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Existence and Nonexistence Results for Weighted Fourth Order Eigenvalue Problems With Variable Exponent

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ABSTRACT: This paper presents sufficient conditions for the existence and nonexistence of eigenvalues for a $p(x)$-biharmonic equation with Navier boundary conditions and weight function on a bounded domain in $\mathbb{R}^N$. Our approach is mainly based on a adequate variational techniques.

Key Words: Fourth order elliptic equation, Variable exponent, Indefinite weight, Eigenvalue problem.

Contents

1. Introduction and statement of main results

The study of problems of elliptic equations and variational problems with p(x)-growth condition has attracted more and more attention in recent years. It possesses a solid background in physics and originates from the study on electrorheological fluids (see [16]) and elastic mechanics (see [18]). It also has wide applications in different research fields, such as image processing model (see e.g. [11,6], stationary thermorheological viscous flows (see [2]) and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium (see [1]).

In the present study, we deal with the following nonlinear eigenvalue problem with indefinite weight

$$\begin{aligned} \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big) = \lambda V(x)|u|^{q(x)-2}u \qquad \text{in } \Omega, \\ u = \triangle u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $V \in L^{r(x)}(\Omega)$ is an indefinite weight which can change sign in Ω and $p, q, r \in C_+(\bar{\Omega}) := \Big\{h; h \in C(\overline{\Omega})$ and $h(x) > 1$ for all $x \in \overline{\Omega}\Big\}$.

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The interest in analyzing this kind of problems is motivated by some recent advances in the study of fourth order nonlinear eigenvalue problems involving variable exponents in the last few years. We refer especially to the results in [3,4,10,13,14].

For instance, the case $V \equiv 1$ and $p(x) = q(x)$ has been studied in [3] and in [4] when $p(x) \neq q(x)$. In particular, in [3], by the Ljusternik-Schnirelmann principle on C^1-manifolds, the authors proved among others things the existence of a sequence of eigenvalues and that $\sup \Lambda = +\infty$, where Λ is the set of all nonnegative eigenvalues. In [4], using the mountain pass lemma and Ekeland's variational principle, they established several existence criteria for eigenvalues.

Motived by the above-mentioned papers, our purpose in this paper is to extend the results of [5] to a fourth order nonlinear problem with sign-changing potential. Our approach follows closely the one in the mentioned paper.

Hereafter, we analyze the problem (1.1) under the following assumptions::

H(p,q,r) $\quad p^+ < q^- \leq q^+ < p_2^*(x)$ and

$$r(x) > \frac{p_2^*(x)}{p_2^*(x) - q(x)}, \quad \text{for all } x \in \overline{\Omega}, \tag{1.2}$$

where

$$h^+ = \max_{\overline{\Omega}} h(x), h^- = \min_{\overline{\Omega}} h(x), \text{ for any } h \in C_+(\overline{\Omega})$$

and

$$\text{for every } x \in \overline{\Omega}, \quad p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ +\infty & \text{if } p(x) \geq \frac{N}{2}. \end{cases}$$

Here, we seek solutions for problem (1.1) belonging to the space $X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ in the sense below.

Definition 1.1. *By a weak solution for* (1.1) *we understand a function* $u \in X$ *such that*

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v \, dx - \lambda \int_\Omega V(x)|u|^{q(x)-2} uv \, dx = 0, \quad \forall v \in X.$$

Moreover, we say that $\lambda \in \mathbb{R}$ *is an eigenvalue of problem 1.1 if the weak solution* u *defined above is not trivial.*

We point out that, in the case of positive weight V, any possible eigenvalue of problem 1.1 is necessarily positive.

Define the functionals $\Phi, \Psi, I_\lambda : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} \, dx, \quad \Psi(u) = \int_\Omega \frac{V(x)}{q(x)} |u|^{q(x)} \, dx.$$

The energy functional corresponding to problem 1.1 is defined as $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

Standard arguments imply that $I_\lambda \in C^1(X, \mathbb{R})$ and for all $u, v \in X$, we have

$$\langle I_\lambda^{'}(u), v\rangle = \langle \Phi^{'}(u), v\rangle - \lambda\langle \Psi^{'}(u), v\rangle.$$

Thus, the weak solutions of (1.1) are exactly the critical points of I_λ.

