Bol. Soc. Paran. Mat. (3s.) **v. 37** 3 (2019): 159–194.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i3.34223

An Inverse Problem for One-dimensional Diffusion Equation in Optical Tomography

Mohamed ADDAM

ABSTRACT: In this paper, we study the one-dimensional inverse problem for the diffusion equation based optical tomography. The objective of the present work is a mathematical and numerical analysis concerning one-dimensional inverse problem. In the first stage, the forward diffusion equation with boundary conditions is solved using an intermediate elliptic equation. We give the existence and the uniqueness results of the solution. An approximation of the photon density in frequency-domain is proposed using a Splines Galerkin method. In the second stage, we give theoretical results such as the stability and lipschitz-continuity of the forward solution and the Fréchet differentiability of the Dirichlet-to-Neumann nonlinear map with respect to the optical parameters. The Fréchet derivative is used to linearize the considered inverse problem. The Newton method based on the regularization technique will allow us to compute the approximate solutions of the inverse problem. Several test examples are used to verify high accuracy, effectiveness and good resolution properties for smooth and discontinuous optical property solutions.

Key Words: Diffusion transport problem, Fourier transform, Nonlinear inverse problem, Newton method, Tikhonov regularization, Spline basis functions, Optical tomography.

Contents

2010 *Mathematics Subject Classification:* 31A20, 49N45, 42A38, 35K55, 65N15, 41A15, 41A25, 65D10

Submitted November 14, 2016. Published April 25, 2017

1. Introduction

This paper deals with the inverse problem in one-dimensional optical tomography. This problem consists of the estimation of the optical properties, namely the absorption coefficient μ_a, the diffusion coefficient D and the refractive index ν, from the flow measurements at the boundary of the computational domain. The signal of measurements is assumed to be measured at the boundary using the photon density solution of the forward diffusion equation with the modified Robin boundary conditions. The estimation of the solution of the inverse problem requires an investigation of the forward problem. Such investigations must be of physical and mathematical aspects, numerical methods, software and computer simulations. In the corresponding forward problem, we consider the diffusion transport equation. This equation arises in many scientific applications such as in optical tomography. The principle of the optical tomography is to use multiple movable light sources and detectors attached to the tissue surface, such as human tissue, to collect information on light attenuation and to reconstruct the internal absorption and scattering distributions. The time-dependent diffusion transport equations have been studied by several authors in the recent years; see [12,13,14,17]. The radiative transfer transport equation (also derived from the Boltzmann equation) has been used successfully as a standard model for describing light transport in scattering media, see [11,13,14,17]. The variational method proves that the corresponding forward problem has a unique solution for the given optical properties and the sources [2,3]. In the literature, many numerical techniques have been proposed to solve such problem. Our approaches for solving the proposed inverse problem is based on the Tikhonov regularization [18]. In the current study, we present some theoretical results concerning the continuity of the forward solutions and Dirichlet-to-Neumann (DtN) map with respect to the optical parameters, Lipschitz-continuous property of the DtN map, the adjoint diffusion problem and the Fréchet derivative. Such derivative allows us to obtain a linearized problem from the nonlinear inverse problem. So, the discretized problem by using the spline basis functions may be written in the vectorial form and the obtained problem is solved by using the Newton iterations. The goal of the inverse problem is to determine unknown coefficients that appear in the diffusion equation from measurements at the boundary of the solution of the associated forward problem.

The paper is organized as follows. In Section 2, we recall some results of the forward diffusion problem. In Section 3, we set the inverse problem of 1D-diffusion equation. We give some theoretical results concerning the Lipschitz-continuity of the Dirichlet-to-Neumann linear map and of its Fréchet Derivative. This guarantees the stability of the solution of the forward diffusion problem in terms of the Robin boundary conditions. Section 3 is ended by some results concerning the adjoint diffusion problem. In Section 4, using the spline basis functions and the Fréchet derivatives, we formulate the discretized inverse problem and its equivalent optimization problem. The Newton method combined with the Tikhonov regularization technique is used to solve such an optimization problem, which require the computation of the jacobian and the hessian matrices. In Section 5, we give some numerical tests to illustrate our proposed methods.

2. Statement of the forward diffusion problem

2.1. One-dimensional diffusion problem

To facilitate the readability of this paper, we recall some notations and results already given in [3]. Let $\phi(x,t)$ be the photon density at the position $x \in]a,b[\subset \mathbb{R}$ and at the time $t \in [0,T]$, where T is the time period. Let μ_a, μ_s and ν be the absorption coefficient, the scattering coefficient and the refractive index coefficient, respectively. The reduced scattering coefficient is given by $\tilde{\mu}_s(x) = (1-\vartheta)\mu_s(x)$ where $\vartheta \in [0,1]$ is the anisotropic parameter. The diffusion potential function D is related to μ_a and $\tilde{\mu}_s$ by the following relation:

$$D(x) = \frac{1}{\mu_a(x) + \tilde{\mu}_s(x)}, \quad \forall x \in [a,b]. \tag{2.1}$$

Let us denote by $c(x) = c_0/\nu(x)$ the speed of light in the domain $[a,b]$, where c_0 is the speed of light in the vacuum [3,15] and $\nu(x) \geq 1$ for all $x \in [a,b]$. We assume that μ_a, μ_s and ν are real, bounded and nonnegative functions on $[a,b]$. For (x,t) in $]a,b[\times[0,T]$ we denote by $\phi(x,t)$ the photon density located at position x at time t. The time-dependent diffusion equation is described by the following parabolic partial differential equation (see [3,4])

$$\frac{1}{c(x)}\frac{\partial\phi(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(D(x)\frac{\partial\phi(x,t)}{\partial x}\right) + \mu_a(x)\phi(x,t) = f(x,t), \tag{2.2}$$

for every $(x,t) \in]a,b[\times[0,T]$ with the initial condition $\phi(x,0) = \phi_0(x)$ for every $x \in (a,b)$, where ϕ_0 is a given function. The derivative operations given in (2.2) are used in the distribution sense.
Let $\zeta_a \geq 0$ and $\zeta_b \geq 0$ be the refractive indexes mismatched at the boundaries a and b, respectively. We introduce the following constants β_a and β_b given for all $\xi \in \{a,b\}$ by $\beta_\xi = 0$ if $\zeta_\xi = 0$ and $\beta_\xi = \zeta_\xi^{-1}$ if $\zeta_\xi > 0$.
In this work, we consider the Robin boundary conditions given as follows,

$$\begin{aligned} \phi(a,t) - \frac{\beta_a}{2}D(a)\frac{\partial\phi}{\partial x}(a,t) &= g_a(t), \\ \phi(b,t) + \frac{\beta_b}{2}D(b)\frac{\partial\phi}{\partial x}(b,t) &= g_b(t). \end{aligned} \tag{2.3}$$

The purposed forward diffusion problem is (2.2)-(2.3) which consists of finding the photon density $\phi(x,t)$ for a given data f, g_a, g_b, D, μ_a and ν, and the inverse problem for one-dimensional diffusion equation is to compute the optical parameters $\mathrm{s} = (\nu, D, \mu_a)$ for a given source terms f, g_a, g_b and the flow measurements at

the boundary of our computational domain.
Throughout this paper, we assume that the real-valued functions D, μ_a and ν are bounded functions and belongs to $L^\infty([a,b],\mathbb{R})$, and D, μ_a and ν are positive functions and there is $\alpha > 0$ such that:

$$D(x) \geq \alpha, \quad \mu_a(x) \geq 0, \quad \nu(x) \geq 1, \quad \text{almost every in} \quad [a,b]. \tag{2.4}$$

2.2. Frequency-domain framework

To understand the forward problem and its corresponding inverse problem treated in this work, we need to introduce the following classical spaces [1,8]. The space $L^2(]a,b[,\mathbb{C})$ of square integrable complex-valued functions on the open interval $]a,b[\subset \mathbb{R}$ is denoted by $\widehat{L}_2 := L^2(]a,b[,\mathbb{C})$. The space $\widehat{L}_2$ is endowed by its natural topology defined by the following inner product and its associated norm

$$(u|v)_2 = \int_a^b u(x)\overline{v(x)}dx, \quad \|u\|_2 = \sqrt{(u|u)_2}, \quad \forall u, v \in \widehat{L}_2. \tag{2.5}$$

For every $k \in \mathbb{N}$, let $\widehat{H}_k = H^k(]a,b[,\mathbb{C}) := W^{k,2}(]a,b[,\mathbb{C})$ be the usual Sobolev space, with the classical norm $\|\,.\,\|_{\widehat{H}_k}$ given by

$$\|u\|_{\widehat{H}_k} = \left(\sum_{0\leq \ell \leq k} \left\| \frac{\partial^\ell u}{\partial x^\ell} \right\|_2^2 \right)^{\frac{1}{2}}, \quad \forall u \in \widehat{H}_k \tag{2.6}$$

where $\frac{\partial^\ell u}{\partial x^\ell}$ stands for the derivative of u, in the distribution sense, of order ℓ. The classical space $L^2(]a,b[,\mathbb{R})$ of square integrable real-valued functions on the open interval $]a,b[$ is denoted by $L_2 := L^2(]a,b[,\mathbb{R})$ and H_k denotes the usual Sobolev space $H_k = H^k(]a,b[,\mathbb{R}) := W^{k,2}(]a,b[,\mathbb{R})$ of real-valued functions. The space H_k is equipped with the classical norm $\|\,.\,\|_{H_k}$ given as in (2.6).
For $1 \leq p < \infty$, the spaces $L^p(\mathbb{R};\widehat{H}_k)$ and $L^p(\mathbb{R};H_k)$ endowed with the norms

$$\|u\|_{L^p(\mathbb{R};\widehat{H}_k)} = \left(\int_{-\infty}^{+\infty} \|u(.,\omega)\|_{\widehat{H}_k}^p \, d\omega \right)^{\frac{1}{p}}, \|\phi\|_{L^p(\mathbb{R};H_k)} = \left(\int_{-\infty}^{+\infty} \|\phi(.,t)\|_{H_k}^p \, dt \right)^{\frac{1}{p}}, \tag{2.7}$$

respectively, are Banach spaces.

