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The Backward Operator of Double Almost $(\lambda_m\mu_n)$ Convergence in χ^2-Riesz Space Defined By a Musielak-Orlicz Function

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ABSTRACT: In this paper we introduce the backward operator is ∇ and study the notion backward operator of $\nabla-$ statistical convergence and backward operator of $\nabla-$ statistical Cauchy sequence using by almost $(\lambda_m\mu_n)$ convergence in χ^2-Riesz space and also some inclusion theorems are discussed.

Key Words: Aanalytic sequence, Musielak-Orlicz function, Double sequences, Chi sequence,Lambda, Riesz space, strongly, Statistical convergent.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Tripathy [1] and Mursaleen [2] and Mursaleen and Edely [3,4], Subramanian and Misra [5], Pringsheim [6], Moricz and Rhoades [7], Robison [8], Savas et al. [9], Raj et al. [10], Francesco Tulone [11] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

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$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, ... \right\}, \tag{1.1}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if

$$((m + n)! |x_{mn}|)^{\frac{1}{m+n}} \to 0,$$

as $m, n \to \infty$. That is, $|x_{mn}| \to 0$. The double gai sequences will be denoted by χ^2.

2. Definitions and Preliminaries

A double sequence $x = (x_{mn})$ has limit 0 (denoted by $P - limx = 0$) (i.e) $((m + n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. (i.e) $|x_{mn}| \to 0$. We shall write more briefly as $P - convergent\ \ to\ \ 0$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Lemma 2.1. *Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

A double sequence $M = (M_{mn})$ of Orlicz function is called a Musielak-Orlicz function [see [12]]. A double sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (M_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a sequence of Musielak-Orlicz M. For a given sequence of Musielak-Orlicz function M, the Musielak-Orlicz sequence space t_M is defined as follows

$$t_M = \left\{x \in w^2 : I_M(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|)^{1/m+n}.$$

Definition 2.2. *A double sequence $x = (x_{mn})$ of real numbers is called almost $P-$ convergent to a limit 0 if*

$$P - lim_{p,q\to\infty} sup_{r,s\geq 0} \frac{1}{pq} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} ((m+n)!\,|x_{mn}|)^{1/m+n} \to 0.$$

that is, the average value of (x_{mn}) taken over any rectangle $\{(m, n) : r \leq m \leq r+p-1, s \leq n \leq s+q-1\}$ tends to 0 as both p and q to ∞, and this $P-$ convergence is uniform in r and s. Let denote the set of sequences with this property as $\left[\widehat{\chi^2}\right]$.

Definition 2.3. *Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and*
$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1,\ \ \mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$
Let $I_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$.
For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number

$$\delta_{\lambda,\mu}(K) = lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, (i,j) \in K\}|,$$

is called the $(\lambda, \mu)-$ density of the set K provided the limit exists.

Definition 2.4. *A double sequence $x = (x_{mn})$ of numbers is said to be $(\lambda, \mu)-$ statistical convergent to a number ξ provided that for each $\epsilon > 0$,*
$lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}| = 0$,
that is, the set $K(\epsilon) = \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}|$ has $(\lambda, \mu)-$ density zero. In this case the number ξ is called the $(\lambda, \mu)-$ statistical limit of the sequence $x = (x_{mn})$ and we write $St_{(\lambda,\mu)} lim_{m,n\to\infty} = \xi$.

Definition 2.5. *The double sequence $\theta_{i,\ell} = \{(m_i, n_\ell)\}$ is called double lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$

Let $m_{i,\ell} = m_i n_\ell, h_{i,\ell} = h_i \overline{h_\ell}$, *and* $\theta_{i,\ell}$ *is determine by*
$I_{i,\ell} = \{(m,n) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}.$

Definition 2.6. *Let* M *be an Orlicz function and* $P = (p_{mn})$ *be any factorable double sequence of strictly positive real numbers, we define the following sequence space:*

$$\chi^2_M \left[AC_{\lambda_m \mu_n}, P\right] =$$
$$\left\{(x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m \mu_n} \sum_{(m,n)\in I_{r,s}} \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0\right\},$$

uniformly in r *and* s.