In the sequel, for the sake of convenience, we put

$$\phi(u) = \int_\Omega |\Delta u|^{p(x)}\, dx, \quad \psi(u) = \int_\Omega V(x)|u|^{q(x)}\, dx, \text{ for every } u \in X.$$

Define

$$\lambda^* = \inf\left\{\frac{\Phi(u)}{\Psi(u)}, u \in X \text{ and } \Psi(u) > 0\right\} \text{ and } \lambda_* = \inf\left\{\frac{\phi(u)}{\psi(u)}, u \in X \text{ and } \psi(u) > 0\right\}.$$

The main results of this work are the following.

Theorem 1.1. *Suppose $V > 0$ on Ω. Then, under assumption **H(p,q,r)** and satisfy*

$$q^+ - \frac{1}{2}p^- < q^-, \tag{1.3}$$

we have

(i) $0 < \lambda_* \leq \lambda^*$,

(ii) λ^* *is an eigenvalue of problem* (1.1),

(iii) any $\lambda > \lambda^*$ *is an eigenvalue of problem* (1.1) *while any* $\lambda < \lambda_*$ *is not an eigenvalue.*

In the case when V is a sign-changing function, we define

$$X^+ = \left\{u \in X : \int_\Omega V(x)|u|^{q(x)}\, dx > 0\right\} \quad \text{and}$$

$$X^- = \left\{u \in X : \int_\Omega V(x)|u|^{q(x)}\, dx < 0\right\}.$$

$$\alpha^* = \inf_{u\in X^+} \frac{\Phi(u)}{\Psi(u)}, \quad \alpha_* = \inf_{u\in X^+} \frac{\phi(u)}{\psi(u)}, \tag{1.4}$$

$$\beta^* = \inf_{u\in X^-} \frac{\Phi(u)}{\Psi(u)}, \quad \beta_* = \inf_{u\in X^-} \frac{\phi(u)}{\psi(u)}. \tag{1.5}$$

So, we have

Theorem 1.2. *Suppose that **H(p,q,r)** and*

$$\left|\left\{x \in \Omega : V(x) > 0\right\}\right| \neq 0 \tag{1.6}$$

hold. Then, we have

(i) $\beta^* \leq \beta_* < 0 < \alpha_* \leq \alpha^*$,

(ii) α^* *(resp.* β^**) is a positive (resp. negative) eigenvalue of problem* (1.1),

(iii) any $\lambda \in (-\infty, \beta^*) \cup (\alpha^*, \infty)$ *is an eigenvalue of problem* (1.1) *while any* $\lambda \in (\beta_*, \alpha_*)$ *is not an eigenvalue.*

This article is composed of three sections. Section 2 contains some useful results on Sobolev spaces with variable exponents. The proofs are given in Section 3.

2. Preliminary results

In order to guarantee the integrity of the paper, we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For details, see [8,9,15].

For $p \in C_+(\Omega)$, define the space

$$L^{p(x)}(\Omega) = \{u;\ \text{measurable real-valued function and } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

Equipped with the so-called Luxemburg norm

$$|u|_{p(x)} := \inf\{\mu > 0 : \quad \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1\},$$

$L^{p(x)}(\Omega)$ becomes a separable, reflexive and Banach space. An important role in manipulating the generalized Lebesgue spaces is played by the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$, called the *modular* of the $L^{p(x)}(\Omega)$ space, defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

We recall the following, (see [8,15],

Proposition 2.1. *For all* $u \in L^{p(x)}(\Omega)$, *we have*

1. $|u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}$ *if* $|u|_{p(x)} > 1$.
2. $|u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}$ *if* $|u|_{p(x)} \leq 1$.

