$ is called *convergent* in the *Pringsheim's*

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sense to the limit $L \in \mathbb{C}$ (shortly, $p-$convergent to L) if for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$ for all $k, l > N$; where $\mathbb{C}$ denotes the complex field. The space of all $p-$convergent double sequences is denoted by $\mathcal{C}_p$. In addition to $x \in \mathcal{C}_p$, if $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to* L *in Pringsheim's sense* (shortly, $bp-$convergent to L) and the set of all such sequences is denoted by $\mathcal{C}_{bp}$.

The main drawback of the Pringsheim's convergence is that a $p-$convergent double sequence need not be bounded. In [4], Hardy lacked this disadvantage by giving the definition of regular convergence. A double sequence $x = (x_{kl}) \in \mathcal{C}_p$ is said to be *regularly convergent* to the limit $L \in \mathbb{C}$ (shortly, $r-$convergent to L) if the limits $I_k := \lim_{l\to\infty} x_{kl}$ $(k \in \mathbb{N})$ and $J_l := \lim_{k\to\infty} x_{kl}$ $(l \in \mathbb{N})$ exist. Note that, in this case we have that the limits $\lim_{k\to\infty} I_k = L$ and $\lim_{l\to\infty} J_l = L$, where L is the $p-$limit of x. As seen from the definition, in addition to the Pringsheim's convergence, the regular convergence requires the convergence of rows and columns of a double sequence, and so it is bounded. The space of all such sequences is denoted by $\mathcal{C}_r$ and the inclusion $\mathcal{C}_r \subset \mathcal{C}_{bp}$ strictly holds.

The spaces of all null sequences contained in the sequence spaces $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ are denoted by $\mathcal{C}_{p0}$, $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$; respectively. Móricz [5] proved that $\mathcal{C}_{bp}$, $\mathcal{C}_r$, $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|\cdot\|_\infty$. Also, he defined the pseudonorm

$$\|x\|_{\mathcal{C}_p} = \lim_{N\to\infty} \sup_{k,l\geq N} |x_{kl}|$$

for $\mathcal{C}_p$, and showed that $\mathcal{C}_p$ is complete under $\|\cdot\|_{\mathcal{C}_p}$. Moreover, he remarked that $\|x\|_{\mathcal{C}_p} = 0$ holds identically for any $x \in \mathcal{C}_{p0}$. Note that because of the topological structure of the spaces $\mathcal{C}_p$ and $\mathcal{C}_{p0}$, we can not define a norm for them.

For short, throughout the text the summations without limits run from 0 to ∞, for example $\sum_{k,l}$ means that $\sum_{k,l=0}^{\infty}$, and we assume that ϑ denotes any of the convergence rule symbols p, bp or r.

Let λ be a double sequence space converging with respect to some linear convergence rule $\vartheta - \lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{k,l} x_{kl}$ with respect to this rule is defined by $\vartheta - \sum_{k,l} x_{kl} = \vartheta - \lim_{m,n\to\infty} \sum_{k,l=0}^{m,n} x_{kl}$. If there is no confusion, we use $\sum_{k,l} x_{kl}$ instead of $\vartheta - \sum_{k,l} x_{kl}$.

The space of all absolutely summable double sequences is denoted by $\mathcal{L}_u$, that is

$$\mathcal{L}_u := \left\{ x = (x_{kl}) \in \Omega : \|x\|_{\mathcal{L}_u} = \sum_{k,l} |x_{kl}| < \infty \right\},$$

which is a Banach space with the norm $\|\cdot\|_{\mathcal{L}_u}$ (see [6]).

Let X be a real or complex linear space, g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) or shortly X is called a paranormed space and g is a paranorm for X, if the following axioms are satisfied for all elements $x, y \in X$:

(i) $g(x) \geq 0$.

(ii) $g(x) = 0$ if $x = \theta$, where θ is the zero vector in X.

(iii) $g(x) = g(-x)$.

(iv) $g(x + y) \leq g(x) + g(y)$.

(v) Scalar multiplication is continuous, i.e., $|\alpha_i - \alpha| \to 0$ and $g(x^i - x) \to 0$ imply $g(\alpha_i x^i - \alpha x) \to 0$, as $i \to \infty$, for all α's in $\mathbb{R}$ and all x's in X.

Throughout the paper, $t = (t_{kl})$ will denote a double sequence of strictly positive real numbers (not necessarily bounded, in general) and we shall write for simplicity in notation that $\inf t_{kl} = \inf_{k,l\in\mathbb{N}} t_{kl}$ and $\sup t_{kl} = \sup_{k,l\in\mathbb{N}} t_{kl}$.