We assume that the signal function $t \longmapsto \phi(x,t)$ and the derivative function $t \longmapsto \dfrac{\partial \phi}{\partial x}(x,t)$ have extensions to $\mathbb{R}$ belonging to the space $L^1(\mathbb{R},\mathbb{R})$, almost everywhere in $[a,b]$. We also assume that the given sources terms f, g_a and g_b have extensions, which are still denoted by f, g_a and g_b such that $f \in L^\infty\big(]a,b[;L^1(\mathbb{R},\mathbb{R})\big)$ and $g_a, g_b \in L^1(\mathbb{R},\mathbb{R})$.
We recall that the Fourier transform $\widehat{g}$ of a function g in $L^1(\mathbb{R},\mathbb{R})$ is given by

$$\widehat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt, \quad i = \sqrt{-1}.$$

By using the Lebesgue theorem of the derivation under the integral sign, we obtain

$$\widehat{\frac{\partial \phi}{\partial x}}(x,\omega) = \frac{\partial \widehat{\phi}}{\partial x}(x,\omega), \quad \forall (x,\omega) \in]a,b[\times \mathbb{R}.$$

We apply the Fourier transform to the equation (2.2) together with the non-homogeneous Robin boundary conditions (2.3) and we use the classical relation $\widehat{\frac{\partial \mathrm{g}}{\partial t}} = \mathrm{i}\omega\widehat{\mathrm{g}}$. Then we obtain the following Sturm-Liouville equation satisfied by the photon density $u(x,\omega)$ in the frequency domain

$$-\frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,\omega)}{\partial x}\right)+\left(\mu_a(x)+\frac{\mathrm{i}\omega}{c_0}\nu(x)\right)u(x,\omega) = F(x,\omega), \quad \forall (x,\omega) \in]a,b[\times \mathbb{R}, \tag{2.8}$$

with the following non-homogeneous boundary conditions

$$\begin{aligned} u(a,\omega) - \frac{\beta_a}{2}D(a)\frac{\partial u}{\partial x}(a,\omega) &= G_a(\omega),\\ u(b,\omega) + \frac{\beta_b}{2}D(b)\frac{\partial u}{\partial x}(b,\omega) &= G_b(\omega), \end{aligned} \tag{2.9}$$

where $u(.,\omega) = \widehat{\phi}(.,\omega)$, $F(.,\omega) = \widehat{f}(.,\omega)$, $G_a(\omega) = \widehat{g}_a(\omega)$ and $G_b(\omega) = \widehat{g}_b(\omega)$, are the Fourier transforms with respect to the time variable of $\phi(x,.)$, $f(x,.)$, g_a and g_b, respectively. In the literature, u is called the photon density in frequency domain and F, G_a and G_b are also called the sources terms in frequency domain.

2.3. Weak variational formulation

Let $u_*(.,\omega)$ be a sufficiently smooth function satisfying the problem (2.8)-(2.9), for instance $u_*(.,\omega)$ is in $\widehat{\mathrm{H}}_1$. Multiplying the diffusion equation (2.8) by an arbitrary function $\overline{\mathrm{v}} \in \widehat{\mathrm{H}}_1$, and integrating over $[a,b]$ with respect to the non-homogeneous Robin boundary conditions (2.9), provides the following weak variational formulation:

$$\mathcal{A}_\omega\left(u_*(.,\omega),\mathrm{v}\right) = \mathscr{L}_\omega\left(\mathrm{v}\right), \quad \forall \mathrm{v} \in \widehat{\mathrm{H}}_1, \tag{2.10}$$

where the sesquilinear form $\mathcal{A}_\omega : \widehat{\mathrm{H}}_1 \times \widehat{\mathrm{H}}_1 \to \mathbb{C}$ and the semi-linear form $\mathscr{L}_\omega : \widehat{\mathrm{H}}_1 \to \mathbb{C}$, are defined for all $u, \mathrm{v} \in \widehat{\mathrm{H}}_1$ by

$$\begin{aligned}\mathcal{A}_\omega\left(u,\mathrm{v}\right) = \int_a^b D(x)\frac{\partial u}{\partial x}(x)\overline{\frac{\partial \mathrm{v}}{\partial x}}(x)dx + \int_a^b \left(\mu_a(x)+\frac{\mathrm{i}\omega}{c_0}\nu(x)\right)u(x)\overline{\mathrm{v}(x)}dx\\ +2\left(\zeta_a u(a)\overline{\mathrm{v}(a)} + \zeta_b u(b)\overline{\mathrm{v}(b)}\right),\end{aligned}$$

and

$$\mathscr{L}_\omega\left(\mathrm{v}\right) = 2\left(\zeta_a G_a(\omega)\overline{\mathrm{v}(a)} + \zeta_b G_b(\omega)\overline{\mathrm{v}(b)}\right) + \int_a^b F(x,\omega)\overline{\mathrm{v}(x)}dx,$$

respectively. Therefore, the problem (2.8)-(2.9) is equivalent to the weak variational formulation (2.10).

From the Sobolev continuous embedding theorem, there exists a constant $C > 0$ (depending only on a and b) such that

$$\sup_{x\in[a,b]} |u(x)| \leq C \, \|u\|_{\widehat{\mathrm{H}}_1}, \quad \forall u \in \widehat{\mathrm{H}}_1. \tag{2.11}$$

Using the Schwarz inequality and the Sobolev continuous embedding theorems, it follows that there exists a constant $C_0 > 0$ (depending only on a and b) such that

$$|\mathscr{L}_\omega(\mathrm{v})| \leq M(\omega) \, \|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}, \quad \forall \mathrm{v} \in \widehat{\mathrm{H}}_1, \tag{2.12}$$

where $M(\omega) = C_0 \left(\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} + |G_a(\omega)| + |G_b(\omega)| \right)$.

Therefore, the operator $\mathscr{L}_\omega : \widehat{\mathrm{H}}_1 \to \mathbb{C}$ is a continuous semi-linear form on $\widehat{\mathrm{H}}_1$. Since the functions D, μ_a and ν satisfies (2.4) and using again the Sobolev continuous imbedding theorem, it follows that there exists a constant $C_1 > 0$ (not depending on the frequency ω) such that

$$|\mathcal{A}_\omega\,(u,\mathrm{v})\,| \leq C_1 \sqrt{1+\omega^2} \|u\|_{\widehat{\mathrm{H}}_1} \|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}, \quad \forall u, \mathrm{v} \in \widehat{\mathrm{H}}_1. \tag{2.13}$$

It is obvious to obtain the coercivity bound

$$\mathrm{Re}\,(\mathcal{A}_\omega\,(u,u)) \geq \alpha \|u\|^2_{\widehat{\mathrm{H}}_1}, \quad \forall u \in \widehat{\mathrm{H}}_1, \tag{2.14}$$

where α is the positive constant given in (2.4).

Hence, from (2.13) and (2.14), the sesquilinear form $\mathcal{A}_\omega : \widehat{\mathrm{H}}_1 \times \widehat{\mathrm{H}}_1 \to \mathbb{C}$ is continuous and coercive.

A well-posedness of the problem (2.8)-(2.9) is established in [3,4] by the following theorem

Theorem 2.1. *For a given frequency $\omega \in \mathbb{R}$, the problem (2.8)-(2.9) has a unique solution $u_*(.,\omega)$ in $\widehat{\mathrm{H}}_1$ satisfying the following upper bound*

$$\|u_*(.,\omega)\|_{\widehat{\mathrm{H}}_1} \leq \frac{M(\omega)}{\alpha}, \tag{2.15}$$

where the positive constants α and $M(\omega)$ are given in (2.4) and (2.12), respectively. Furthermore, if $D \in \mathcal{C}^1([a,b])$, μ_a, ν and $F(.,\omega)$ are in $\mathcal{C}([a,b])$, then $u_*(.,\omega) \in \mathcal{C}^2([a,b])$ is the unique solution of the problem (2.8)-(2.9) in the usual sense.

2.4. Discretized forward diffusion problem

We discretize the interval $[a,b]$ using equally spaced knots $x_i = a + ih$, for $i = 0, 1, 2 \ldots, n$ with $x_0 = a, x_n = b$ and $h = (b-a)/n$. For $i = -3, \ldots, n+1$, let us consider a partition of $(n+5)$ nodes of the interval $[a-3h, b+h]$

$$x_{-3} < x_{-2} < x_{-1} < a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < b = x_n < x_{n+1}. \tag{2.16}$$

So, for a fixed frequency ω in $\mathbb{R}$, let us denote by $\widehat{\mathbb{V}}_h$ the finite dimensional subspace of $\widehat{\mathrm{H}}_2$, given by

$$\widehat{\mathbb{V}}_h = \{u_h(.,\omega) \in \widehat{\mathrm{H}}_1 : \ u_h(.,\omega)|_{[x_i,x_{i+1}[} \in \mathbb{P}_3, \ \text{ for } \ 0 \le i \le n-1\}. \tag{2.17}$$

The classical Galerkin approximation consists of finding an approximation $u_h(.,\omega)$ of the analytic solution $u_*(.,\omega)$ as a solution in $\widehat{\mathbb{V}}_h$ of the following discrete variational problem:

$$\mathcal{A}_\omega(u_h(.,\omega), \mathrm{v}_h) = \mathscr{L}_\omega(\mathrm{v}_h) \ , \ \forall \mathrm{v}_h \in \widehat{\mathbb{V}}_h. \tag{2.18}$$