Proposition 2.2. *For all* $u_n, u \in L^{p(x)}(\Omega)$, *we have*

1. $|u|_{p(x)} = a \Leftrightarrow \rho(\frac{u}{a}) = 1,$ *for* $u \neq 0$ *and* $a > 0$.
2. $|u|_{p(x)} > 1 (= 1; < 1) \Leftrightarrow \rho(u) > 1 (= 1; < 1)$.
3. $|u|_{p(x)} \to 0 (\text{ resp. } \to +\infty) \Leftrightarrow \rho(u) \to 0 (\text{ resp. } \to +\infty)$.
4. *The following statements are equivalent each other:*

   (a) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,

(b) $\lim_{n\to\infty} \rho(u_n - u) = 0$,

(c) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

As in the constant exponent case, for any positive integer k, set

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq k\}.$$

We can define the norm on $W^{k,p(x)}(\Omega)$ by

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}$$

and $W^{k,p(x)}(\Omega)$ also becomes a separable, reflexive and Banach space. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Definition 2.1. *Assume that spaces* E, F *are Banach spaces, we define the norm on the space* $E \cap F$ *as* $\|u\| = \|u\|_E + \|u\|_F$.

From Definition 2.1, we can know that for any $u \in X$, $\|u\|_X = |u|_{1,p(x)} + |u|_{2,p(x)}$, thus $\|u\|_X = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$.

In the Zang and Fu's paper [17], the equivalence of the norms was proved, and it was even proved that the norm $|\triangle u|_{p(x)}$ is equivalent to the norm $\|u\|_X$ (see [17, Theorem 4.4]).

Let us choose on X the norm $\|.\|$ defined by

$$\|u\| = |\Delta u|_{p(x)}.$$

Note that, $(X, \|.\|)$ is also a separable and reflexive Banach space.

Similar to Proposition 2.2, we have the following.

Proposition 2.3. *Denote* $I(u) = \int |\Delta u(x)|^{p(x)} dx$ *then,*

1. *For* $u \in X$ *and* $\|u\| = a$, *we have*
   (a) $a < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1 > 1)$;
   (b) $a \geq 1 \Rightarrow a^{p^-} \leq I(u) \leq a^{p^+}$;
   (c) $a \leq 1 \Rightarrow a^{p^+} \leq I(u) \leq a^{p^-}$.
2. *If* $u, u_n \in X, n = 1, 2, ...,$ *then the following statements are equivalent each other:*
   (i) $\lim_{n\to\infty} \|u_n - u\| = 0$;
   (ii) $\lim_{n\to\infty} I(u_n - u) = 0$;
   (iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} I(u_n) = I(u)$.

The following result (see [3, Theorem 3.2]), which will be used later, is an embedding result between the spaces X and $L^{q(x)}(\Omega)$.

Theorem 2.1. *Let $p, q \in C_+(\Omega)$. Assume that*

$$p(x) < \frac{N}{2} \quad \textit{and} \ q(x) < p_2^*(x).$$

Then, there is a continuous and compact embedding X into $L^{q(x)}(\Omega)$.

We recall also the following proposition, which will be needed later.

Proposition 2.4. *([7]) Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and*
$1 \le p(x)q(x) \le \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \le \left| |u|^{p(x)} \right|_{q(x)} \le |u|^{p^-}_{p(x)q(x)},$$

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \le \left| |u|^{p(x)} \right|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}.$$

Before given the proofs, we need to establish the following auxiliary result which will be used later.

Proposition 2.5. *(i) Φ is weakly lower semi-continuous, namely, $u_n \rightharpoonup u$ implies that $\Phi(u) \le \liminf \Phi(u_n)$.*

(ii) Ψ is a weakly-strongly continuous functional, namely, $u_n \rightharpoonup u$ implies $\Psi(u_n) \to \Psi(u)$.

Proof.

(i) The convexity of Φ ensures this assertion.

(ii) Let (u_n) be a sequence in X such that $u_n \rightharpoonup u$ in X. Denote by $r^{'}(x)$ the conjugate exponent of the function $r(x)$ $\left(r^{'}(x) = \frac{r(x)}{r(x)-1} \right)$. Then, as $q(x)r^{'}(x) < p_2^*(x)$, Theorem 2.1 implies $u_n \rightharpoonup u$ in $L^{q(x)r^{'}(x)}(\Omega)$. This, together with the continuity of the Nemytski operator $\mathcal{N}_{V,q}$ defined by

$$\mathcal{N}_{V,q}(u)(x) = V(x)|u(x)|^{q(x)} \text{ if } u \neq 0[\text{ and } \mathcal{N}_{V,q}(u)(x) = 0 \text{ otherwise},$$

yields that $\Psi(u_n) \to \Psi(u)$. The proof is complete.