In the present study; we define the paranormed double sequence space $\mathcal{M}_u(t)$ of all bounded double sequences, as follows:

$$\mathcal{M}_u(t) := \left\{ x = (x_{kl}) \in \Omega : \sup_{k,l\in\mathbb{N}} |x_{kl}|^{t_{kl}} < \infty \right\}.$$

Let $t = (t_{kl}) \in \mathcal{M}_u$ and $M = \max\{1, \sup t_{kl}\}$. Now, one can easily observe by similar approach used in single sequences that the set $\mathcal{M}_u(t)$ is complete paranormed space with the paranorm

$$g(x) = \sup_{k,l\in\mathbb{N}} |x_{kl}|^{t_{kl}/M}$$

if and only if $\inf t_{kl} > 0$. When all terms of $t = (t_{kl})$ are constant and equal to $q > 0$, then $\mathcal{M}_u(t) = \mathcal{M}_u$.

By $\mathbf{e^{kl}} = \left(\mathbf{e^{kl}_{\mathit{ij}}}\right)$, we mean the double sequence defined by

$$\mathbf{e}^{\mathbf{kl}}_{ij} := \begin{cases} 1 & , \quad (i,j) = (k,l), \\ 0 & , \quad (i,j) \neq (k,l) \end{cases}$$

for each $k, l \in \mathbb{N}$. All considered spaces contain Φ, the set of all finitely non-zero double sequences, i.e.,

$$\begin{aligned} \Phi &:= \left\{ x = (x_{kl}) \in \Omega : \exists\, N \in \mathbb{N} \ni \forall\, (k,l) \in \mathbb{N}^2 \backslash [0,N]^2,\ x_{kl} = 0 \right\} \\ &:= \operatorname{span}\left\{ \mathbf{e^{kl}} : k,l \in \mathbb{N} \right\}. \end{aligned}$$

Also, we use $\mathbf{e}$, $\mathbf{e_k}$ and $\mathbf{e^l}$ given by

$\mathbf{e} := \sum_{k,l} \mathbf{e^{kl}}$; the double sequence that all terms are one,

$\mathbf{e_k} := \sum_l \mathbf{e^{kl}}$; the double sequence that all terms of k-th row are one and other terms are zero,

$\mathbf{e^l} := \sum_k \mathbf{e^{kl}}$; the double sequence that all terms of l-th column are one and other terms are zero.

It is easy to see that the set $\left\{ \mathbf{e^{kl}}, \mathbf{e_k}, \mathbf{e^l}, \mathbf{e}; k,l \in \mathbb{N} \right\}$ generates a subspace of the space $\mathcal{M}_u(t)$.

2. The Sequence Space $\mathcal{M}_u(t)$

In this section, we give inclusion theorems related to the space $\mathcal{M}_u(t)$, and examine some topological properties of this space.

Theorem 2.1. *Let* $\inf t_{kl} > 0$*. Then, the set* $\mathcal{M}_u(t)$ *is overlap with* $\mathcal{C}_p$ *and* $\mathcal{C}_{p0}$ *but neither contains the other.*

Proof: It is trivial that $\mathbf{e^{kl}} \in \mathcal{C}_{p0} \cap \mathcal{M}_u(t)$ and $\{(-1)^{k+l}\} \in \mathcal{M}_u(t) \setminus \mathcal{C}_p$. Define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} l & , \quad k = 0 \text{ and } l \in \mathbb{N}, \\ 0 & , \quad k \geq 1 \text{ and } l \in \mathbb{N}. \end{cases}$$

Then, $x \in \mathcal{C}_{p0} \setminus \mathcal{M}_u(t)$. Hence, $\mathcal{M}_u(t)$ is overlap with both the spaces $\mathcal{C}_p$ and $\mathcal{C}_{p0}$ but neither contains the other. □

Theorem 2.2. *The following statements hold:*

(i) $\mathcal{M}_u \subset \mathcal{M}_u(t)$ *if and only if* $\sup t_{kl} < \infty$.

(ii) $\mathcal{M}_u(t) \subset \mathcal{M}_u$ *if and only if* $\inf t_{kl} > 0$.

(iii) $\mathcal{M}_u(t) = \mathcal{M}_u$ *if and only if* $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$.