The solution $u_h(.,\omega)$ of the problem (2.18) is written in the following form

$$u_h(x,\omega) = \sum_{i=1}^{N} \mathrm{z}_{h,i}(\omega) B_i(x), \tag{2.19}$$

where the $\mathrm{z}_{h,i}(\omega)$ are unknown complex coefficients depending on the frequency ω and $(B_i)_{1\le i\le n}$ are the functions given by

$$B_i(x) = \mathcal{B}_3\left(\frac{x - x_{i-4}}{h}\right), \quad i = 1, \dots, N,$$

with $\mathcal{B}_3$ is the classical cubic Spline with the support embedding in $[0, 4]$. For more details on the Spline we refer to [2,5]. The space $\widehat{\mathbb{V}}_h$ of solution given in (2.17) is spanned by the Spline family $(B_i)_{1\le i\le N}$.
By using the test function $\mathrm{v}_h = B_j$ in the weak variational formulation (2.18), we obtain

$$\sum_{i=1}^{N} \mathrm{z}_{h,i}(\omega) \mathcal{A}_\omega\left(B_i, B_j\right) = \mathscr{L}_\omega(B_j), \ j = 1, \dots, N, \tag{2.20}$$

with

$$\mathcal{A}_\omega\left(B_i, B_j\right) = \frac{1}{h^2}\int_a^b D(x) B_i'(x) B_j'(x) dx + \int_a^b \left(\mu_a(x) + \frac{\mathrm{i}\omega}{c_0}\nu(x)\right) B_i(x) B_j(x) dx$$
$$+ 2\left(\zeta_a B_i(a) B_j(a) + \zeta_b B_i(b) B_j(b)\right),$$

and

$$\mathscr{L}_\omega(B_j) = 2\left(\zeta_a G_a(\omega) B_j(a) + \zeta_b G_b(\omega) B_j(b)\right) + \int_a^b F(x,\omega) B_j(x) dx,$$

for all $(i,j) \in \{1, \dots, N\}^2$. Let $A(\omega) = [A_{ij}(\omega)]$ be the $N \times N$ matrix whose entries are the complex coefficients

$$A_{ij}(\omega) = \mathcal{A}_\omega\left(B_i, B_j\right)$$

, let $\mathrm{b}_h(\omega) = (\mathrm{b}_{h,1}(\omega), \dots, \mathrm{b}_{h,N}(\omega))^T$ be the complex vector whose coefficients are $\mathrm{b}_{h,j}(\omega) = \mathscr{L}_\omega(B_j)$ and let $\mathrm{z}_h(\omega) = (\mathrm{z}_{h,1}(\omega), \dots, \mathrm{z}_{h,N}(\omega))^T$ be the vector formed by the unknown complex coefficients appearing in the expression (2.19) of the approximate solution u_{*h}. The relation (2.20) leads to the following $N \times N$ linear system

$$\mathrm{A}(\omega)\mathrm{z}_h(\omega) = \mathrm{b}_h(\omega). \tag{2.21}$$

where for all frequency values ω, the coefficients $\mathrm{z}_{h,i}(\omega)$ appearing in the expression (2.19) of the approximate solution u_h are obtained by solving the linear system (2.21).

3. Inverse problem for diffusion problem in optical tomography

In the following sections, we consider the Schwartz space $\mathcal{S}(\mathbb{R};\mathrm{L}_2)$ of functions $\phi : t \in \mathbb{R} \longrightarrow \phi(.,t)$ belonging to L_2 and satisfying the conditions:

(i) $\phi(.,t) \in \mathcal{C}^\infty(\mathbb{R};\mathrm{L}_2), \quad \forall t \in \mathbb{R}$,

(ii) $\mathcal{N}_{\ell,m}(\phi) := \sup_{k\leq \ell}\sup_{t\in\mathbb{R}}\Big[(1+t^2)^m \Big\| \dfrac{\partial^k \phi(.,t)}{\partial t^k}\Big\|_{\mathrm{L}_2}\Big] < \infty, \quad \ell, m \in \mathbb{N}.$

The space $\mathcal{S}(\mathbb{R};\mathrm{L}_2)$ is endowed with the topology defined by the set of the semi-norms family $(\mathcal{N}_{\ell,m})_{\ell,m}$. The topological dual space $\mathcal{S}'(\mathbb{R};\mathrm{L}_2)$ of $\mathcal{S}(\mathbb{R};\mathrm{L}_2)$ is the Schwartz space of tempered distributions, namely the space of linear continuous forms from $\mathcal{S}(\mathbb{R};\mathrm{L}_2)$ into $\mathbb{R}$.
For every $\mathrm{s} \in \mathbb{R}$, the non-homogeneous Sobolev spaces $\mathrm{H}^{\mathrm{s}}(\mathbb{R};\mathbb{R})$ defined by

$$\mathrm{H}^{\mathrm{s}}(\mathbb{R};\mathbb{R}) := \{\phi \in \mathcal{S}'(\mathbb{R};\mathbb{R}) : \ \widehat{\phi} \in \mathrm{L}^2_{loc}(\mathbb{R};\mathbb{C}), \int_{-\infty}^{+\infty} (1+\omega^2)^{\mathrm{s}} |\widehat{\phi}(.,\omega)|^2 d\omega < \infty\}, \tag{3.1}$$

is endowed with the topology defined by the norm given by

$$\|\phi\|_{\mathrm{H}^s(\mathbb{R};\mathbb{R})} = \left(\int_{-\infty}^{+\infty} (1+\omega^2)^{\mathrm{s}} |\widehat{\phi}(.,\omega)|^2 d\omega\right)^{\frac{1}{2}}. \tag{3.2}$$

The non-homogeneous Sobolev spaces $\mathrm{H}^{\mathrm{s}}(\mathbb{R};\mathrm{L}_2)$ given by

$$\mathrm{H}^{\mathrm{s}}(\mathbb{R};\mathrm{L}_2) := \{\phi \in \mathcal{S}'(\mathbb{R};\mathrm{L}_2) : \ \widehat{\phi} \in \mathrm{L}^2_{loc}(\mathbb{R};\widehat{\mathrm{L}}_2), \int_{-\infty}^{+\infty} (1+\omega^2)^{\mathrm{s}} \|\widehat{\phi}(.,\omega)\|^2_{\widehat{\mathrm{L}}_2} d\omega < \infty\},$$

is endowed with the topology defined by the norm given by

$$\|\phi\|_{\mathrm{H}^s(\mathbb{R};\mathrm{L}_2)} = \left(\int_{-\infty}^{+\infty} (1+\omega^2)^{\mathrm{s}} \|\widehat{\phi}(.,\omega)\|^2_{\widehat{\mathrm{L}}_2} d\omega\right)^{\frac{1}{2}}. \tag{3.3}$$

For every $(\mathrm{r},\mathrm{s}) \in \mathbb{R}^2$, the non-homogeneous Sobolev spaces $\mathrm{H}^{\mathrm{s}}(\mathbb{R};\mathrm{H}_{\mathrm{r}})$ is defined by replacing the space L_2 and the norm (3.3) by H_{r} and

$$\|\phi\|_{\mathrm{H}^s(\mathbb{R};\mathrm{H}_{\mathrm{r}})} = \left(\int_{-\infty}^{+\infty} (1+\omega^2)^{\mathrm{s}} \|\widehat{\phi}(.,\omega)\|^2_{\widehat{\mathrm{H}}_{\mathrm{r}}} d\omega\right)^{\frac{1}{2}}, \tag{3.4}$$

respectively.

The product space $L^\infty(a,b)\times L^\infty(a,b)\times L^\infty(a,b)$ endowed with the L^∞-topology defined by the norm given by

$$\|(\mathrm{u},\mathrm{v},\mathrm{w})\|_\infty = \|\mathrm{u}\|_\infty + \|\mathrm{v}\|_\infty + \|\mathrm{w}\|_\infty \tag{3.5}$$

is a Banach space.

In the following sections, we will study the inverse problem which consists of the estimation of the optical properties from the flow measurements of analytic and computed solutions of the direct problem (2.8)-(2.9), at the boundary of the domain. For given sources functions f, g_a and g_b, we denote then by $\phi(x,\mathbf{s},t) := \phi[\mathbf{s}](x,t)$ the solution of the forward diffusion problem (2.2)-(2.3) and by $u(x,\mathbf{s},\omega) := u[\mathbf{s}](x,\omega)$ the solution of the forward diffusion problem in frequency domain (2.8)-(2.9), with respect to the optical parameters $\mathbf{s} = (\nu, D, \mu_a)$. We will demonstrate theoretical results concerning the inverse problem and describe a numerical approximation of the parameters $\mathbf{s} = (\nu, D, \mu_a)$ using the values of the signal at the boundary of our computational domain.

3.1. Dirichlet-to-Neumann map

Let $\mathbb{B}$ be a subspace of elements $Y = (\varphi, \mathrm{r}_a, \mathrm{r}_b)$ in $\mathrm{H}_1 \times \mathbb{R} \times \mathbb{R}$ satisfying the following conditions

$$\begin{aligned} \varphi(a) - \frac{\beta_a}{2} D(a) \frac{\partial \varphi}{\partial x}(a) &= \mathrm{r}_a, \\ \varphi(b) + \frac{\beta_b}{2} D(b) \frac{\partial \varphi}{\partial x}(b) &= \mathrm{r}_b, \end{aligned} \tag{3.6}$$

where $\frac{\partial \varphi}{\partial x}$ is the derivative of $\varphi \in \mathrm{H}_1$ in the distribution sense. The subspace $\mathbb{B}$ is equipped by the inner product and its associated norm induced by the Hilbert space $\mathrm{H}_1 \times \mathbb{R} \times \mathbb{R}$. Also, for $Y = (\varphi, \mathrm{r}_a, \mathrm{r}_b) \in \mathbb{B}$, the norm of Y in space $\mathbb{B}$ is given by

$$\|Y\|_{\mathbb{B}} = \left(\|\varphi\|^2_{\mathrm{H}_1} + |\mathrm{r}_a|^2 + |\mathrm{r}_b|^2\right)^{1/2}.$$

Let $\widehat{\mathbb{B}}$ be the subspace of elements $\widehat{Y} = (\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b)$ in $\widehat{\mathbb{B}}$ satisfying the following conditions

$$\begin{aligned} \mathrm{v}(a) - \frac{\beta_a}{2} D(a) \frac{\partial \mathrm{v}}{\partial x}(a) &= \mathrm{z}_a, \\ \mathrm{v}(b) + \frac{\beta_b}{2} D(b) \frac{\partial \mathrm{v}}{\partial x}(b) &= \mathrm{z}_b. \end{aligned} \tag{3.7}$$

The subspace $\widehat{\mathbb{B}}$ is equipped by the inner product and its associated norm induced by the Hilbert space $\widehat{\mathrm{H}}_1 \times \mathbb{C} \times \mathbb{C}$. So, for $\widehat{Y} = (\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b) \in \widehat{\mathbb{B}}$, the norm of $\widehat{Y}$ in the space $\widehat{\mathbb{B}}$ is given by

$$\|\widehat{Y}\|_{\widehat{\mathbb{B}}} = \left(\|\mathrm{v}\|^2_{\widehat{\mathrm{H}}_1} + |\mathrm{z}_a|^2 + |\mathrm{z}_b|^2\right)^{1/2}.$$