We shall denote $\chi^2_M \left[AC_{\lambda_m \mu_n}, P\right]$ *as* $\chi^2 \left[AC_{\lambda_m \mu_n}\right]$ *respectively when* $p_{mn} = 1$ *for all* m *and* n. *If* x *is in* $\chi^2 \left[AC_{\lambda_m \mu_n}, P\right]$, *we shall say that* x *is almost* $(\lambda_m \mu_n)$ *in* χ^2 *strongly* $P-$*convergent with respect to the Orlicz function* M. *Also note if* $M(x) = x, p_{mn} = 1$ *for all* m, n *and* k *then* $\chi^2_M \left[AC_{\lambda_m \mu_n}, P\right] = \chi^2 \left[AC_{\lambda_m \mu_n}, P\right]$, *which are defined as follows:*

$$\chi^2 \left[AC_{\lambda_m \mu_n}, P\right] =$$
$$\left\{(x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m \mu_n} \sum_{(m,n)\in I_{r,s}} \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0\right\},$$

uniformly in r *and* s.
Again note if $p_{mn} = 1$ *for all* m *and* n *then* $\chi^2_M \left[AC_{\lambda_m \mu_n}, P\right] = \chi^2_M \left[AC_{\lambda_m \mu_n}\right]$. *We define*

$$\chi^2_M \left[AC_{\lambda_m \mu_n}, P\right] =$$
$$\left\{(x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m \mu_n} \sum_{m,n\in I_{r,s}} \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0\right\},$$

uniformly in r *and* s.

Definition 2.7. *Let* M *be an Orlicz function and* $P = (p_{mn})$ *be any factorable double sequence of strictly positive real numbers, we define the following sequence space:*

$$\chi^2_M [P] =$$
$$\left\{(x_{mn}) : P - lim_{p,q\to\infty} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0\right\},$$

uniformly in r *and* s.

If we take $M(x) = x, p_{mn} = 1$ *for all* m *and* n *then* $\chi^2_M [P] = \chi^2$.

Definition 2.8. *The double number sequence x is $\widehat{S_{\lambda_m\mu_n}} - P-$ convergent to* 0 *then*
$\left\{(x_{mn}) : P - lim_{m,n} \frac{1}{\lambda_m\mu_n} \max_{r,s} |A| = 0\right\}$, *where*

$$A = \left\{(m,n) \in I_{r,s} : \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}\right\}.$$

In this case we write

$$\widehat{S_{\lambda_m\mu_n}} - lim\left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0.$$

3. The Backward operator of convergence of double almost $(\lambda_m\mu_n)$ in χ^2 Riesz space

Let X, Y be a real vector space of dimension w, where $n \leq m$. A real valued function $F(d_p(x_1,\ldots,x_n),t) = F(\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,t)$ on X satisfying the following conditions:
(i) $F(\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,t) = 0$ if and and only if $F(d_1(x_1,0),\ldots,d_n(x_n,0),t)$ are linearly dependent,
(ii) $F(\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,t)$ is invariant under permutation,
(iii) $F(\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p,t) = F(|\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,t)$, $\alpha \in \mathbb{R}$
(iv)

$$F\left(d_p((x,y),t) = F(d_X(x_1,x_2,\cdots x_n)^p,t) + F(d_Y(y_1,y_2,\cdots y_n)^p)^{1/p},t\right),$$

where $(x,y) = ((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n))$, for $1 \leq p < \infty$; (or)
(v)

$$F(d(x,y),t) := \sup F(\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\},t),$$

for $(X \times X \times \cdots \times X, F, *)$ is called the p product metric of the Cartesian product of n metric spaces.

Definition 3.1. *Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties :*
(i) If $x, y \in L$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in L$.
(ii) If $x, y \in L$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.
If in addition L is a lattice with respect to the partial ordering, then L is said to be Riesz space.
A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.
A linear topology τ on a Riesz space X is said to be locally solid if τ has a base at zero consisting of solid sets.

Definition 3.2. *Let* $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of points in* χ^2 *is said to be* $S(\tau)-$ *convergent in* $(X \times X \times \cdots \times X, F, *)$ *if for each* $t > 0,\ \theta \in (0,1)$ *and for non zero* $z \in X$ *such that*

$$\delta\left(\left\{m, n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\theta\right\}\right)\right) = 0$$

that is ,

$$\left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F(w,z,t) \leq 1-\theta\right\}\right) = 0,\ \textit{where}$$

$$w = \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}.$$

In this case we write

$$S(\tau) - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F(w,z,t)\right\}\right) = 1.$$

Definition 3.3. *Let* $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of points in* χ^2 *is said to be backward operator of* $\nabla-$ *convergent in* $(X \times X \times \cdots \times X, F, *)$ *if for each* $t > 0,\ \beta \in (0,1)$ *there exists an positive integer* n_0 *such that*

$$F\left\{\left[M\left(\frac{((m+n)!|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right\} > 1-\beta.$$

whenever $m, n \geq n_0$ *and for non zero* $z \in X$.

