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Certain Families of Integral Formulas Involving Struve Functions

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ABSTRACT: Recently, a large number of integral formulas involving Bessel functions and their extensions have been investigated. The objective of this paper is to establish four classes of integral formulas associated with the Struve functions, which are expressed in terms of the Fox-Wright function. Among a variety of special cases of the main results, we present only six integral formulas involving trigonometric and hyperbolic functions.

Key Words: Gamma function, Lavoie-Trottier Integral formula, Bessel functions, Generalized hypergeometric series, Fox-Wright function, Struve functions, Trigonometric functions, Hyperbolic functions.

Contents

1. Introduction and Preliminaries

A remarkably large number of integral formulas involving a variety of special functions have been presented (see, *e.g.*, [3,6,8,14,15,22,29,30]; see also [20]). Among those special functions, due mainly to the greater abstruseness of their properties, Bessel functions have found many applications in various problems of mathematical physics (see, *e.g.*, [28]). Recently, a large number of integral formulas involving Bessel functions and their various extensions (or generalizations) have been investigated (see, *e.g.*, [1,4,5,11,12,16,17]). Throughout this paper, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive and non-positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The Struve functions (see [18, Chapter 11]), which are related to Bessel functions and other functions (see [27, pp. 44-45]), have appeared in various problems in physics and applied mathematics, for example, water-wave and surface-wave problems [2,9], unsteady aerodynamics [24], particle quantum dynamical studies of spin decoherence [23] and nanotubes [19]. The Struve functions $\mathtt{H}_\nu(z)$ of order

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ν are defined by (see [18, 288]; see also [27, pp. 44-45])

$$\begin{aligned}\mathtt{H}_\nu(z) :&= \left(\frac{1}{2}\,z\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+\nu+\frac{3}{2})\Gamma(k+\frac{3}{2})}\left(\frac{z}{2}\right)^{2k}\\ &= \frac{\left(\frac{1}{2}\,z\right)^{\nu+1}}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left[\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ -\frac{z^2}{4}\right],\end{aligned}\tag{1.1}$$

where $\Gamma(\lambda)$ is the familiar Gamma function (see, *e.g.*, [25, Section 1.1]) and ${}_pF_q$ is the generalized hypergeometric series defined by (see, *e.g.*, [21, p. 73]):

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{matrix}\ z\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\\ &= {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z),\end{aligned}\tag{1.2}$$

$(\lambda)_n$ being the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by (see [25, p. 2 and p. 5]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N})\end{cases}\\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\quad (\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{aligned}\tag{1.3}$$

The modified Struve functions $\mathtt{L}_\nu(z)$ is defined by (see [18, 288])

$$\begin{aligned}\mathtt{L}_\nu(z) :&= \left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^{2k}\\ &= \frac{\left(\frac{1}{2}\,z\right)^{\nu+1}}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left[\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\\ &= \exp\left[-\frac{1}{2}(\nu+1)\,i\,\pi\right]\,\mathtt{H}_\nu(iz)\ \ (i=\sqrt{-1}).\end{aligned}\tag{1.4}$$

Here, in this paper, we aim at establishing four classes of integral formulas involving the Struve functions (1.1) and (1.4), which are expressed in terms of the well-known Fox-Wright function ${}_p\Psi_q$ defined by (see, for details, [26, p. 21]; see also [13, p. 56])

$$\begin{aligned}{}_p\Psi_q\,[z] &= {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1)\,,\cdots,(a_p,\alpha_p)\,;\\ (b_1,\beta_1)\,,\cdots,\left(b_q,\beta_q\right);\end{matrix}\ z\right] = {}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\,;\\ \left(b_j,\beta_j\right)_{1,q}\,;\end{matrix}\ z\right]\\ &= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma\left(a_i+\alpha_i n\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_j n\right)}\,\frac{z^n}{n!},\end{aligned}\tag{1.5}$$

where the coefficients $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R}^+$ such that

$$1+\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geqq 0. \tag{1.6}$$

By comparing the definitions (1.2) and (1.5), we find

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),\ldots,(\alpha_p,1);\\(\beta_1,1),\ldots,(\beta_q,1);\end{array} z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\, {}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\\beta_1,\ldots,\beta_q;\end{array} z\right]. \tag{1.7}$$

For more detailed properties of ${}_p\Psi_q$ including its asymptotic behavior, one may refer to works (for example) [29,30,10,11,12].