It is clear that the subspaces $(\mathbb{B}, \|\,.\,\|_{\mathbb{B}})$ and $(\widehat{\mathbb{B}}, , \|\,.\,\|_{\widehat{\mathbb{B}}})$ are Hilbert spaces.
Define the nonlinear operator Λ by

$$\Lambda : \mathbf{s} \in (\mathrm{L}^\infty([a,b]))^3 \longrightarrow \Lambda_{\mathbf{s}} \in \mathscr{L}(\mathrm{L}_2 \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^2), \tag{3.8}$$

where $\mathscr{L}(\mathrm{L}_2\times\mathbb{R}\times\mathbb{R};\mathbb{R}^2)$ is the space of linear operators from $\mathrm{L}_2\times\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}^2$, and $\Lambda_{\mathbf{s}}$ is the Dirichlet-to-Neumann (DtN) linear map corresponding to $\mathbf{s}=(\nu,D,\mu_a)$ given by

$$\begin{aligned}\Lambda_{\mathbf{s}} \quad &: \quad \mathrm{L}_2\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}^2,\\ &\quad (f(.,t),g_a(t),g_b(t))\longmapsto\begin{pmatrix} D(a)\dfrac{\partial\phi}{\partial x}(a,\mathbf{s},t)\\ -D(b)\dfrac{\partial\phi}{\partial x}(b,\mathbf{s},t)\end{pmatrix},\end{aligned}\tag{3.9}$$

where $t\in[0,T]$. So, by using the boundary conditions (2.3) and (3.9) we obtain

$$\Lambda_{\mathbf{s}}(f(.,t),g_a(t),g_b(t))=\begin{pmatrix}2\zeta_a\left(\phi(a,\mathbf{s},t)-g_a(t)\right)\\ 2\zeta_b\left(\phi(b,\mathbf{s},t)-g_b(t)\right)\end{pmatrix}.\tag{3.10}$$

Now, define the non-linear operator $\widehat{\Lambda}$ by

$$\widehat{\Lambda}:\mathbf{s}\in\left(\mathrm{L}^\infty([a,b],\mathbb{R})\right)^3\longrightarrow\widehat{\Lambda}_{\mathbf{s}}\in\mathscr{L}(\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C};\mathbb{C}^2),\tag{3.11}$$

where $\mathscr{L}(\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C};\mathbb{C}^2)$ is the space of linear operators from $\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C}$ to $\mathbb{C}^2$, and the Dirichlet-to-Neumann linear map $\widehat{\Lambda}_{\mathbf{s}}$ corresponding to $\mathbf{s}=(\nu,D,\mu_a)$ is given by

$$\begin{aligned}\widehat{\Lambda}_{\mathbf{s}} \quad &: \quad \widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{C}^2,\\ &\quad (F(.,\omega),G_a(\omega),G_b(\omega))\longmapsto\begin{pmatrix} D(a)\dfrac{\partial u}{\partial x}(a,\mathbf{s},\omega)\\ -D(b)\dfrac{\partial u}{\partial x}(b,\mathbf{s},\omega)\end{pmatrix},\end{aligned}\tag{3.12}$$

where $\omega\in\mathbb{R}$ be a frequency value and $u(.,\mathbf{s},.)$ denote the solution the problem (2.8)-(2.9). So, by using (3.12) and the boundary conditions (2.9) we prove that

$$\widehat{\Lambda}_{\mathbf{s}}(F(.,\omega),G_a(\omega),G_b(\omega))=\begin{pmatrix}2\zeta_a\left(u(a,\mathbf{s},\omega)-G_a(\omega)\right)\\ 2\zeta_b\left(u(b,\mathbf{s},\omega)-G_b(\omega)\right)\end{pmatrix}.\tag{3.13}$$

The linear map $\widehat{\Lambda}_{\mathbf{s}}$ is an analogue operator of the linear operator $\Lambda_{\mathbf{s}}$ given in (3.9). It describes Fourier's law in the frequency domain. The operators $\Lambda_{\mathbf{s}}$ and $\widehat{\Lambda}_{\mathbf{s}}$ associated to $\mathbf{s}=(\nu,D,\mu_a)$ are the detectors of the output flow measurements at the boundary $\{a,b\}$ of the domain $[a,b]$.

Proposition 3.1. *For a given frequency $\omega\in\mathbb{R}$, the linear map $\widehat{\Lambda}_{\mathbf{s}}$ is characterized by the following duality product*

$$\begin{aligned}\langle\widehat{\Lambda}_{\mathbf{s}}(\widehat{X}(\omega)),(\mathrm{v},\mathrm{z}_a,\mathrm{z}_b)\rangle &= \int_a^b F(x,\omega)\overline{\mathrm{v}(x)}dx-\int_a^b D(x)\frac{\partial u}{\partial x}(x,\mathbf{s},\omega)\overline{\frac{\partial\mathrm{v}}{\partial x}(x)}dx\\ &-\int_a^b\left(\mu_a(x)+\frac{\mathrm{i}\omega}{c_0}\nu(x)\right)u(x,\mathbf{s},\omega)\overline{\mathrm{v}(x)}dx+D(a)\left(G_a(\omega)-u(a,\mathbf{s},\omega)\right)\overline{\frac{\partial\mathrm{v}}{\partial x}(a)}\\ &+D(b)\left(u(b,\mathbf{s},\omega)-G_b(\omega)\right)\overline{\frac{\partial\mathrm{v}}{\partial x}(b)},\end{aligned}\tag{3.14}$$

for all $(\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b) \in \widehat{\mathbb{B}}$ *and for all source functions* $\widehat{X}(\omega) = (F(.,\omega), G_a(\omega), G_b(\omega)) \in \widehat{\mathrm{L}}_2 \times \mathbb{C} \times \mathbb{C}$*, where* $u(.,\mathbf{s},\omega) \in \widehat{\mathrm{H}}_1$ *is the solution of* (2.8) – (2.9) *corresponding to the source functions* $F(.,\omega)$*,* $G_a(\omega)$ *and* $G_b(\omega)$*.*

Proof. Let us denote by $\mathcal{G}$, the following duality product

$$\begin{aligned}
\mathcal{G} &:= \langle \widehat{\Lambda}_{\mathbf{s}}\left(F(.,\omega), G_a(\omega), G_b(\omega)\right), (\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b)\rangle \\
&= D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\overline{\mathrm{z}_a} - D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\overline{\mathrm{z}_b}.
\end{aligned}$$

By using the boundary conditions (3.7), we obtain the following equality

$$\begin{aligned}
\mathcal{G} &= D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\left(\overline{\mathrm{v}(a)} - \frac{\beta_a}{2}D(a)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)}\right) \\
&\quad -D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\left(\overline{\mathrm{v}(b)} + \frac{\beta_b}{2}D(b)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)}\right) \\
&= D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\overline{\mathrm{v}(a)} - D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\overline{\mathrm{v}(b)} \\
&\quad -D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\frac{\beta_a}{2}D(a)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)} \qquad (3.15)\\
&\quad -D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\frac{\beta_b}{2}D(b)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)}. \qquad (3.16)
\end{aligned}$$

From the variational formulation (2.10) we have

$$\begin{aligned}
D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\overline{\mathrm{v}(a)} &- D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\overline{\mathrm{v}(b)} \qquad (3.17)\\
&= -\int_a^b \left(\mu_a(x) + \frac{\mathrm{i}\omega}{c_0}\nu(x)\right)u(x,\mathbf{s},\omega)\overline{\mathrm{v}(x)}dx \\
&\quad -\int_a^b D(x)\frac{\partial u}{\partial x}(x,\mathbf{s},\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(x)}dx \qquad (3.18)\\
&\quad +\int_a^b F(x,\omega)\overline{\mathrm{v}(x)}dx. \qquad (3.19)
\end{aligned}$$

Using the boundary conditions corresponding to the solution $u(.,\mathbf{s},\omega)$, we get

$$\begin{aligned}
D(a)\frac{\partial u}{\partial x}(a,\mathbf{s},\omega)\frac{\beta_a}{2}D(a)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)} &= D(a)\left(u(a,\mathbf{s},\omega) - G_a(\omega)\right)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)}, \\
D(b)\frac{\partial u}{\partial x}(b,\mathbf{s},\omega)\frac{\beta_b}{2}D(b)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)} &= D(b)\left(G_b(\omega) - u(b,\mathbf{s},\omega)\right)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)}.
\end{aligned} \qquad (3.20)$$

Substituting the identities (3.17) and (3.20) into (3.15), the expected result (3.14) holds. □

In the following theorem, we will show the continuity of the linear operators $\Lambda_{\mathbf{s}}$ and $\widehat{\Lambda}_{\mathbf{s}}$ with respect to optical parameters $\mathbf{s} = (\nu, D, \mu_a)$. Let us denote it by

$$\mathscr{L}_2 = \mathscr{L}\left(F, G\right),$$

where

$$F = \mathrm{H}^{-1}(\mathbb{R}; \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R})$$

and

$$G = \mathrm{H}^{-1}(\mathbb{R}; \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}),$$

the space of the linear and continuous operators from

$$\mathrm{H}^{-1}(\mathbb{R}; \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}) \quad \text{to} \quad \mathrm{H}^{-1}(\mathbb{R}; \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{R}),$$

and

$$\widehat{\mathscr{L}_2} = \mathscr{L}\left(\widehat{F}, \widehat{G}\right),$$

where

$$\widehat{F} = \mathrm{H}^{-1}(\mathbb{R}; \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C})$$

and

$$\widehat{G} = \mathrm{H}^{-1}(\mathbb{R}; \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}),$$

the space of the linear and continuous operators from

$$\mathrm{H}^{-1}(\mathbb{R}; \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}) \quad \text{to} \quad \mathrm{H}^{-1}(\mathbb{R}; \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}; \mathbb{C}).$$