Proof: (i) Sufficiency part is trivial. So, we omit it. Suppose that $\mathcal{M}_u \subset \mathcal{M}_u(t)$ but $\sup t_{kl} = \infty$. Then, there exist the index sequences (k_i) and (l_i) of natural numbers, at least one of them is strictly increasing, such that $t_{k_i,l_i} > i + 1$. We define the sequence $x = (x_{kl}) \in \mathcal{M}_u$ by

$$x_{kl} := \begin{cases} 2 & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$. Then, one can easily see that

$$\sup_{k,l \in \mathbb{N}} |x_{kl}|^{t_{kl}} = \sup_{i \in \mathbb{N}} 2^{t_{k_i,l_i}} > \sup_{i \in \mathbb{N}} 2^{i+1} = \infty$$

which gives the fact $x \in \mathcal{M}_u \setminus \mathcal{M}_u(t)$. This contradicts $\mathcal{M}_u \subset \mathcal{M}_u(t)$. Hence, $\sup t_{kl}$ must be finite.

(ii) We only show the necessity part. Let $\mathcal{M}_u(t) \subset \mathcal{M}_u$ but $\inf t_{kl} = 0$. Then, again there exist the index sequences (k_i) and (l_i) of natural numbers, at least one of them is strictly increasing, such that $t_{k_i,l_i} < 1/(i+1)$. We define $x = (x_{kl}) \notin \mathcal{M}_u$ by

$$x_{kl} := \begin{cases} i+1 & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$ which gives that

$$\sup_{k,l\in\mathbb{N}} |x_{kl}|^{t_{kl}} = \sup_{i\in\mathbb{N}} (i+1)^{t_{k_i,l_i}} \leq \sup_{i\in\mathbb{N}} (i+1)^{1/(i+1)} \leq 2,$$

i.e., $x \in \mathcal{M}_u(t) \setminus \mathcal{M}_u$, a contradiction. Therefore, $\inf t_{kl}$ must be positive.

(iii) This is trivial from Parts (i) and (ii) of the present theorem. □

Theorem 2.3. *$\mathcal{C}_{bp} \subset \mathcal{M}_u(t)$ if and only if $t \in \mathcal{M}_u$.*

Proof: We only consider the necessity part. Let $\mathcal{C}_{bp} \subset \mathcal{M}_u(t)$ but $t \notin \mathcal{M}_u$. Then, the sequence $x = (x_{kl})$ with $x_{kl} := 2$ for all $k, l \in \mathbb{N}$, is in the set $\mathcal{C}_{bp} \setminus \mathcal{M}_u(t)$, a contradiction. Hence, t must be in $\mathcal{M}_u$. □

Now, one can easily derive the following corollary:

Corollary 2.4. *$\mathcal{C}_{bp0} \subset \mathcal{M}_u(t)$ if and only if $t \in \mathcal{M}_u$.*

Theorem 2.5. *The set $\mathcal{M}_u(t)$ is a linear space if and only if $t \in \mathcal{M}_u$.*

Proof: We only show the necessity part. Let $\mathcal{M}_u(t)$ be a linear space but $t \notin \mathcal{M}_u$. Consider the sequences x and t used in the proof of Part (i) of Theorem 2.2. Then, the sequence $x/2 \in \mathcal{M}_u(t)$ while $x \notin \mathcal{M}_u(t)$. However, this contradicts the linearity of the set. Therefore, t must be in the space $\mathcal{M}_u$. □

Definition 2.6. [8, p. 225] *A double sequence space λ containing Φ is said to be monotone if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all double sequences consisting of 0's and 1's.*

If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta(p)} = \lambda^{\beta(bp)} = \lambda^{\beta(r)}$. But the converse is not true, in general.

Theorem 2.7. *The space $\mathcal{M}_u(t)$ is monotone for all t's.*

Proof: Let $x = (x_{kl}) \in \mathcal{M}_u(t)$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Since $|x_{kl}u_{kl}|^{t_{kl}} \leq |x_{kl}|^{t_{kl}}$ for all $k, l \in \mathbb{N}$, it is clear that $xu \in \mathcal{M}_u(t)$. Hence, $\mathcal{M}_u(t)$ is monotone. □

3. Dual Spaces of $\mathcal{M}_u(t)$

In this section, we determine the dual spaces of the set $\mathcal{M}_u(t)$ for all t's. In the rest of the study, ζ denotes any of the symbols α, $\beta(\vartheta)$ or γ, and $\lambda^{n\zeta} = \left\{\lambda^{(n-1)\zeta}\right\}^{\zeta}$ for any space λ and an integer $n \in \mathbb{N}_1$, the set of positive integers.