3. Proofs

At first, we start with the following Lemma which plays a crucial role for proving Theorem1.1.

Lemma 3.1. *Suppose that assumptions* $\boldsymbol{H}(p,q,r)$ *and* (1.3) *hold, then*

$$\lim_{\|u\|\to 0} \frac{\Phi(u)}{\Psi(u)} = \infty, \tag{3.1}$$

and

$$\lim_{\|u\|\to \infty} \frac{\Phi(u)}{\Psi(u)} = \infty. \tag{3.2}$$

Proof: Applying the Hölder's inequality, we obtain

$$|\Psi(u)| \leq \frac{2}{q^-}|V|_{r(x)}\Big||u|^{q(x)}|\Big|_{r'(x)}.$$

By help of proposition 2.4, it follows

$$|\Psi(u)| \leq \frac{2}{q^-}|V|_{r(x)}|u|^{q^i}_{q(x)r'(x)}, \tag{3.3}$$

where $i = +$ if $|u|_{q(x)r'(x)} > 1$ and $i = -$ if $|u|_{q(x)r'(x)} < 1$.

On the other hand, from (1.2), we have $p(x) < q(x)r^{'}(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$. So, in view proposition 2.1, X is continuously embedded in $L^{q(x)r^{'}(x)}(\Omega)$. Then, there exists $c > 0$ such that

$$|\Psi(u)| \leq \frac{2c}{q^-}|V|_{r(x)}\|u\|^{q^i}. \tag{3.4}$$

For any $u \in X$ with $\|u\| \leq 1$ small enough, by relations (3.3) and (3.4), we infer

$$\frac{\Phi(u)}{\Psi(u)} \geq \frac{\frac{1}{p^+}\|u\|^{p^+}}{\frac{2c}{q^-}|V|_{r(x)}\|u\|^{q^i}}. \tag{3.5}$$

Since $p^+ < q^- \leq q^+$, passing to the limit as $\|u\| \to 0$ in the above inequality, we deduce that the assertion (3.1) holds true.

Next, we show that relation the assertion (3.2) holds. From (1.3), there exists a positive constant δ such that $q^+ - \frac{1}{2}p^- < \delta < q^-$, and thus we have

$$p^- > 2(q^+ - \delta) > 2(q^- - \delta). \tag{3.6}$$

Let $s(x)$ be any measurable function $s(x)$ such that

$$\frac{p_2^*(x)}{p_2^*(x) + \delta - q(x)} \leq s(x) \leq \frac{p_2^*(x)r(x)}{p_2^*(x) + \delta r(x)}, \tag{3.7}$$

holds for almost all $x \in \Omega$ and

$$\delta\Big(\frac{s^+}{s^-} + 1\Big) \leq q^-. \tag{3.8}$$

Clearly, $s \in L^{\infty}(\Omega)$ and $1 < s(x) < r(x)$. Moreover, It is easy to see

$$\delta t(x) \le p_2^*(x) \text{ and } (q(x)-\delta)s^{'}(x) \le p_2^*(x), \text{ for all } x \in \overline{\Omega}, \tag{3.9}$$

where $t(x) := \frac{r(x)s(x)}{r(x)-s(x)}$ and $s^{'}(x) = \frac{s(x)}{s(x)-1}$.

Let $u \in X$ with $\|u\| > 1$. Thanks to Hölder's inequality again, we have

$$\left|\Psi(u)\right| \le \frac{2}{q^-}\left|V|u|^{\delta}\right|_{s(x)}\left||u|^{q(x)-\delta}\right|_{s'(x)}. \tag{3.10}$$

Without loss of generality we may suppose that $\left|V|u|^{\delta}\right|_{s(x)} > 1$. Then, from Proposition 2.1 and using Hölder's inequality, we get

$$\begin{aligned}\left|\Psi(u)\right| &\le \frac{2}{q^-}\Big(\rho_{s(x)}(V|u|^{\delta})\Big)^{\frac{1}{s^-}}\left||u|^{q(x)-\delta}\right|_{s'(x)}\\ &= \frac{2}{q^-}\Big(\int_{\Omega}|V|^{s(x)}u|^{\delta s(x)}\,dx)\Big)^{\frac{1}{s^-}}\left||u|^{q(x)-\delta}\right|_{s'(x)}\\ &\le \frac{4}{q^-}\left||V|^{s(x)}\right|^{\frac{1}{s^-}}_{\frac{r(x)}{s(x)}}\left||u|^{\delta s(x)}\right|^{\frac{1}{s^-}}_{\frac{r(x)}{r(x)-s(x)}}\left||u|^{q(x)-\delta}\right|_{s'(x)}.\end{aligned} \tag{3.11}$$