We denote by $\| \, . \, \|_{\mathscr{L}_2}$ and $\| \, . \, \|_{\widehat{\mathscr{L}_2}}$ the usual norms of $\mathscr{L}_2$ and $\widehat{\mathscr{L}_2}$, respectively.
So, we introduce the following notations. Let $X = (f, g_a, g_b) \in \mathrm{H}^{-1}(\mathbb{R}, \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})$, we set $X(t) = (f(., t), g_a(t), g_b(t))$ for all $t \in [0, T]$. Denote by

$$\|X\| = \left(\|f\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \mathrm{L}_2)} + \|g_a\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})} + \|g_b\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})} \right)^{1/2}$$

the norm of X the space $\mathrm{H}^{-1}(\mathbb{R}, \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})$. Analogously, let $\widehat{X} = (F, G_a, G_b)$ be an element in the product space $\mathrm{H}^{-1}(\mathbb{R}, \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{C})$, and for ω in $\mathbb{R}$, we set $\widehat{X}(\omega) = (F(., \omega), G_a(\omega), G_b(\omega))$. The norm of $\widehat{X}$ in the space $\mathrm{H}^{-1}(\mathbb{R}, \widehat{\mathrm{L}}_2) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{C}) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{C})$ is given by

$$|||\widehat{X}||| = \left(\|F\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \widehat{\mathrm{L}}_2)} + \|G_a\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \mathbb{C})} + \|G_b\|^2_{\mathrm{H}^{-1}(\mathbb{R}, \mathbb{C})} \right)^{1/2}.$$

Theorem 3.2. *Let $f \in \mathrm{H}^{-1}(\mathbb{R}, \mathrm{L}_2)$ and $g_a, g_b \in \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})$ be given source functions. Let $\mathbf{s}_1 = (\nu_1, D_1, \sigma_{a1})$ and $\mathbf{s}_2 = (\nu_2, D_2, \sigma_{a2})$ be two functions in $(\mathrm{L}^\infty([a, b], \mathbb{R}))^3$ with their components satisfying the condition (2.4) and let $u_1 := \widehat{\phi}_1$ and $u_2 := \widehat{\phi}_2$ be the solutions of the diffusion problem (2.8) − (2.9) with respect to the parameters $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. Then, there exists a constant $C > 0$, independent of $\mathbf{s}_1$ and $\mathbf{s}_2$, such that* *(i)* $\|u_2 - u_1\|_{\mathrm{H}^{-2}(\mathbb{R},\widehat{\mathrm{H}}_1)} \leq C\|X\| \, \|\mathbf{s}_2 - \mathbf{s}_1\|_\infty$

(ii) $\|\phi_2 - \phi_1\|_{\mathrm{H}^{-2}(\mathbb{R},\mathrm{H}_1)} \leq C\|X\| \, \|\mathbf{s}_2 - \mathbf{s}_1\|_\infty$

(iii) $\|\Lambda_{\mathbf{s}_1} - \Lambda_{\mathbf{s}_2}\|_{\mathscr{L}_2} \leq \|\widehat{\Lambda}_{\mathbf{s}_1} - \widehat{\Lambda}_{\mathbf{s}_2}\|_{\widehat{\mathscr{L}_2}} \leq C\|\mathbf{s}_1 - \mathbf{s}_2\|_\infty$.

Proof. Let ω be a given real frequency.

(i) We consider two solutions $u_1(.,\omega)$ and $u_2(.,\omega)$ in $\widehat{\mathrm{H}}_1$ of the diffusion problem (2.8) − (2.9) with respect to the parameters $\mathbf{s}_1 = (\nu_1, D_1, \sigma_{a1})$ and $\mathbf{s}_2 = (\nu_2, D_2, \sigma_{a2})$, respectively. The difference $\mathrm{w}(.,\omega) := u_2(.,\omega) - u_1(.,\omega)$ is the weak solution of the following problem

$$-\frac{\partial}{\partial x}\left(D_2(x)\frac{\partial \mathrm{w}}{\partial x}(x,\omega)\right) + \left(\sigma_{a2}(x) + \frac{\mathrm{i}\omega}{c_0}\nu_2(x)\right)\mathrm{w}(x,\omega) =$$

$$\frac{\partial}{\partial x}\left((D_2 - D_1)(x)\frac{\partial u_1}{\partial x}(x,\omega)\right) \tag{3.21}$$

$$-\left((\sigma_{a2} - \sigma_{a1})(x) + \frac{\mathrm{i}\omega}{c_0}(\nu_2 - \nu_1)(x)\right)u_1(x,\omega), \tag{3.22}$$

with the following non-homogeneous boundary conditions

$$\begin{aligned} \mathrm{w}(a,\omega) - \frac{\beta_a}{2}D_2(a)\frac{\partial \mathrm{w}}{\partial x}(a,\omega) &= \frac{\beta_a}{2}\left(D_2(a) - D_1(a)\right)\frac{\partial u_1}{\partial x}(a,\omega), \\ \mathrm{w}(b,\omega) + \frac{\beta_b}{2}D_2(b)\frac{\partial \mathrm{w}}{\partial x}(b,\omega) &= -\frac{\beta_b}{2}\left(D_2(b) - D_1(b)\right)\frac{\partial u_1}{\partial x}(b,\omega). \end{aligned} \tag{3.23}$$

Multiplying the equation (3.21) by $\overline{\mathrm{w}(.,\omega)}$ and integrating over $[a,b]$, by using the conditions (3.23), we obtain the following equation

$$\begin{aligned} \mathcal{B}_1(\mathrm{w},\mathrm{w}) \quad := \quad & \int_a^b D_2(x)\left|\frac{\partial \mathrm{w}}{\partial x}(x,\omega)\right|^2 dx + \int_a^b \sigma_{a2}(x)\left|\mathrm{w}(x,\omega)\right|^2 dx \\ & + \frac{\mathrm{i}\omega}{c_0}\int_a^b \nu_2(x)\left|\mathrm{w}(x,\omega)\right|^2 dx + 2\left(\zeta_a\left|\mathrm{w}(a,\omega)\right|^2 + \zeta_b\left|\mathrm{w}(b,\omega)\right|^2\right) \\ = \quad & -\int_a^b (D_2 - D_1)(x)\frac{\partial u_1}{\partial x}(x,\omega)\overline{\frac{\partial \mathrm{w}}{\partial x}(x,\omega)}dx \\ & -\int_a^b (\sigma_{a2} - \sigma_{a1})(x)u_1(x,\omega)\overline{\mathrm{w}(x,\omega)}dx \\ & -\frac{\mathrm{i}\omega}{c_0}\int_a^b (\nu_2 - \nu_1)(x)u_1(x,\omega)\overline{\mathrm{w}(x,\omega)}dx, \end{aligned}$$

we have

$$\begin{aligned} |\mathcal{B}_1(\mathrm{w},\mathrm{w})| \leq & \int_a^b |(D_2 - D_1)(x)|\left|\frac{\partial u_1}{\partial x}(x,\omega)\right|\left|\frac{\partial \mathrm{w}}{\partial x}(x,\omega)\right| dx \\ & + \int_a^b \left|(\sigma_{a2} - \sigma_{a1})(x) - \frac{\mathrm{i}\omega}{c_0}(\nu_2 - \nu_1)(x)\right| |u_1(x,\omega)|\,|\mathrm{w}(x,\omega)|\,dx. \end{aligned}$$

The inequality $t_1t_2 + t_3t_4 \leq (t_1 + t_3)(t_2 + t_4)$ for positive reals numbers $t_i(i = 1, \ldots, 4)$ implies that there exists a constant $C_1 > 0$ independent of ω such that

$$|\mathcal{B}_1(\mathrm{w}, \mathrm{w})| \leq C_1 \left(\|D_2 - D_1\|_\infty + \|\nu_2 - \nu_1\|_\infty + \|\sigma_{a2} - \sigma_{a1}\|_\infty\right) \sqrt{1 + \omega^2} \times \int_a^b \left(\left|\frac{\partial u_1}{\partial x}(x, \omega)\right| + |u_1(x, \omega)|\right) \left(\left|\frac{\partial \mathrm{w}}{\partial x}(x, \omega)\right| + |\mathrm{w}(x, \omega)|\right) dx.$$

The Cauchy-Schwartz inequality gives

$$|\mathcal{B}_1(\mathrm{w}, \mathrm{w})| \leq 2C_1 \sqrt{1 + \omega^2} \left(\|D_2 - D_1\|_\infty + \|\nu_2 - \nu_1\|_\infty + \|\sigma_{a2} - \sigma_{a1}\|_\infty\right) \times \|\mathrm{w}(., \omega)\|_{\widehat{\mathrm{H}}_1} \|u_1(., \omega)\|_{\widehat{\mathrm{H}}_1}. \quad (3.24)$$

Also, we have

$$|\mathcal{B}_1(\mathrm{w}, \mathrm{w})| \geq \int_a^b D_2(x) \left|\frac{\partial \mathrm{w}}{\partial x}(x, \omega)\right|^2 dx + \int_a^b \sigma_{a2}(x) |\mathrm{w}(x, \omega)|^2 dx + 2\left(\zeta_a |\mathrm{w}(a, \omega)|^2 + \zeta_b |\mathrm{w}(b, \omega)|^2\right) \geq \alpha \|\mathrm{w}(x, \omega)\|^2_{\widehat{\mathrm{H}}_1}, \quad (3.25)$$

where α is the constant appearing in (2.4). From (3.24) and (3.25), it follows that

$$\alpha \|\mathrm{w}(., \omega)\|_{\widehat{\mathrm{H}}_1} \leq 2C_1 \sqrt{1 + \omega^2} \left(\|D_2 - D_1\|_\infty + \|\nu_2 - \nu_1\|_\infty\right) \|u_1(., \omega)\|_{\widehat{\mathrm{H}}_1} + 2C_1 \sqrt{1 + \omega^2} \left(\|\sigma_{a2} - \sigma_{a1}\|_\infty\right) \|u_1(., \omega)\|_{\widehat{\mathrm{H}}_1}. \quad (3.26)$$

Combining with the inequality (2.15) of Theorem 2.1 and (3.26) we get the following inequality

$$\|\mathrm{w}(., \omega)\|_{\widehat{\mathrm{H}}_1} \leq \frac{2C_0C_1}{\alpha^2} \sqrt{1 + \omega^2} \left(\|F(., \omega)\|_{\widehat{\mathrm{L}}_2} + |G_a(\omega)| + |G_b(\omega)|\right) \times \|\mathbf{s}_2 - \mathbf{s}_1\|_\infty, \quad (3.27)$$

where C_0 is the positive constant appearing in the expression of $M(\omega)$ given by (2.12) and (2.15). Thus, using the inequality $(t_1 + t_2 + t_3)^2 \leq 3(t_1^2 + t_2^2 + t_3^2)$ for reals numbers $t_i(i = 1, 2, 3)$, we obtain

$$\|u_2(., \omega) - u_1(., \omega)\|^2_{\widehat{\mathrm{H}}_1} \leq 3 \left(\frac{2C_0C_1}{\alpha^2}\right)^2 (1 + \omega^2) |||\widehat{X}(\omega)|||^2 \|\mathbf{s}_2 - \mathbf{s}_1\|^2_\infty, \quad (3.28)$$

and this implies the following equivalent inequality

$$(1 + \omega^2)^{-2} \|u_2(., \omega) - u_1(., \omega)\|^2_{\widehat{\mathrm{H}}_1} \leq 3 \left(\frac{2C_0C_1}{\alpha^2}\right)^2 \|\mathbf{s}_2 - \mathbf{s}_1\|^2_\infty \times (1 + \omega^2)^{-1} |||\widehat{X}(\omega)|||^2. \quad (3.29)$$

To achieve the proof of Item i), we integrate the inequations (3.29) over $\mathbb{R}$ with respect to ω by taking into account the definition of the norms in (3.2), (3.3) and (3.5).