Definition 3.4. *Let* $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of points in* χ^2 *is said to be backward operator of* $\nabla-$ *Cauchy in* $(X \times X \times \cdots \times X, F, *)$ *if for each* $t > 0,\ \beta \in (0,1)$ *there exists an positive integer* $n_0 = n_0(\epsilon)$ *such that*

$$F\left\{\left[M\left(\frac{((m+n)!|x_{mn}-x_{rs}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right\} < 1-\theta.$$

whenever $m, n, r, s \geq n_0$ *and for non zero* $z \in X$.

Definition 3.5. *Let* $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of points in* χ^2 *is said to be* $S(\tau)-$ *convergent in* $(X \times X \times \cdots \times X, F, *)$ *if for each* $t > 0,\ \beta \in (0,1)$ *and for non zero* $z \in X$ *such that*

$$\delta_\nabla\left(\left\{m, n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\beta\right\}\right)\right) = 0$$

that is ,
$\left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z; t) \leq 1-\beta\right\}\right) = 0$, *with*

$$w_0 = \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}.$$

In this case we write
$S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z; t)\right\}\right) = 1.$

Definition 3.6. *Let* $\chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of points in* χ^2 *is said to be backward operator of* $\nabla-$ *Cauchy in* $(X \times X \times \cdots \times, F, *)$ *if for each* $t > 0$, $\beta \in (0,1)$ *there exists an positive integer* $n_0 = n_0(\epsilon)$ *such that*

$$\delta_\nabla\left(\left\{m,n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{mn}-x_{rs}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\beta\right\}\right)\right) = 0$$

or equivalently,

$$\delta_\nabla\left(\left\{m,n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{mn}-x_{rs}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) > 1-\beta\right\}\right)\right) = 1.$$

4. Main Results

Proposition 4.1. *Let* $\chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *of* χ^2 *in* $(X \times X \times \cdots \times X, F, *)$ *if for each* $t > 0$, $\beta \in (0,1)$ *and for non zero* $z \in X$, *then the following statements are equivalent*
(i)$\delta_\nabla\left(\left\{m,n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{mn}-x_{rs}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\beta\right\}\right)\right) = 0$
(ii)$\delta_\nabla\left(\left\{m,n \in \mathbb{N}^2 : F\left(\left(\left[M\left(\frac{((m+n)!|x_{mn}-x_{rs}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) > 1-\beta\right\}\right)\right) = 1.$
(iii)$S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z, t)\right\}\right) = 1.$

Theorem 4.2. *Let* $\chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \in S(\tau)_\nabla$ *and* $c \in \mathbb{R}$ *be a almost* $(\lambda_m\mu_n)$ *Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *in* $(X \times X \times \cdots \times X, F, *)$ *then*
(i)$S(\tau)_\nabla - \left((x_{mn}) : P - c\lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z, t)\right\}\right) =$
$c\,S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z, t)\right\}\right).$
(ii)
$S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_1, z, t)\right\}\right)$
$= S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z, t)\right\}\right) +$
$S(\tau)_\nabla - \left((x_{mn}) : P - \lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{\sum_{(m,n)\in I_{r,s}} F(w_2, z, t)\right\}\right)$, *where*

$$w_0 = \left[M\left(\frac{((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},$$

$$w_1 = \left[M \left(\frac{((m+n)! \, |x_{m+r,n+s} + y_{m+r,n+s} - 0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}},$$

and

$$w_2 = \left[M \left(\frac{((m+n)! \, |y_{m+r,n+s} - 0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}}.$$

Proof. The proof of this theorem is straightforward, and thus will be omitted. □

Theorem 4.3. *Let* $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *be a almost* $(\lambda_m\mu_n)$ *Riesz space of Musielak-Orlicz function. A double sequence* (x_{mn}) *analytic in* $(X \times X \times \cdots \times X, F, *)$ *then*
(a)$\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_\nabla$ *implies*
$\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to S(\tau)_\nabla$.
(b)$\Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to S(\tau)_\nabla$ *imply*
$\Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_\nabla$.
(c)$S(\tau)_\nabla \bigcap \Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$W(\tau)_\nabla \bigcap \Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.