For the present investigation, we also need the following integral formula (see [14]):

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.8}$$

provided $\min\{\Re(\alpha), \Re(\beta)\} > 0$.

2. Main results

Here we present four classes of integral formulas associated with the Struve functions (1.1) and (1.4) by mainly using the integral formula (1.8). The results are expressed in terms of the Fox-Wright function ${}_p\Psi_q$ in (1.5) and given in the following four theorems. Here and in what follows, all involved complex powers are, for simplicity, assumed to have principal values.

Theorem 2.1. *Let $\xi, \eta, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$ and $\min\{\Re(\xi+\eta), \Re(\xi+\nu)\} > 0$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\xi+\eta-1}(1-x)^{2\xi-1}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-1} \mathtt{H}_\nu\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right] dx\\&= \left(\frac{y}{2}\right)^{\nu+1}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)}\\&\quad\times {}_2\Psi_3\left[\begin{array}{c}(\xi+\nu+1,2),(1,1);\\(2\xi+\nu+\eta+1,2),(3/2,1),(\nu+3/2,1);\end{array} -\left(\frac{y}{2}\right)^2\right].\end{aligned} \tag{2.1}$$

Proof: Let $\mathcal{L}$ be the left-hand side of (2.1). Using the series representation of the Struve function (1.1) in the integrand of (2.1), and, then, interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we have

$$\begin{aligned}\mathcal{L} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+\frac{3}{2})\Gamma(k+\frac{3}{2})} \left(\tfrac{y}{2}\right)^{2k+\nu+1} \\ &\quad \times \int_0^1 x^{\xi+\eta-1} (1-x)^{2(\xi+\nu+2k)-1} \left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1} \left(1-\frac{x}{4}\right)^{\xi+\nu+2k-1} dx.\end{aligned}$$

It is noted that, under given conditions,

$$\Re(\xi+\eta) > 0 \quad \text{and} \quad \Re(\xi+\nu+2k) > 0 \quad (k \in \mathbb{N}_0).$$

Then, we can apply the integral formula (1.8) to the last resulting integrals to yield

$$\begin{aligned}\mathcal{L} &= (y/2)^{\nu+1}\,\Gamma(\xi+\eta) \left(\frac{2}{3}\right)^{2(\xi+\eta)} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(\xi+2k+\nu+1)}{\Gamma(2\xi+\eta+\nu+1+2k)\,\Gamma(k+\nu+\frac{3}{2})\Gamma(k+\frac{3}{2})} \left(\frac{y^2}{4}\right)^k,\end{aligned}$$

which, upon using (1.5), is seen to lead to the right-hand side of (2.1). □

The following integral formulas given in Theorems 2.2-2.4 can be established by using a similar argument as in the proof of Theorem 2.1. So the details of their proofs are omitted.

Theorem 2.2. *Let ξ, η, $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$ and $\min\{\Re(\xi+\eta), \Re(\xi+\nu)\} > 0$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\xi-1} (1-x)^{2(\xi+\eta)-1} \left(1-\frac{x}{3}\right)^{2\xi-1} \left(1-\frac{x}{4}\right)^{(\xi+\eta)-1} \mathtt{H}_\nu \left[yx\left(1-\frac{x}{3}\right)^2\right] dx \\ &= \left(\frac{y}{2}\right)^{\nu+1} \Gamma(\xi+\eta) \left(\frac{2}{3}\right)^{\xi+\nu+1} \\ &\quad \times {}_2\Psi_3 \left[\begin{matrix} (\xi+\nu+1, 2), (1,1); \\ (2\xi+\nu+\eta+1, 2), (\nu+3/2, 1)\,(3/2, 1); \end{matrix} \; -\left(\frac{2y}{9}\right)^2 \right].\end{aligned} \tag{2.2}$$