(ii) Multiplying the inequality (3.28) by the weight function $\omega \longmapsto (1+\omega^2)^{-2}$ and integrating over $\mathbb{R}$ with respect to ω, we obtain,

$$\|\phi_2 - \phi_1\|_{\mathrm{H}^{-2}(\mathbb{R},\mathrm{H}_1)} \leq \left(\frac{2\sqrt{3}C_0C_1}{\alpha^2}\right) \|X\| \, \|\mathbf{s}_2 - \mathbf{s}_1\|_\infty ,$$

and which proves Item (ii).

(iii) As in Item (i), we consider two solutions $u_2(.,\omega)$ and $u_1(.,\omega)$ in $\widehat{\mathrm{H}}_1$ of the diffusion problem (2.8) – (2.9) with respect to the parameters $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. Let $\widehat{Y} = (\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b)$ where v is a function in $\widehat{\mathrm{H}}_1$ satisfying the trace boundary conditions (3.7). According to Proposition 3.1, we get

$$\begin{aligned}
\langle (\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega), \widehat{Y}\rangle = &- \int_a^b (D_2 - D_1)(x)\frac{\partial u_2}{\partial x}(x,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(x)}dx \\
&- \int_a^b (\mu_{a2} - \mu_{a1})(x) u_2(x,\omega)\overline{\mathrm{v}(x)}dx \\
&- \frac{\mathrm{i}\omega}{c_0}\int_a^b (\nu_2 - \nu_1)(x) u_2(x,\omega)\overline{\mathrm{v}(x)}dx \\
&- \int_a^b D_1(x)\frac{\partial (u_2 - u_1)}{\partial x}(x,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(x)}dx \\
&- \int_a^b \sigma_{a1}(x)(u_2 - u_1)(x,\omega)\overline{\mathrm{v}(x)}dx \\
&- \frac{\mathrm{i}\omega}{c_0}\int_a^b \nu_1(x)(u_2 - u_1)(x,\omega)\overline{\mathrm{v}(x)}dx \\
&+ D_1(b)(u_1 - u_2)(b,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)} \\
&- D_1(a)(u_1 - u_2)(a,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)} \\
&+ (D_1(b) - D_2(b))\,(u_2(b,\omega) - G_b(\omega))\overline{\frac{\partial \mathrm{v}}{\partial x}(b)} \\
&+ (D_1(a) - D_2(a))\,(G_a(\omega) - u_2(a,\omega))\overline{\frac{\partial \mathrm{v}}{\partial x}(a)}.
\end{aligned}$$

Taking into account of (2.11), there exists a constant $C_1 > 0$ such that

$$\left| D_1(b)(u_1 - u_2)(b,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(b)} \right| \leq C_1 \|D_1\|_\infty \|(u_2 - u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1} \|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} ,$$

$$\left|D_1(a)(u_1-u_2)(a,\omega)\overline{\frac{\partial \mathrm{v}}{\partial x}(a)}\right| \leq C_1\|D_1\|_\infty\|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\,,$$

$$\begin{aligned}\left|(D_1(b)-D_2(b))\,(u_2(b,\omega)-G_b(\omega))\overline{\frac{\partial \mathrm{v}}{\partial x}(b)}\right| \;\leq\; & C_1\|D_2-D_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\\ & +C_1\|D_2-D_1\|_\infty|G_b(\omega)|\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\,,\end{aligned}$$

$$\begin{aligned}\left|(D_1(a)-D_2(a))\,(G_a(\omega)-u_2(a,\omega))\overline{\frac{\partial \mathrm{v}}{\partial x}(a)}\right| \;\leq\; & C_1\|D_2-D_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\\ & +C_1\|D_2-D_1\|_\infty|G_a(\omega)|\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\,.\end{aligned}$$

It follows that

$$\begin{aligned}|\langle(\widehat{\Lambda}_{\mathbf{s}_2}-\widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega),\widehat{Y}\rangle| \leq & \|D_2-D_1\|_\infty\int_a^b\left|\frac{\partial u_2}{\partial x}(x,\omega)\right|\left|\frac{\partial \mathrm{v}}{\partial x}(x)\right|dx\\ & +\|\sigma_{a2}-\sigma_{a1}\|_\infty\int_a^b|u_2(x,\omega)||\mathrm{v}(x)|dx\\ & +\sqrt{1+\omega^2}\|\nu_2-\nu_1\|_\infty\int_a^b|u_2(x,\omega)||\mathrm{v}(x)|dx\\ & +\|D_1\|_\infty\int_a^b\left|\frac{\partial(u_2-u_1)}{\partial x}(x,\omega)\right|\left|\frac{\partial \mathrm{v}}{\partial x}(x)\right|dx\\ & +\|\sigma_{a1}\|_\infty\int_a^b|(u_2-u_1)(x,\omega)|\,|\mathrm{v}(x)|\,dx\\ & +\sqrt{1+\omega^2}\|\nu_1\|_\infty\int_a^b|(u_2-u_1)(x,\omega)|\,|\mathrm{v}(x)|\,dx\\ & +2C_1\|D_1\|_\infty\|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\\ & +C_1\|D_2-D_1\|_\infty 2\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\\ & +C_1\|D_2-D_1\|_\infty\,(|G_a(\omega)|+|G_b(\omega)|)\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\,.\end{aligned}$$

Since ν, D and μ_a belong to $\mathrm{L}^\infty([a,b],\mathbb{R})$, then there exists $\varrho>0$ such that

$$\varrho=\max\{\|\nu\|_\infty,\|D\|_\infty,\|\sigma_a\|_\infty\},$$

and using the norm (3.5) we obtain

$$
\begin{aligned}
|\langle(\widehat{\Lambda}_{\mathbf{s}_2}-\widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega),\widehat{Y}\rangle| \quad \leq \quad & \sqrt{1+\omega^2}\|\mathbf{s}_2-\mathbf{s}_1\|_\infty \int_a^b \left|\frac{\partial u_2}{\partial x}(x,\omega)\right| \left|\frac{\partial \mathrm{v}}{\partial x}(x)\right| dx \\
& +\sqrt{1+\omega^2}\|\mathbf{s}_2-\mathbf{s}_1\|_\infty \int_a^b |u_2(x,\omega)||\mathrm{v}(x)|dx \\
& +\varrho\sqrt{1+\omega^2}\int_a^b \left|\frac{\partial(u_2-u_1)}{\partial x}(x,\omega)\right| \left|\frac{\partial \mathrm{v}}{\partial x}(x)\right| dx \\
& +\varrho\sqrt{1+\omega^2}\int_a^b |(u_2-u_1)(x,\omega)|\,|\mathrm{v}(x)|\,dx \\
& +2C_1\varrho\|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +2C\|D_2-D_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +C_1\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\sqrt{1+\omega^2}\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +C_1\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\sqrt{1+\omega^2}|G_a(\omega)|\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +C_1\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\sqrt{1+\omega^2}|G_b(\omega)|\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}
\end{aligned}
$$

Using again the inequality $t_1t_2 + t_3t_4 \leq (t_1 + t_3)(t_2 + t_4)$ for positive real numbers $t_i (i = 1, \dots, 4)$, we obtain

$$
\begin{aligned}
& |\langle(\widehat{\Lambda}_{\mathbf{s}_2}-\widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega),\widehat{Y}\rangle| \leq \\
& \sqrt{1+\omega^2}\|\mathbf{s}_2-\mathbf{s}_1\|_\infty \int_a^b \left(\left|\frac{\partial u_2}{\partial x}(x,\omega)\right| + |u_2(x,\omega)|\right)\left(\left|\frac{\partial \mathrm{v}}{\partial x}(x)\right| + |\mathrm{v}(x)|\right) dx \\
& +\varrho\sqrt{1+\omega^2}\int_a^b \left(\left|\frac{\partial(u_2-u_1)}{\partial x}(x,\omega)\right| + |(u_2-u_1)(x,\omega)|\right) \\
& \times \left(\left|\frac{\partial \mathrm{v}}{\partial x}(x)\right| + |\mathrm{v}(x)|\right) dx \\
& +2C_1\varrho\sqrt{1+\omega^2}\|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +2C_1\sqrt{1+\omega^2}\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +C_1\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\sqrt{1+\omega^2}\left(\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} + |G_a(\omega)| + |G_b(\omega)|\right)\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}.
\end{aligned}
$$

From the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|\langle(\widehat{\Lambda}_{\mathbf{s}_2}-\widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega),\widehat{Y}\rangle| \leq\ & 2(1+C_1)\sqrt{1+\omega^2}\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +\varrho(1+2C_1)\sqrt{1+\omega^2}\|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1} \\
& +C_1\|\mathbf{s}_2-\mathbf{s}_1\|_\infty\sqrt{1+\omega^2}\left(\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} + |G_a(\omega)| + |G_b(\omega)|\right)\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}.
\end{aligned}
$$

Let $C_2 = \max\{2(1+C_1), \varrho(1+2C_1)\}$, then we have

$$\begin{aligned}|\langle(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega), \widehat{Y}\rangle| \leq & C_2\sqrt{1+\omega^2}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\|u_2(.,\omega)\|_{\widehat{\mathrm{H}}_1} \\ & + \|(u_2-u_1)(.,\omega)\|_{\widehat{\mathrm{H}}_1} \\ & + C_2\sqrt{1+\omega^2}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} \\ & + C_2\sqrt{1+\omega^2}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\left(|G_a(\omega)| + |G_b(\omega)|\right).\end{aligned}$$