The α-*dual* λ^{α}, the $\beta(\vartheta)$-*dual* $\lambda^{\beta(\vartheta)}$ and γ-*dual* λ^{γ} of a double sequence space λ are respectively defined by

$$\begin{aligned}
\lambda^{\alpha} &:= \left\{a=(a_{kl})\in\Omega : \sum_{k,l}|a_{kl}x_{kl}|<\infty \text{ for all } x=(x_{kl})\in\lambda\right\},\\
\lambda^{\beta(\vartheta)} &:= \left\{a=(a_{kl})\in\Omega : \left(\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\right)_{m,n\in\mathbb{N}}\in\mathcal{C}_{\vartheta} \text{ for all } x=(x_{kl})\in\lambda\right\},\\
\lambda^{\gamma} &:= \left\{a=(a_{kl})\in\Omega : \sup_{m,n\in\mathbb{N}}\left|\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\right|<\infty \text{ for all } x=(x_{kl})\in\lambda\right\}.
\end{aligned}$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^{\zeta}\subset\lambda^{\zeta}$ whenever $\lambda\subset\mu$. Also, $\lambda^{\alpha}\subset\lambda^{\beta(\vartheta)}$ and $\lambda^{\alpha}\subset\lambda^{\gamma}$. Further, $\lambda^{\beta(\eta)}\subset\lambda^{\gamma}$ for $\eta\in\{bp,r\}$, and if λ is monotone then $\lambda^{\alpha}=\lambda^{\beta(\vartheta)}\subset\lambda^{\gamma}$.

Now, we define the sets $\mathcal{M}_{\infty}^{(1)}(t)$ and $\mathcal{M}_{0}^{(1)}(t)$, as follows:

$$\begin{aligned}
\mathcal{M}_{\infty}^{(1)}(t) &:= \bigcap_{N=2}^{\infty}\left\{a=(a_{kl})\in\Omega : \sum_{k,l}|a_{kl}|N^{1/t_{kl}}<\infty\right\},\\
\mathcal{M}_{0}^{(1)}(t) &:= \bigcup_{N=2}^{\infty}\left\{a=(a_{kl})\in\Omega : \sup_{k,l\in\mathbb{N}}|a_{kl}|N^{-1/t_{kl}}<\infty\right\}.
\end{aligned}$$

One can see that the sets $\mathcal{M}_{\infty}^{(1)}(t)$, $\mathcal{M}_{0}^{(1)}(t)$ are monotone spaces for all t's and

(i) $\mathcal{M}_{\infty}^{(1)}(t)=\mathcal{L}_{u}$ if and only if $\inf t_{kl}>0$,

(ii) $\mathcal{M}_{0}^{(1)}(t)=\mathcal{M}_{u}$ if and only if $\inf t_{kl}>0$.

Theorem 3.1. $\{\mathcal{M}_{u}(t)\}^{\zeta}=\mathcal{M}_{\infty}^{(1)}(t)$.

Proof: This is similar to the proof given for $\alpha-$ and $\gamma-$duals of the spaces of single sequences. Additionally, since $\mathcal{M}_{u}(t)$ is monotone; we get $\{\mathcal{M}_{u}(t)\}^{\alpha}=\{\mathcal{M}_{u}(t)\}^{\beta(\vartheta)}=\mathcal{M}_{\infty}^{(1)}(t)$. □

Theorem 3.2. $\left\{\mathcal{M}_{\infty}^{(1)}(t)\right\}^{\zeta}=\mathcal{M}_{0}^{(1)}(t)$.

Proof: $\mathcal{M}_{0}^{(1)}(t)\subset\left\{\mathcal{M}_{\infty}^{(1)}(t)\right\}^{\zeta}$: Let $a=(a_{kl})\in\mathcal{M}_{0}^{(1)}(t)$ and $x=(x_{kl})\in\mathcal{M}_{\infty}^{(1)}(t)$. Then, we get for some integer $N>1$ that

$$\begin{aligned}
\sum_{k,l}|a_{kl}x_{kl}| &= \sum_{k,l}\left|a_{kl}N^{-1/t_{kl}}x_{kl}N^{1/t_{kl}}\right|\\
&\leq \sup_{k,l\in\mathbb{N}}|a_{kl}|N^{-1/t_{kl}}\sum_{k,l}|x_{kl}|N^{1/t_{kl}}<\infty
\end{aligned}$$

which gives the fact $a \in \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\alpha$. Furthermore, since $\mathcal{M}_\infty^{(1)}(t)$ is monotone, we have the inclusions

$$\mathcal{M}_0^{(1)}(t) \subset \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\alpha = \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^{\beta(\vartheta)} \subset \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\gamma. \tag{3.1}$$