Proof. Let $\epsilon > 0$ and $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to W(\tau)_\nabla$ for all $r, s \in \mathbb{N}$, we have
$\left(\lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{ \sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|xy_{m+r,n+s}-0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}}, z; t \right) \right\} \right) \geq \epsilon$
$\left\{ \sum_{(m,n)\in I_{r,s}} F(w_0, z; t) \right\} \geq$
$\left| \left(\lim_{mn} \frac{1}{\lambda_m\mu_n} \left\{ \sum_{(m,n)\in I_{r,s}} F(w_0, z; t) \geq \epsilon \right\} \right) \right| \cdot min\left(\epsilon^h, \epsilon^H\right)$.
Hence $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to S(\tau)_\nabla$.

Proof (b): Suppose that $\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ $\in S(\tau)_\nabla \bigcap \Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Since
$\chi_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \in$
$\Lambda_M^{2\tau}\left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, we write
$\left\{ \sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}}, z, t \right) \right\} \leq T$, for all $r, s \in \mathbb{N}^2$, let
$G_{rs} = \left| \left(\frac{1}{\lambda_m\mu_n} \left\{ \sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}}, z, t \right) \geq \epsilon \right\} \right) \right|$
and
$H_{rs} = \left| \left(\frac{1}{\lambda_m\mu_n} \left\{ \sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho} \right) \right]^{p_{mn}}, z, t \right) < \epsilon \right\} \right) \right|$.
Then we have

$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right) =$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in G_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right) +$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in H_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right) \leq$
$max\left(T^h,T^H\right)G_{rs} + max\left(\epsilon^h,\epsilon^H\right)$. Taking the limit as $\epsilon \to 0$ and $r,s\to\infty$, it follows that $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right] \in W(\tau)_\nabla$.
(c) Follows from (a) and (b). □

Theorem 4.4. *If* $\texttt{lim inf}_{rs}\left(\frac{\lambda_r\mu_s}{rs}\right) > 0$, *then* $S(\tau)\subset S(\tau)_\nabla$

Proof. Let $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right] \in S(\tau)$. For given $\epsilon>0$, we get
$\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\geq\epsilon\right\}\right)\right| \supset G_{rs}$
where G_{rs} is in the theorem of 4.3.(b). Thus,
$\left|\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\geq\epsilon\right\}\right)\right| \geq G_{rs}$
$=\frac{\lambda_r\mu_s}{rs}$. Taking limit as $r,s\to\infty$ and using $\texttt{lim inf}_{rs}\left(\frac{\lambda_r\mu_s}{rs}\right)>0$, we get
$\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right] \in S(\tau)_\nabla$. □

Theorem 4.5. *Let* $0<u_{mn}\leq v_{mn}$ *and* $\left(u_{mn}v^{-1}_{mn}\right)$ *be double analytic. Then* $W(\tau,v)_\nabla \subset w(\tau,u)_\nabla$

.

Proof. Let $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right] \in W(\tau,v)_\nabla$. Let $W(\tau)_\nabla = \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F(w_0,z,t)\right\}\right)$ for all $r,s\in\mathbb{N}^2$ and $\lambda_m\mu_n = u_{mn}v^{-1}_{mn}$ for all $m,n\in\mathbb{N}^2$. Then $0<\lambda_m\mu_n\leq 1$ for all $m,n\in\mathbb{N}^2$. Let b be a constant such that $0<b\leq\lambda_m\mu_n\leq 1$ for all $m,n\in\mathbb{N}^2$.

Define the double sequences (k_{mn}) and (ℓ_{mn}) as follows:
For $W(\tau)_\nabla \geq 1$, let $(k_{mn}) = (W(\tau)_\nabla)$ and $\ell_{mn}=0$ and for $W(\tau)_\nabla<1$, let $k_{mn}=0$ and $\ell_{mn}=W(\tau)_\nabla$. Then it is clear that for all $m,n\in\mathbb{N}$, we have $W(\tau)_\nabla = k_{mn}+\ell_{mn}$ and $W(\tau)^{\lambda_m\mu_n}_\nabla = k^{\lambda_m\mu_n}_{mn}+\ell^{\lambda_m\mu_n}_{mn}$. Now it follows that $k^{\lambda_m\mu_n}_{mn}\leq k_{mn}\leq W(\tau)_\nabla$ and $\ell^{\lambda_m\mu_n}_{mn}\leq\ell^{\lambda\mu}_{mn}$. Therefore

$$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|W(\tau)^{\lambda_m\mu_n}_\nabla\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right) =$$
$$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|(k_{mn}+\ell_{mn})^{\lambda_m\mu_n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right) =$$
$$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|W(\tau)^{\lambda_m\mu_n}_\nabla\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right) +$$