Using (2.12), (2.15) and (3.27) of Item (i), there exists a constant $C_3 > 0$ such that

$$\begin{aligned}|\langle(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega), \widehat{Y}\rangle| \leq & C_3(1+\omega^2)\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} \\ & + C_3(1+\omega^2)\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\left(|G_a(\omega)| + |G_b(\omega)|\right) \\ \leq & C_3(1+\omega^2)\left(\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2}\right)\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\|\widehat{Y}\|_{\widehat{\mathbb{B}}} \\ & + C_3(1+\omega^2)\left(|G_a(\omega)| + |G_b(\omega)|\right)\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty\|\widehat{Y}\|_{\widehat{\mathbb{B}}}.\end{aligned}$$

Then, the following inequality

$$\begin{aligned}\left\|(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega)\right\|_{\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C}} \leq \quad & C_3\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty(1+\omega^2)\|F(.,\omega)\|_{\widehat{\mathrm{L}}_2} \\ & C_3\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty(1+\omega^2)\left(|G_a(\omega)| + |G_b(\omega)|\right)\end{aligned}$$

holds. Using again the inequality $(t_1+t_2+t_3)^2 \leq 3(t_1^2+t_2^2+t_3^2)$ for reals numbers $t_i (i=1,2,3)$, it follows that

$$(1+\omega^2)^{-2}\left\|(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}(\omega)\right\|^2_{\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C}} \leq 3C_3^2\|\widehat{X}(\omega)\|\,\|\mathbf{s}_2 - \mathbf{s}_1\|^2_\infty. \quad (3.30)$$

Integrating the inequality (3.30) over $\mathbb{R}$ with respect to ω holds the following inequality

$$\left\|(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}\right\| \leq C_3\sqrt{3}\|X\|\,\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty.$$

Now, by using the Parseval identity, we may write

$$\|(\Lambda_{\mathbf{s}_2} - \Lambda_{\mathbf{s}_1})X\| = \frac{1}{\sqrt{2\pi}}\left\|(\widehat{\Lambda}_{\mathbf{s}_2} - \widehat{\Lambda}_{\mathbf{s}_1})\widehat{X}\right\| \leq C_3\sqrt{\frac{3}{2\pi}}\,\|X\|\,\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty. \quad (3.31)$$

Thus, there exists a constant $C = C_3\sqrt{\dfrac{3}{2\pi}}$ such that

$$\|\Lambda_{\mathbf{s}_1} - \Lambda_{\mathbf{s}_2}\|_{\mathscr{L}_2} = \sup_{\|X\|=1}\|(\Lambda_{\mathbf{s}_2} - \Lambda_{\mathbf{s}_1})X\| \leq C\,\|\mathbf{s}_2 - \mathbf{s}_1\|_\infty, \quad (3.32)$$

which concludes the proof of Item (iii). □

3.2. Fréchet derivatives and adjoint diffusion problem

Now, we will study the Fréchet derivatives of the DtN. Let $\mathbf{s} = (\nu, D, \sigma_a)$, $h = (h_1, h_2, h_3)$ and $\hat{\mathbf{s}} = (\nu_0, D_0, \sigma_{a_0})$ are available such that $\mathbf{s} = \hat{\mathbf{s}} + h$. Let's consider ϕ and ϕ_0 the analytic solutions of the problem (2.2)-(2.3) with respect to the parameters $\mathbf{s}$ and $\hat{\mathbf{s}}$, respectively, and with the source functions $X(t) = (f(.,t), g_a(t), g_b(t))$. The difference $\psi = \phi - \phi_0$ belongs to $\mathrm{H}_2 \subset \mathrm{H}_1$. Let's consider u and u_0 the exact solutions of the problem (2.8)-(2.9) with respect to the parameters $\mathbf{s}$ and $\hat{\mathbf{s}}$, respectively, and with the source functions $\widehat{X}(\omega) = (F(.,\omega), G_a(\omega), G_b(\omega))$. So, the difference $\mathrm{w} = u - u_0$ belongs to $\widehat{\mathrm{H}}_2 \subset \widehat{\mathrm{H}}_1$. Let us consider $Y = (\varphi, \mathrm{r}_a, \mathrm{r}_b)$ and $\widehat{Y} = (\mathrm{v}, \mathrm{z}_a, \mathrm{z}_b)$ belongs to $\mathbb{B}$ and $\widehat{\mathbb{B}}$ respectively. We consider the non-linear maps $\Upsilon : \mathbf{s} \in (\mathrm{L}^\infty([a,b], \mathbb{R}))^3 \longrightarrow \Upsilon_{\mathbf{s}} \in \mathscr{L}(\mathrm{L}_2 \times \mathbb{R}^2, \mathbb{R}^2)$ and $\widehat{\Upsilon} : \mathbf{s} \in (\mathrm{L}^\infty([a,b], \mathbb{R}))^3 \longrightarrow \widehat{\Upsilon}_{\mathbf{s}} \in \mathscr{L}(\widehat{\mathrm{L}}_2 \times \mathbb{C}^2, \mathbb{C}^2)$ satisfying the duality properties

$$\begin{aligned}\langle (\Upsilon_{\mathbf{s}} h)\, X(t), Y \rangle = \quad & \left(D(a) \frac{\partial \psi}{\partial x}(a, \mathbf{s}, t) + h_2(a) \frac{\partial \phi_0(a,t)}{\partial x} \right) \mathrm{r}_a \\ & - \left(D(b) \frac{\partial \psi}{\partial x}(b, \mathbf{s}, t) + h_2(b) \frac{\partial \phi_0(b,t)}{\partial x} \right) \mathrm{r}_b, \qquad (3.33)\end{aligned}$$

and

$$\begin{aligned}\langle \left(\widehat{\Upsilon}_{\mathbf{s}} h \right) \widehat{X}(\omega), \widehat{Y} \rangle = \quad & \left(D(a) \frac{\partial \mathrm{w}}{\partial x}(a, \mathbf{s}, \omega) + h_2(a) \frac{\partial u_0(a,\omega)}{\partial x} \right) \overline{\mathrm{z}_a} \\ & - \left(D(b) \frac{\partial \mathrm{w}}{\partial x}(b, \mathbf{s}, \omega) + h_2(b) \frac{\partial u_0(b,\omega)}{\partial x} \right) \overline{\mathrm{z}_b}, \qquad (3.34)\end{aligned}$$

respectively. We have

$$\begin{aligned}\langle \left(\widehat{\Upsilon}_{\mathbf{s}} h \right) \widehat{X}(\omega), \widehat{Y} \rangle \quad = \quad & - \int_a^b D(x) \frac{\partial \mathrm{w}}{\partial x}(x,\omega) \overline{\frac{\partial \mathrm{v}}{\partial x}(x)} dx \\ & - \int_a^b \left(\mu_a(x) + \frac{\mathrm{i}\omega}{c_0} \nu(x) \right) \mathrm{w}(x,\omega) \overline{\mathrm{v}(x)} dx \\ & - \int_a^b \left(\frac{\mathrm{i}\omega}{c_0} h_1(x) + h_3(x) \right) u_0(x,\omega) \overline{\mathrm{v}(x)} dx \\ & - \int_a^b h_2(x) \frac{\partial u_0}{\partial x}(x,\omega) \overline{\frac{\partial \mathrm{v}}{\partial x}(x)} dx \\ & + D(b) \mathrm{w}(b,\omega) \overline{\frac{\partial \mathrm{v}}{\partial x}(b)} \\ & - D(a) \mathrm{w}(a,\omega) \overline{\frac{\partial \mathrm{v}}{\partial x}(a)}.\end{aligned}$$

We notice that the function $\mathrm{w} = u - u_0 \in \widehat{\mathrm{H}}_2$ satisfies the non-homogeneous equation

$$\begin{aligned} & -\frac{\partial}{\partial x}\left(D(x)\frac{\partial \mathrm{w}}{\partial x}(x,\omega)\right) + \left(\sigma_a(x) + \frac{\mathrm{i}\omega}{c_0}\nu(x)\right)\mathrm{w}(x,\omega) \\ & = -\frac{\mathrm{i}\omega}{c_0}h_1(x)u_0(x,\omega) + \frac{\partial}{\partial x}\left(h_2(x)\frac{\partial u_0}{\partial x}(x,\omega)\right) - h_3(x)u_0(x,\omega), \end{aligned} \tag{3.35}$$

with the non-homogeneous Robin boundary conditions

$$\begin{aligned} \mathrm{w}(a,\omega) - \frac{\beta_a}{2}D(a)\frac{\partial \mathrm{w}}{\partial x}(a,\omega) &= \frac{\beta_a}{2}h_2(a)\frac{\partial u_0}{\partial x}(a,\omega), \\ \mathrm{w}(b,\omega) + \frac{\beta_b}{2}D(b)\frac{\partial \mathrm{w}}{\partial x}(b,\omega) &= -\frac{\beta_b}{2}h_2(b)\frac{\partial u_0}{\partial x}(b,\omega). \end{aligned} \tag{3.36}$$

Theorem 3.3. *Let ω be a real frequency. Let $X = (f, g_a, g_b)$ be in $\mathrm{H}^{-1}(\mathbb{R}, \mathrm{L}_2) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R}) \times \mathrm{H}^{-1}(\mathbb{R}, \mathbb{R})$, with the notations and assumptions of Proposition 3.1 and Theorem 3.2. There exists a constant $C > 0$ such that*

(i) $\left\| \left(\widehat{\Lambda}_{\mathbf{s}} - \widehat{\Lambda}_{\hat{\mathbf{s}}} - \widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega)\right\|_{\widehat{\mathrm{L}}_2\times\mathbb{C}\times\mathbb{C}} \leq C\|h\|_\infty\sqrt{1+\omega^2}\ \|(u-u_0)(.,\omega)\|_{\widehat{\mathrm{H}}_1}.$

(ii) $\|\Lambda_{\mathbf{s}} - \Lambda_{\hat{\mathbf{s}}} - \Upsilon_{\hat{\mathbf{s}}}h\|_{\mathscr{L}_2} = \frac{1}{\sqrt{2\pi}}\left\|\widehat{\Lambda}_{\mathbf{s}} - \widehat{\Lambda}_{\hat{\mathbf{s}}} - \widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right\|_{\widehat{\mathscr{L}_2}} \leq \frac{C}{\sqrt{2\pi}}\|h\|_\infty^2.$
Furthermore, $\widehat{\Upsilon}_{\hat{\mathbf{s}}} := \widehat{\Lambda}'_{\hat{\mathbf{s}}}$ is the Fréchet derivative at $\hat{\mathbf{s}} = (\nu_0, D_0, \sigma_{a0})$ of $\widehat{\Lambda}_{\mathbf{s}}$ with respect to (ν, D, σ_a).