$\left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\zeta \subset \mathcal{M}_0^{(1)}(t)$: Let $\zeta = \gamma$ and suppose that $a = (a_{kl}) \in \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\gamma \setminus \mathcal{M}_0^{(1)}(t)$. Then, we have for any $x = (x_{kl}) \in \mathcal{M}_\infty^{(1)}(t)$ and all integers $N > 1$ that

$$\sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl}x_{kl} \right| < \infty \quad \text{and} \quad \sup_{k,l\in\mathbb{N}} |a_{kl}|N^{-1/t_{kl}} = \infty.$$

Then, we may find the index sequences (k_i) and (l_i) of natural numbers, at least one of them is strictly increasing such that

$$|a_{k_i,l_i}|(i+2)^{-1/t_{k_i,l_i}} > (i+2)^2.$$

We define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} (i+2)^{-2}(i+2)^{-1/t_{kl}}\, sgn\ a_{kl} & , \quad k = k_i \ \text{ and } \ l = l_i, \\ 0 & , \quad k \neq k_i \ \text{ or } \ l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$. Then, we obtain for any integer $N > 1$ that

$$\begin{aligned} \sum_{k,l} |x_{kl}|N^{1/t_{kl}} &= \sum_i \frac{N^{1/t_{k_i,l_i}}}{(i+2)^2(i+2)^{1/t_{k_i,l_i}}} \\ &\leq \sum_{i=0}^{N-3} \frac{N^{1/t_{k_i,l_i}}}{(i+2)^2(i+2)^{1/t_{k_i,l_i}}} + \sum_{i=N-2}^{\infty} \frac{1}{(i+2)^2} < \infty, \end{aligned}$$

i.e., $x \in \mathcal{M}_\infty^{(1)}(t)$, and additionally

$$\sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl}x_{kl} \right| > \sum_i 1 = \infty,$$

i.e., $a \notin \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\gamma$, a contradiction. Hence, a must be in $\mathcal{M}_0^{(1)}(t)$. Thus, we get the inclusions

$$\left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\alpha = \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^{\beta(\vartheta)} \subset \left\{\mathcal{M}_\infty^{(1)}(t)\right\}^\gamma \subset \mathcal{M}_0^{(1)}(t). \tag{3.2}$$

Therefore, the desired result follows by combining (3.1) and (3.2). □

Theorem 3.3. $\left\{\mathcal{M}_0^{(1)}(t)\right\}^\zeta = \mathcal{M}_\infty^{(1)}(t)$.

Proof: $\mathcal{M}_\infty^{(1)}(t) \subset \left\{\mathcal{M}_0^{(1)}(t)\right\}^\zeta$: Now, this part is trivial from the proof of first inclusion of Theorem 3.2.

$\left\{\mathcal{M}_0^{(1)}(t)\right\}^\zeta \subset \mathcal{M}_\infty^{(1)}(t)$: Let $\zeta = \gamma$ and suppose that $a = (a_{kl}) \in \left\{\mathcal{M}_0^{(1)}(t)\right\}^\gamma \setminus \mathcal{M}_\infty^{(1)}(t)$. Then, we have for some integer $N > 1$ that $\sum_{k,l} |a_{kl}| N^{1/t_{kl}} = \infty$. We define $x = (x_{kl})$ by $x_{kl} := N^{1/t_{kl}} sgn\; a_{kl}$ for all $k, l \in \mathbb{N}$ and some integer $N > 1$. Then, we get $\sup_{k,l\in\mathbb{N}} |x_{kl}| N^{-1/t_{kl}} = 1$, that is, $x \in \mathcal{M}_0^{(1)}(t)$ but

$$\sup_{m,n\in\mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl} x_{kl} \right| = \sum_{k,l} |a_{kl}| N^{1/t_{kl}} = \infty,$$

i.e., $a \notin \left\{\mathcal{M}_0^{(1)}(t)\right\}^\gamma$, a contradiction. Hence, a must be in $\mathcal{M}_\infty^{(1)}(t)$. Thus, we obtain the inclusions

$$\left\{\mathcal{M}_0^{(1)}(t)\right\}^\alpha = \left\{\mathcal{M}_0^{(1)}(t)\right\}^{\beta(\vartheta)} \subset \left\{\mathcal{M}_0^{(1)}(t)\right\}^\gamma \subset \mathcal{M}_\infty^{(1)}(t).$$

This step completes the proof. □

Now, one can easily derive the following corollary proved by mathematical induction for all $k \in \mathbb{N}_1$.