In view of proposition 2.4, we write

$$\begin{aligned}\left||u|^{\delta s(x)}\right|^{\frac{1}{s^-}}_{\frac{r(x)}{r(x)-s(x)}} &\le \left|u\right|^{\delta\frac{s^+}{s^-}}_{\delta t(x)} + \left|u\right|^{\delta}_{\delta t(x)}\\ \left||u|^{q(x)-\delta}\right|_{s'(x)} &\le \left|u\right|^{q^+-\delta}_{(q(x)-\delta)s'(x)} + \left|u\right|^{q^--\delta}_{(q(x)-\delta)s'(x)}\\ \left||V|^{s(x)}\right|^{\frac{1}{s^-}}_{\frac{r(x)}{s(x)}} &\le |V|^{\nu}_{r(x)},\end{aligned}$$

with $\nu = \begin{cases}\frac{s^+}{s^-} & \text{if } |V|_{r(x)} > 1,\\ 1 & \text{if } |V|_{r(x)} \le 1.\end{cases}$

Hence, substituting the above inequalities into (3.10) and thanks to Young's inequality, it follows

$$\begin{aligned}\left|\Psi(u)\right| &\le \frac{4}{q^-}|V|^{\nu}_{r(x)}\bigg(\left|u\right|^{\delta\frac{s^+}{s^-}}_{\delta t(x)} + \left|u\right|^{\delta}_{\delta t(x)}\bigg)\bigg(\left|u\right|^{q^+-\delta}_{(q(x)-\delta)s'(x)} + \left|u\right|^{q^--\delta}_{(q(x)-\delta)s'(x)}\bigg)\\ &\le \frac{4}{q^-}|V|^{j}_{r(x)}\Bigg(\left|u\right|^{2\delta\frac{s^+}{s^-}}_{\delta t(x)} + \left|u\right|^{2\delta}_{\delta t(x)} + \left|u\right|^{2(q^+-\delta)}_{(q(x)-\delta)s'(x)} + \left|u\right|^{2(q^--\delta)}_{(q(x)-\delta)s'(x)}\Bigg)\end{aligned} \tag{3.12}$$

From (3.9), we infer by Theorem 2.1 that X is continuously embedded both in $L^{\delta\left(\frac{r(x)}{s(x)}\right)'}(\Omega)$ and $L^{(q(x)-\delta)s'(x)}(\Omega)$. Therefore, there exists positive constants c such that

$$\left|\Psi(u)\right| \le \frac{4c}{q^-}|V|^{\nu}_{r(x)}\bigg(\|u\|^{2\delta\frac{s^+}{s^-}} + \|u\|^{2\delta} + \|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)}\bigg). \tag{3.13}$$

Finally, we obtain

$$\frac{\Phi(u)}{\Psi(u)} \geq \frac{q^- \|u\|^{p^-}}{4cp^+|V|^{\nu}_{r(x)}\left(\|u\|^{2\delta\frac{s^+}{s^-}} + \|u\|^{2\delta} + \|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)}\right)}.$$

Combining (3.6) and (3.8), we deduce $p^- > 2(q^+ - \delta) > 2(q^- - \delta) > 2\delta\frac{s^+}{s^-} > 2\delta$. Then, passing to the limit as $\|u\| \to \infty$ in the above inequality we conclude that relation (3.2) holds true. This ends the proof of lemma 3.1. □

Proof of Theorem1.1.

(i) Observe that $\lambda_* \geq 0$ and $\frac{q^-}{p^+}\lambda_* \leq \lambda^* \leq \frac{q^+}{p^-}\lambda_*$. Then, $\lambda_* \leq \lambda^*$ because $p^+ < q^-$.