$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|(\ell_{mn})^{\lambda_m\mu_n}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right)$.
Now for each r,s,
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|(\ell_{mn})^{\lambda\mu}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right)=$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|\left((\ell_{mn})^{\lambda\mu}\left(\frac{1}{\lambda_m\mu_n}\right)^{1-\lambda\mu}\right)\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right)^{\lambda\mu}$
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\frac{\left[M\left((m+n)!\left|\left(((\ell_{mn})^{\lambda\mu})^{\lambda\mu}\right)^{1/\lambda\mu}\right|-0\right)^{1/m+n}\right]^{p_{mn}},z,t}{\rho}\right)\right\}\right)$.

Theorem 4.6.

$$\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]=W\left(\tau,\Lambda^2\right)_\nabla,$$

where
$W\left(\tau,\Lambda^2\right)_\nabla = sup\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F(w_0,z,t)<\infty\right\}\right)$

Proof. Let $x=(x_{mn})\in W\left(\tau,\Lambda^2\right)_\nabla$. Then there exists a constant $T_1>0$ such that
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right)\leq$
$sup\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right)\leq T_1$ for all $r,s\in\mathbb{N}$. Therefore we have
$x=(x_{mn})\in\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$. Conversely, let $x=(x_{mn})\in\Lambda^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$. Then there exists a constant $T_2>0$ such that
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right)\leq T_2$ for all m,n and r,s. So,
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,t\right)\right\}\right)\leq$
$T_2\,\frac{1}{\lambda_m\mu_n}\sum_{(m,n)\in I_{r,s}} 1\leq T_2$, for all m,n and r,s. Thus $x=(x_{mn})\in W\left(\tau,\Lambda^2\right)_\nabla$.
□

Theorem 4.7. $\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ *be a almost $(\lambda_m\mu_n)$ Riesz space of Musielak-Orlicz function. A double sequence (x_{mn}) in $(X\times X\times\cdots\times X,F,*)$ is backward operator of $\nabla-$ statistically convergent if and only if it is backwards operator of $\nabla-$statistically Cauchy*

Proof. Let $x=(x_{mn})$ be a backwards operator of $\nabla-$statistically convergent sequence in

$\chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Let $\epsilon > 0$ be given. Choose $s > 0$ such that

$$(1-s)*(1-s) > 1-\epsilon \tag{4.1}$$

is satisfied. For $t > 0$ and non-zero

$$z \in \chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$$

Define
$A(s,t) = \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(w_0, z, \frac{t}{2}\right) \leq 1-s\right\}\right)$ and
$A^c(s,t) = \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F\left(w_0, z, \frac{t}{2}\right) > 1-s\right\}\right)$. It follows that $\delta_\nabla(A(s,t)) = 0$ and consequently $\delta_\nabla(A^c(s,t)) = 1$. Let $\eta \in A^c(s,t)$. Then

$$F\left(\left[M\left(\frac{((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, \frac{t}{2}\right) \leq 1-s \tag{4.2}$$

$B(\epsilon,t) = \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}} F(w_0, z, t) \leq 1-\epsilon\right\}\right)$. It is enough to prove that $B(\epsilon,t) \subseteq A(s,t)$. Let $a, b \in B(\epsilon,t)$, then for non-zero

$$z \in \chi^{2\tau}_M \left[AC_{\lambda_m\mu_n}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$$

$$\frac{1}{\lambda_m\mu_n}\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!\,|x_{a+r,b+s}-x_{c+r,d+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\epsilon. \tag{4.3}$$