Corollary 3.4. $$\{\mathcal{M}_u(t)\}^{n\zeta} := \begin{cases} \mathcal{M}_\infty^{(1)}(t) & , \quad n = 2k-1, \\ \mathcal{M}_0^{(1)}(t) & , \quad n = 2k. \end{cases}$$

If we choose $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$, then we have the following conclusion for all $k \in \mathbb{N}_1$:

Corollary 3.5. $$\mathcal{M}_u^{n\zeta} := \begin{cases} \mathcal{L}_u & , \quad n = 2k-1, \\ \mathcal{M}_u & , \quad n = 2k. \end{cases}$$

Definition 3.6. [9, p. 342] Let λ be a sequence space. Then, λ is called a $\zeta-$*space* if $\lambda = \lambda^{2\zeta}$. Further, an $\alpha-$space is also called *Köthe space* or *perfect sequence space*.

Theorem 3.7. *$\mathcal{M}_u(t)$ is perfect if and only if $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$.*

Proof: Let $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$. Then, we have $\mathcal{M}_u(t) = \mathcal{M}_u = \mathcal{M}_0^{(1)}(t)$. Hence, $\mathcal{M}_u(t)$ is perfect.

Conversely, suppose that $\mathcal{M}_u(t)$ is perfect, i.e., $\mathcal{M}_u(t) = \mathcal{M}_0^{(1)}(t)$, but $\inf t_{kl} = 0$ or $\sup t_{kl} = \infty$.

(i) Let $\inf t_{kl} = 0$. Then, we put $t_{k_i,l_i} < 1/(i+1)$ and define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} (i+1)N^{1/t_{kl}} & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$ and for some integer $N > 1$. Therefore, we obtain

$$\sup_{k,l\in\mathbb{N}} |x_{kl}|^{t_{kl}} = N \sup_{i\in\mathbb{N}} (i+1)^{t_{k_i,l_i}} \leq N \sup_{i\in\mathbb{N}} (i+1)^{1/(i+1)} \leq 2N,$$

i.e., $x \in \mathcal{M}_u(t)$ but

$$\sup_{k,l\in\mathbb{N}} |x_{kl}| N^{-1/t_{kl}} = \sup_{i\in\mathbb{N}} (i+1) = \infty,$$

i.e., $x \notin \mathcal{M}_0^{(1)}(t)$, a contradiction. Hence, $\inf t_{kl}$ must be positive.

(ii) Let $\sup t_{kl} = \infty$. Then, we put $t_{k_i,l_i} > i+1$ and define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} 2N^{1/t_{kl}} & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$ and for some integer $N > 1$. Therefore, it is trivial that $x \in \mathcal{M}_0^{(1)}(t) \setminus \mathcal{M}_u(t)$, a contradiction. Hence, $\sup t_{kl}$ must be finite. □

Since there are various convergence rules for double sequences, we give a new definition for $\beta-$space.

Definition 3.8. *Let λ be a double sequence space and the symbols ϑ, ν denote any convergence rule. Then, we call that λ is a $\beta(\vartheta,\nu)-$space if $\lambda = \left\{\lambda^{\beta(\vartheta)}\right\}^{\beta(\nu)}$ for fixed ϑ, ν's and is a $\beta-$space if $\lambda = \left\{\lambda^{\beta(\vartheta)}\right\}^{\beta(\nu)}$ for all ϑ, ν's. In this section, we only use this definition for $\vartheta, \nu \in \{p, bp, r\}$.*

We can give the following theorem without proof since it can be proved in the similar way used in the proof of Theorem 3.7:

Theorem 3.9. *The following statements hold:*

(a) $\mathcal{M}_u(t)$ *is a* $\beta-$*space if and only if* $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$.

(b) $\mathcal{M}_u(t)$ *is a* $\gamma-$*space if and only if* $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$.

Now, we have the following corollary:

Corollary 3.10. $\mathcal{M}_u(t)$ *is a* $\zeta-$*space if and only if* $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$.