Theorem 2.3. *Let ξ, η, $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$ and $\min\{\Re(\xi+\eta), \Re(\xi+\nu)\} >$*

0. *Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}\int_0^1 & x^{\xi+\eta-1}(1-x)^{2\xi-1}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-1}\mathtt{L}_\nu\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx\\ &=\left(\frac{y}{2}\right)^{\nu+1}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)}\\ &\quad\times {}_2\Psi_3\left[\begin{matrix}(\xi+\nu+1,2),(1,1);\\(2\xi+\nu+\eta+1,2),(3/2,1),(\nu+3/2,1);\end{matrix}\ \frac{y^2}{4}\right].\end{aligned} \tag{2.3}$$

Theorem 2.4. *Let ξ, η, $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$ and $\min\{\Re(\xi+\eta), \Re(\xi+\nu)\} > 0$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}\int_0^1 & x^{\xi-1}(1-x)^{2(\xi+\eta)-1}\left(1-\frac{x}{3}\right)^{2\xi-1}\left(1-\frac{x}{4}\right)^{(\xi+\eta)-1}\mathtt{L}_\nu\left[yx\left(1-\frac{x}{3}\right)^2\right]dx\\ &=\left(\frac{y}{2}\right)^{\nu+1}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{\xi+\nu+1}\\ &\quad\times {}_2\Psi_3\left[\begin{matrix}(\xi+\nu+1,2),(1,1);\\(2\xi+\nu+\eta+1,2),(\nu+3/2,1)(3/2,1);\end{matrix}\ \left(\frac{2y}{9}\right)^2\right].\end{aligned} \tag{2.4}$$

3. Special Cases

Here we consider some special cases of the main results in Section 2. The Struve functions $\mathtt{H}_\nu(z)$ in (1.1) are specialized to connect with circular functions (see, *e.g.*, [18, p. 291]):

$$\mathtt{H}_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z \tag{3.1}$$

and

$$\mathtt{H}_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}(1-\cos z). \tag{3.2}$$

Setting $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ in the results in Theorems 2.1 and 2.2, we obtain some interesting integral formulas involving trigonometric functions which, without proof, are given in Corollaries 3.1–3.4.

Corollary 3.1. *Let $\xi, \eta \in \mathbb{C}$ with $\Re(\xi+\eta) > 0$ and $\Re(\xi) > \frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}\int_0^1 & x^{\xi+\eta-1}(1-x)^{2\xi-1}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-1}\sin\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx\\ &=\frac{\sqrt{\pi}y}{2}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)}{}_1\Psi_2\left[\begin{matrix}(\xi+1,2);\\(2\xi+\eta+1,2),(3/2,1);\end{matrix}\ -\left(\frac{y}{2}\right)^2\right].\end{aligned} \tag{3.3}$$

Corollary 3.2. *Let ξ, $\eta \in \mathbb{C}$ with $\Re(\xi+\eta) > 0$ and $\Re(\xi) > \frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\int_0^1 x^{\xi-3/2}(1-x)^{2(\xi+\eta)-1}\left(1-\frac{x}{3}\right)^{2(\xi-1)}\left(1-\frac{x}{4}\right)^{(\xi+\eta)-1}\sin\left[xy\left(1-\frac{x}{3}\right)^2\right]dx$$
$$= \frac{\sqrt{\pi}y}{2}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2\left(\xi+\frac{1}{2}\right)} {}_1\Psi_2\left[\begin{matrix}\left(\xi+\frac{1}{2},2\right); \\ \left(2\xi+\eta+\frac{1}{2},2\right),(3/2,1);\end{matrix} \; -\left(\frac{2y}{9}\right)^2\right]. \tag{3.4}$$