If

$$\frac{1}{\lambda_m\mu_n}\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{a+r,b+s}-x_{c+r,d+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \leq 1-\epsilon.$$

then we have

$$\frac{1}{\lambda_m\mu_n}\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!|x_{a+r,b+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, \frac{t}{2}\right) \leq 1-s$$

and therefore $a, b \in A(s,t)$. As otherwise that is if
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((a+b)!|x_{a+r,b+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, \frac{t}{2}\right) > 1-s\right\}\right)$
then by (4.1),(4.2) and (4.3) we get

$$\begin{aligned}
1-\epsilon \geq & \frac{1}{\lambda_m\mu_n}\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((m+n)!\,|x_{a+r,b+s}-x_{c+r,d+s}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}}, z, t\right) \\
\geq & \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(a,b)\in I_{r,s}} F\left(\left[M\left(\frac{((a+b)!\,|x_{a+r,b+s}-0|)^{1/a+b}}{\rho}\right)\right]^{p_{ab}}, z, \frac{t}{2}\right) > 1-s\right\}\right)* \\
& \left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(c,d)\in I_{r,s}} F\left(\left[M\left(\frac{((c+d)!\,|x_{c+r,d+s}-0|)^{1/c+d}}{\rho}\right)\right]^{p_{cd}}, z, \frac{t}{2}\right) > 1-s\right\}\right) \\
\geq & (1-s)*(1-s) \\
> & 1-\epsilon,
\end{aligned}$$

which is not possible. Thus $B(\epsilon,t)\subset A(s,t)$. Since $\delta_{\nabla}(A(s,t))=0$, it follows that $\delta_{\nabla}(B(\epsilon,t))=0$. This shows that (x_{mn}) is $\nabla-$statistically Cauchy.

Conversely, suppose (x_{mn}) is backward operator of $\nabla-$statistically Cauchy not in $\nabla-$statistically convergent. Then there exists positive integer η and for non-zero $z\in\chi^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ such that if we take

$$A(\epsilon,t)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(a,b)\in I_{r,s}}F\left(\left[M\left(\frac{|x_{a+r,b+s}-x_{c+r,d+s}|^{1/a+b}}{\rho}\right)\right]^{p_{ab}},z,t\right)\leq 1-\epsilon\right\}\right)$$

and
$B(\epsilon,t)=\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}}F\left(w_0,z,\frac{t}{2}\right)>1-\epsilon\right\}\right).$
then

$$\delta_{\nabla}(A(\epsilon,t))=0=\delta_{\nabla}(B(\epsilon,t))$$

consequently

$$\delta_{\nabla}(A^c(\epsilon,t))=1=\delta_{\nabla}(B^c(\epsilon,t)). \tag{4.4}$$

Since
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(a,b)\in I_{r,s}}F\left(\left[M\left(\frac{|x_{a+r,b+s}-x_{c+r,d+s}|^{1/a+b}}{\rho}\right)\right]^{p_{ab}},z,t\right)\right\}\right)\geq$
$2\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}}F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,\frac{t}{2}\right)\right\}\right)>1-\epsilon,$
if
$\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(m,n)\in I_{r,s}}F\left(\left[M\left(\frac{((m+n)!|x_{m+r,n+s}-0|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}},z,\frac{t}{2}\right)\right\}\right)>\frac{1-\epsilon}{2}$
then we have

$$\delta_{\nabla}\left(\frac{1}{\lambda_m\mu_n}\left\{\sum_{(a,b)\in I_{r,s}}F\left(\left[M\left(\frac{(|x_{a+r,n+s}-x_{c+r,d+s}|)^{1/a+b}}{\rho}\right)\right]^{p_{ab}},z,t\right)>1-\epsilon\right\}\right)=0$$

that is $\delta_{\nabla}(A^c(\epsilon,t))=0$, which contradicts (4.4) as $\delta_{\nabla}(A^c(\epsilon,t))=1$. Hence $x=(x_{mn})$ is $\nabla-$statistically convergent. □

5. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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References

1. B.C. Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.
2. M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.
3. M. Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.
4. M. Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.
5. N. Subramanian and U.K.Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).
6. F. Moricz and B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.
7. A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.
8. G.M. Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28**, (1926), 50-73.
9. E. Savas and Richard F. Patterson, On some double almost lacunary sequence spaces defined by Orlicz functions, *Filomat (Niš)*, **19**, (2005), 35-44.
10. Kuldip Raj and Sunil K. Sharma, Lacunary sequence spaces defined by a Musielak Orlicz function , *Le Matematiche*, **Vol. LXVIII-Fasc. I**, (2013), 33-51.
11. Francesco Tulone , Regularity of some method of summation for double sequences , *Le Matematiche*, **Vol. LXV-Fasc. II**, (2010), 45-48.
12. J. Musielak, Orlicz Spaces,Lectures Notes in Math.,1034, *Springer-Verlag*, **1983**.

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