Also, one can easily show the following theorem:

Theorem 3.11. $\mathcal{M}_\infty^{(1)}(t)$ *and* $\mathcal{M}_0^{(1)}(t)$ *are* $\zeta-$*spaces for all* t*'s.*

4. Matrix Transformations

Let λ and μ be two double sequence spaces, and $A = (a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix transformation* from λ into μ and we write $A : \lambda \to \mu$, if for every sequence $x = (x_{kl}) \in \lambda$ the A-transform $Ax = \{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and belongs to μ; where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \quad \text{for each} \quad m, n \in \mathbb{N}. \tag{4.1}$$

We denote the set of all four-dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. Thus, $A \in (\lambda : \mu)$ if and only if the double series on the right side of (4.1) converges in the sense of ϑ, i.e., $A_{mn} \in \lambda^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$, and also we have $Ax \in \mu$ for every $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l\in\mathbb{N}}$ for each $m, n \in \mathbb{N}$. Here, we note that on four-dimensional matrix transformations ϑ must be fixed, in general, otherwise the results may be incorrect. In this paper, we do not fix ϑ since the $\beta(\vartheta)-$duals of corresponding spaces are identical.

For all $m, n, k, l \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both, [10]. Following Adams [10], we also say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$. Referring to Cooke [11, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

Theorem 4.1. *Let $t = (t_{kl})$, $q = (q_{kl})$ be any sequences of strictly positive real numbers and $q \in \mathcal{M}_u$. Then, the necessary and sufficient conditions for $A = (a_{mnkl}) \in (X : Y)$ can be read from the following table:*

Y \ X	$\mathcal{M}_u(t)$	$\mathcal{M}_\infty^{(1)}(t)$	$\mathcal{M}_0^{(1)}(t)$
$\mathcal{M}_u$	**1.**	**2.**	**1.**
$\mathcal{M}_u(q)$	**1.**	**2.**	**1.**

where

1.

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}|N^{1/t_{kl}} < \infty \quad \textit{for all integers } N > 1. \tag{4.2}$$

2.

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}|N^{-1/t_{kl}} < \infty \quad \textit{for some integer } N > 1. \tag{4.3}$$

Proof: ***Necessity.*** We only show the necessity part for the class $(\mathcal{M}_u(t) : \mathcal{M}_u)$, since it is similar for the other classes of four-dimensional matrices.

Let $A = (a_{mnkl}) \in (\mathcal{M}_u(t) : \mathcal{M}_u)$. Then, Ax must exist and belongs to the space $\mathcal{M}_u$ for every $x \in \mathcal{M}_u(t)$. In order to Ax be exist, the double sequence $A_{mn} =$

$(a_{mnkl})_{k,l\in\mathbb{N}}$ must be in $\{\mathcal{M}_u(t)\}^{\beta(\vartheta)} = \mathcal{M}_\infty^{(1)}(t)$ for each $m, n \in \mathbb{N}$. Therefore, the necessity of (4.2) is immediate.

Sufficiency. Here, we only show the sufficiency part for the class $(X : \mathcal{M}_u(q))$, where X denotes any of the spaces $\mathcal{M}_u(t)$ or $\mathcal{M}_\infty^{(1)}(t)$ and $q \in \mathcal{M}_u$. For this, we suppose that the characterizing conditions of the class $(X : \mathcal{M}_u(q))$ are satisfied and $x \in X$.

Let $X = \mathcal{M}_u(t)$. Since $x \in \mathcal{M}_u(t)$, we can choose $N > 1$ such that $|x_{kl}|^{t_{kl}} \leq N$. Then, we get

$$\begin{aligned}\sup_{m,n\in\mathbb{N}} \left|\sum_{k,l} a_{mnkl}x_{kl}\right|^{q_{mn}} &\leq \sup_{m,n\in\mathbb{N}} \left(\sum_{k,l} |a_{mnkl}||x_{kl}|\right)^{q_{mn}} \\ &\leq \sup_{m,n\in\mathbb{N}} \left(\sum_{k,l} |a_{mnkl}|N^{1/t_{kl}}\right)^{q_{mn}} < \infty\end{aligned}$$

which gives $Ax \in \mathcal{M}_u(q)$ under the condition $q \in \mathcal{M}_u$. Therefore, $A \in (\mathcal{M}_u(t) : \mathcal{M}_u(q))$.