Corollary 3.3. *Let ξ, $\eta \in \mathbb{C}$ with $\Re(\xi+\eta) > 0$ and $\Re(\xi) > -\frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\xi+\eta-1}(1-x)^{2(\xi-1)}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-\frac{3}{2}}\\ &\quad\times\left[1-\cos\left\{y\left(1-\frac{x}{4}\right)(1-x)^2\right\}\right]dx\\ &= \frac{\sqrt{\pi}}{4}y^2\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)}\\ &\quad\times {}_2\Psi_3\left[\begin{matrix}(\xi+3/2,2),(1,1); \\ \left(2\xi+\eta+\frac{3}{2},2\right),(3/2,1),(2,1);\end{matrix} \; -\left(\frac{y}{2}\right)^2\right].\end{aligned} \tag{3.5}$$

Corollary 3.4. *Let ξ, $\eta \in \mathbb{C}$ with $\Re(\xi+\eta) > 0$ and $\Re(\xi) > -\frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\xi-1}(1-x)^{2(\xi+\eta)-1}\left(1-\frac{x}{3}\right)^{2(\xi-1)}\left(1-\frac{x}{4}\right)^{\xi+\eta-1}\\ &\quad\times\left[1-\cos\left\{yx\left(1-\frac{x}{3}\right)^2\right\}\right]dx\\ &= \frac{\sqrt{\pi}}{4}y^2\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2\left(\xi+\frac{3}{2}\right)}\\ &\quad\times {}_2\Psi_3\left[\begin{matrix}\left(\xi+\frac{3}{2},2\right),(1,1); \\ \left(2\xi+\eta+\frac{3}{2},2\right),(3/2,1),(2,1);\end{matrix} \; -\left(\frac{2y}{9}\right)^2\right].\end{aligned} \tag{3.6}$$

Further, setting $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ in the result in Theorem 2.3 with the aid of the following known formulas (see [18, p. 291]):

$$\mathtt{L}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sinh z \tag{3.7}$$

and

$$\mathtt{L}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}(\cosh z - 1), \tag{3.8}$$

we obtain some interesting integral formulas involving hyperbolic functions which, without proof, are given in Corollaries 3.5 and 3.6.

Corollary 3.5. *Let ξ, $\eta \in \mathbb{C}$ with $\Re(\xi+\eta)>0$ and $\Re(\xi)>\frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}\int_0^1 & x^{\xi+\eta-1}(1-x)^{2(\xi-1)}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-3/2} \\ & \times \sinh\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right] dx \\ &= \frac{\sqrt{\pi}y}{2}\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)} {}_1\Psi_2\left[\begin{matrix}\left(\xi+\frac{1}{2},2\right); \\ \left(2\xi+\eta+\frac{1}{2},2\right),(3/2,1);\end{matrix}\ \frac{y^2}{4}\right].\end{aligned} \tag{3.9}$$

Corollary 3.6. *Let ξ, $\eta \in \mathbb{C}$ with $\Re(\xi+\eta)>0$ and $\Re(\xi)>-\frac{1}{2}$. Then, for $y \in \mathbb{C}$, the following integral formula holds true:*

$$\begin{aligned}\int_0^1 & x^{\xi+\eta-1}(1-x)^{2(\xi-1)}\left(1-\frac{x}{3}\right)^{2(\xi+\eta)-1}\left(1-\frac{x}{4}\right)^{\xi-3/2} \\ & \times \left[\cosh\left\{y\left(1-\frac{x}{4}\right)(1-x)^2\right\}-1\right] dx \\ &= \frac{\sqrt{\pi}}{4}y^2\Gamma(\xi+\eta)\left(\frac{2}{3}\right)^{2(\xi+\eta)} \\ & \times {}_2\Psi_3\left[\begin{matrix}\left(\xi+\frac{3}{2},2\right),(1,1); \\ \left(2\xi+\eta+\frac{3}{2},2\right),(3/2,1),(2,1);\end{matrix}\ \frac{y^2}{4}\right].\end{aligned} \tag{3.10}$$

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