Suppose, to the contrary, that $\lambda_* = 0$, so $\lambda^* = 0$. Let (u_n) be a sequence in $X \setminus \{0\}$ such that

$$\lim_n \frac{\Phi(u_n)}{\Psi(u_n)} = 0.$$

As in (3.5), we have

$$\frac{\Phi(u_n)}{\Psi(u_n)} \geq C\|u_n\|^{p^+-q^-},$$

for some positive constant C. Since $p^+ < q^-$, we obtain $\|u_n\| \to \infty$. Hence, it follows from 3.1 that

$$\lim_n \frac{\Phi(u_n)}{\Psi(u_n)} = \infty,$$

which contradicts with the hypothesis.

(ii) Let $(u_n) \in X \setminus \{0\}$ be a minimizing sequence for λ^*, that is

$$\lim_n \frac{\Phi(u_n)}{\Psi(u_n)} = \lambda^*. \tag{3.14}$$

By 3.2, (u_n) is bounded in X which is reflexive. Then, there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. This together with proposition 2.5 yields that

$$\Psi(u_n) \to \Psi(u). \tag{3.15}$$

and

$$\liminf \Phi(u_n) \geq \Phi(u). \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain that if $u \neq 0$,

$$\frac{\Phi(u)}{\Psi(u)} = \lambda^*.$$

It remains to show that u is nontrivial. Let suppose by contradiction that $u = 0$. Then, $\lim \Psi(u_n) = 0$ and so, via (3.14), we deduce

$$\lim \Phi(u_n) = \lim \frac{\Phi(u_n)}{\Psi(u_n)} \Psi(u_n) = 0.$$

This fact combined with proposition 2.3 implies that $\|u_n\| \to 0$. According to (3.2), we get

$$\lim \frac{\Phi(u_n)}{\Psi(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \neq 0$.

(iii) Assume that $\lambda > \lambda^*$ is fixed. Let $u \in X$ with $\|u\| > 1$. It follows from inequality (3.13) that

$$I_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda K\left(\|u\|^{2\delta\frac{s^+}{s^-}} + \|u\|^{2\delta} + \|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)}\right).$$

with $K = \frac{4c}{q^-}|V|^{\nu}_{r(x)}$. As $p^- > 2(q^+ - \delta) > 2(q^- - \delta) > 2\delta\frac{s^+}{s^-}$, the inequality above implies that $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$, that is, I_λ is coercive. Also, by proposition 2.5, the functional I_λ is weakly lower semi-continuous. These two facts enable us to apply [12, Proposition 1.2, Chapter 32], there exists a global minimizer u_0 of I_λ in X. Since $\lambda > \lambda^*$, we verify by definition of λ^* that there is an element $v \in X \setminus \{0\}$ such that $\frac{\Phi(v)}{\Psi(v)} < \lambda$. Then, $I_\lambda(v) < 0$ which assures that

$$I_\lambda(u_0) = \inf_{u \in X\setminus\{0\}} I_\lambda(u) < 0.$$

Consequently, we conclude that $u_0 \neq 0$.

Now, assuming by contradiction that there exists $\lambda \in (0, \lambda_*)$ an eigenvalue of problem 1.1. Then, there exists $u_\lambda \in X \setminus \{0\}$ such that

$$\langle \Phi'(u_\lambda), v\rangle = \lambda \langle \Psi'(u_\lambda), v\rangle, \quad \forall v \in X.$$

In particular, for $v = u_\lambda$, we obtain

$$\phi(u_\lambda) = \lambda \psi(u_\lambda).$$

As $u_\lambda \neq 0$, we have $\psi(u_\lambda) > 0$. This, together with the fact $\lambda < \lambda_*$ yields

$$\phi(u_\lambda) > \lambda_* \psi(u_\lambda) > \lambda \psi(u_\lambda) = \phi(u_\lambda)$$

This a contradiction. The theorem 1.1 is proved.

Proof of Theorem1.2.

Precise that if $\lambda > 0$ is an eigenvalue of problem (1.1) with weight V then, $-\lambda$ is an eigenvalue of problem (1.1) with weight $?V$. Hence, it is enough to show Theorem 1.1 only for $\lambda > 0$. So, the problem 1.1 has only to be considered in X^+ and in this case, the proof is similar to that of Theorem 1.1 and thus it will be omitted here.

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