Let $X = \mathcal{M}_\infty^{(1)}(t)$. Then, for some integer $N > 1$ we have

$$\begin{aligned}\sup_{m,n\in\mathbb{N}} \left|\sum_{k,l} a_{mnkl}x_{kl}\right|^{q_{mn}} &\leq \sup_{m,n\in\mathbb{N}} \left(\sum_{k,l} |a_{mnkl}|N^{-1/t_{kl}}|x_{kl}|N^{1/t_{kl}}\right)^{q_{mn}} \\ &\leq \sup_{m,n\in\mathbb{N}} \left(\sup_{k,l\in\mathbb{N}} |a_{mnkl}|N^{-1/t_{kl}} \sum_{k,l} |x_{kl}|N^{1/t_{kl}}\right)^{q_{mn}} \\ &< \infty,\end{aligned}$$

which gives $Ax \in \mathcal{M}_u(q)$ under the condition $q \in \mathcal{M}_u$. Therefore, $A \in (\mathcal{M}_\infty^{(1)}(t) : \mathcal{M}_u(q))$. $\square$

Let $0 < \inf t_{kl} \leq \sup t_{kl} < \infty$ and $q \in \mathcal{M}_u$. Then, we have the following corollary:

Corollary 4.2. *The necessary and sufficient conditions for $A = (a_{mnkl}) \in (X : Y)$ can be read from the following table:*

Y \ X	$\mathcal{M}_u$	$\mathcal{L}_u$	$\mathcal{M}_u$
$\mathcal{M}_u$	**3.**	**4.**	**3.**
$\mathcal{M}_u(q)$	**3.**	**4.**	**3.**

where

3.

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty.$$

4.

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}| < \infty.$$

5. Conclusion

In [7], Gökhan and Çolak introduced the space $\mathcal{M}_u(t)$. They stated by Part (ii) of Theorem 2 that $\mathcal{M}_u(t)$ is a paranormed space if and only if $\inf_{k,l\in\mathbb{N}_1} t_{kl} > 0$, and examined its dual spaces. However, the proofs are very complicated and contain some missing points. For instance, in the proof of Part (i) of Theorem 1 in [7], they wrote for a double sequence (p_{mn}) of strictly positive real numbers that if $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$ then there are two cases:

(a) There exist a strictly increasing sequence $\{m(i)\}$ of positive integers and a fixed positive integer j_0 such that $p_{m(i),n(j_0)} < 1/(i+1)$.

(b) There exist strictly increasing sequences $\{m(i)\}$ and $\{n(j)\}$ of positive integers such that $p_{m(i),n(j)} < 1/(i+j) < 1/i$.

Nevertheless, these cases do not include all possibilities whenever $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$. One can easily observe this by means of the sequence $p = (p_{mn})$ defined by

$$p_{mn} := \begin{cases} 1/m & , \quad m = n, \\ 1 & , \quad m \neq n \end{cases}$$

for all $m, n \in \mathbb{N}_1$. Clearly, $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$ and Part (a) is invalid. To obtain $p_{m(i),n(j)} < 1/i$, we must take $m(i) = n(j)$ for all $i, j \in \mathbb{N}_1$. This implies that $m(i) = n(j) = k$ for all $i, j \in \mathbb{N}_1$ and a fixed integer $k \in \mathbb{N}_1$. Therefore, they are not increasing sequences. Also, even if they are strictly increasing sequences, we get for infinitely many $m(i)$ and $n(j)$'s that $p_{m(i),n(j)} = 1$. Thus, Part (b) is invalid too. In this paper, we repair such mistakes and also calculate $n-$th $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals of the space $\mathcal{M}_u(t)$. Moreover, we characterize some classes of four-dimensional matrix transformations including the space $\mathcal{M}_u(t)$ and its dual spaces. So, the present study may consider as a complement of Gökhan and Çolak [7].

As a natural continuation of this paper, our next works will be devoted for investigation of the paranormed spaces $\mathcal{C}_p(t)$, $\mathcal{C}_{p0}(t)$, $\mathcal{C}_{bp}(t)$, $\mathcal{C}_{bp0}(t)$, $\mathcal{C}_r(t)$ and $\mathcal{C}_{r0}(t)$ corresponding to the spaces $\mathcal{C}_p$, $\mathcal{C}_{p0}$, $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ of all $p-$convergent, $p-$null, $bp-$convergent, $bp-$null, $r-$convergent and $r-$null double sequences, respectively.

Başar et al. [12] and Altay and Başar [13] have introduced the paranormed spaces $bv(u,p)$ and $r^q(p)$ of all sequences $x = (x_k)$ such that $\{u_k(x_k - x_{k-1})\}$ and $\left(\sum_{k=0}^{n} q_k x_k / Q_n\right)$ belong to the space $\ell(p)$, respectively. We note that to obtain more general spaces of double sequences with some algebraic and topological properties, following [12] and [13], one can investigate the domain of certain four dimensional triangles, for example four dimensional backward difference matrix Δ or Riezs mean R^{qs} with respect to the sequences $q = (q_k)$ and $s = (s_l)$ of non-negative numbers which are not all zero, in the space $\mathcal{M}_u(t)$.

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