(iii) *The derivatives $\widehat{\Lambda}'_{\hat{\mathbf{s}}}$ and $\Lambda'_{\hat{\mathbf{s}}}$ have the following Lipschitz-continuous property,*

$$\|\Lambda'_{\mathbf{s}} - \Lambda'_{\hat{\mathbf{s}}}\|_{\mathscr{L}_2} \leq C\sqrt{\frac{2}{\pi}}\ \|\mathbf{s}-\hat{\mathbf{s}}\|_\infty \quad \text{and} \quad \left\|\widehat{\Lambda}'_{\mathbf{s}} - \widehat{\Lambda}'_{\hat{\mathbf{s}}}\right\|_{\widehat{\mathscr{L}_2}} \leq 2C\ \|\mathbf{s}-\hat{\mathbf{s}}\|_\infty.$$

(iv) *If $D(\xi) = D_0(\xi)$, for all ξ in $\{a, b\}$, then the linearization $\widehat{\Lambda}'_{\mathbf{s}}h$ is given by*

$$\left(\widehat{\Lambda}'_{\mathbf{s}}h\right)X(\omega) = \begin{pmatrix} D(a)\frac{\partial \mathrm{w}}{\partial x}(a,\omega) \\ -D(b)\frac{\partial \mathrm{w}}{\partial x}(b,\omega) \end{pmatrix}.$$

where w is the solution of Eq.(3.35)-(3.36).

Proof. Let $\omega \in \mathbb{R}$ be a fixed frequency.

(i) We have

$$\langle\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega),(\mathrm{v},\mathrm{z}_a,\mathrm{z}_b)\rangle=$$
$$-\int_a^b\left(h_3(x)+\frac{\mathrm{i}\omega}{c_0}h_1(x)\right)(u-u_0)(x,\omega)\overline{\mathrm{v}(x)}dx$$
$$-\int_a^b h_2(x)\frac{\partial(u-u_0)}{\partial x}(x,\omega)\overline{\frac{\partial\mathrm{v}}{\partial x}(x)}dx$$
$$+h_2(a)\left(u_0-u-\frac{\beta_a}{2}D_0(a)\frac{\partial u_0}{\partial x}\right)(a,\omega)\overline{\frac{\partial\mathrm{v}}{\partial x}(a)}$$
$$+h_2(b)\left(u-u_0-\frac{\beta_b}{2}D_0(b)\frac{\partial u_0}{\partial x}\right)(b,\omega)\overline{\frac{\partial\mathrm{v}}{\partial x}(b)}.$$

Let $\widetilde{\mathrm{v}}(\xi,\omega)=\frac{\beta_\xi}{2}D_0(\xi)\frac{\partial u_0}{\partial x}(\xi,\omega)$ for $\xi=a$ or b, we use similar arguments as in the proof of Theorem 3.2. We get

$$\left|\langle\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega),\widehat{Y}\rangle\right|\leq\|h_2\|_\infty\left|\left(u-u_0-\widetilde{\mathrm{v}}\right)(b,\omega)\right|\left|\frac{\partial\mathrm{v}}{\partial x}(b)\right|$$
$$+\|h_2\|_\infty\left|\left(u_0-u-\widetilde{\mathrm{v}}\right)(a,\omega)\right|\left|\frac{\partial\mathrm{v}}{\partial x}(a)\right|$$
$$+\|h\|_\infty\sqrt{1+\omega^2}\int_a^b\left|\frac{\partial(u-u_0)}{\partial x}(x,\omega)\right|\left(\left|\frac{\partial\mathrm{v}(x)}{\partial x}\right|+|\mathrm{v}(x)|\right)dx$$
$$+\|h\|_\infty\sqrt{1+\omega^2}\int_a^b|(u-u_0)(x,\omega)|\left(\left|\frac{\partial\mathrm{v}(x)}{\partial x}\right|+|\mathrm{v}(x)|\right)dx.$$

Using the Cauchy-Schwarz inequality and according to (2.11), there exists a constant $C_1>0$ such that

$$\left|\langle\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega),\widehat{Y}\rangle\right|\leq 2\|h\|_\infty\sqrt{1+\omega^2}\,\|(u-u_0)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}$$
$$+C_1\|h\|_\infty\left\|\left(u_0-u-\widetilde{\mathrm{v}}\right)(.,\omega)\right\|_{\widehat{\mathrm{H}}_1}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}$$
$$+C_1\|h\|_\infty\left\|\left(u_0-u-\widetilde{\mathrm{v}}\right)(.,\omega)\right\|_{\widehat{\mathrm{H}}_1}\|\mathrm{v}\|_{\widehat{\mathrm{H}}_1}.$$

According to the inequality (3.26), there exists a constant $C_2>0$ such that

$$\alpha\,\|(u_0-u-\widetilde{\mathrm{v}})(.,\omega)\|_{\widehat{\mathrm{H}}_1}\leq C_2\varrho\sqrt{1+\omega^2}\,\|(u-u_0)(.,\omega)\|_{\widehat{\mathrm{H}}_1}. \tag{3.37}$$

It follows that

$$\left|\langle\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega),\widehat{Y}\rangle\right|\leq C_3\|h\|_\infty f(\omega)\,\|(u-u_0)(.,\omega)\|_{\widehat{\mathrm{H}}_1}\,\|\widehat{Y}\|_{\widehat{\mathrm{L}}_2\times\mathbb{C}^2},$$

where $C_3 = 2\max\{1,\frac{C_1C_2\varrho}{\alpha}\}$ is independent of ω, $\mathbf{s}$ and $\hat{\mathbf{s}}$ and $f(\omega)=\sqrt{1+\omega^2}$. Thus, for $\|\widehat{Y}\|_{\widehat{\mathrm{L}}_2\times\mathbb{C}^2}=1$, we have

$$\left\|\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega)\right\|_{\widehat{\mathrm{L}}_2\times\mathbb{C}^2}\leq C_3\|h\|_\infty\sqrt{1+\omega^2}\,\|(u-u_0)(.,\omega)\|_{\widehat{\mathrm{H}}_1}.$$

(ii) By using the inequality (3.28), we obtain

$$(1+\omega^2)^{-2}\left\|\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}(\omega)\right\|^2_{\widehat{\mathrm{L}}_2\times\mathbb{C}^2} \leq C^2\|h\|^4_\infty\|F(.,\omega)\|^2_{\widehat{\mathrm{L}}_2} + C^2\|h\|^4_\infty|G_a(\omega)|^2 + C^2\|h\|^4_\infty|G_b(\omega)|^2,$$

where $C^2 = 3C_3^2\left(\frac{2C_0C_1}{\alpha^2}\right)^2$. Integrating over $\mathbb{R}$ with respect to ω and using the Parseval identity, we obtain

$$\begin{aligned}\|(\Lambda_{\mathbf{s}}-\Lambda_{\hat{\mathbf{s}}}-\Upsilon_{\hat{\mathbf{s}}}h)\,X\| &= \frac{1}{\sqrt{2\pi}}\left\|\left(\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right)\widehat{X}\right\| \\ &\leq \frac{C}{\sqrt{2\pi}}\|h\|^2_\infty\left(\|f\|^2_{\mathrm{H}^{-1}(\mathbb{R},\mathrm{L}_2)}+g(a,b)\right)^{1/2},\end{aligned}$$

where $g(a,b) = \|g_a\|^2_{\mathrm{H}^{-1}(\mathbb{R},\mathbb{R})} + \|g_b\|^2_{\mathrm{H}^{-1}(\mathbb{R},\mathbb{R})}$.

$$\|\Lambda_{\mathbf{s}}-\Lambda_{\hat{\mathbf{s}}}-\Upsilon_{\hat{\mathbf{s}}}h\|_{\mathscr{L}_2} = \frac{1}{\sqrt{2\pi}}\left\|\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right\|_{\widehat{\mathscr{L}_2}} \leq \frac{C}{\sqrt{2\pi}}\ \|h\|^2_\infty. \tag{3.38}$$

(iii) Set $\mathbf{s}-\hat{\mathbf{s}} = h$, using Item 2, we have

$$\begin{aligned}\|\Lambda'_{\mathbf{s}}h-\Lambda'_{\hat{\mathbf{s}}}h\|_{\mathscr{L}_2} &= \|\Lambda_{\mathbf{s}}-\Lambda_{\hat{\mathbf{s}}}-\Upsilon_{\hat{\mathbf{s}}}h\|_{\mathscr{L}_2}+\|\Lambda_{\hat{\mathbf{s}}}-\Lambda_{\mathbf{s}}-\Upsilon_{\mathbf{s}}h\|_{\mathscr{L}_2} \\ &= \frac{1}{\sqrt{2\pi}}\left\|\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Upsilon}_{\hat{\mathbf{s}}}h\right\|_{\widehat{\mathscr{L}_2}}+\frac{1}{\sqrt{2\pi}}\left\|\widehat{\Lambda}_{\hat{\mathbf{s}}}-\widehat{\Lambda}_{\mathbf{s}}-\widehat{\Upsilon}_{\mathbf{s}}h\right\|_{\widehat{\mathscr{L}_2}} \\ &\leq \frac{C}{\sqrt{2\pi}}\ \|h\|^2_\infty+\frac{C}{\sqrt{2\pi}}\ \|h\|^2_\infty = C\sqrt{\frac{2}{\pi}}\ \|h\|^2_\infty.\end{aligned}$$

It follows that

$$\|\Lambda'_{\mathbf{s}}-\Lambda'_{\hat{\mathbf{s}}}\|_{\mathscr{L}_2} \leq C\sqrt{\frac{2}{\pi}}\ \|\mathbf{s}-\hat{\mathbf{s}}\|_\infty. \tag{3.39}$$

By similar arguments, we prove that

$$\left\|\widehat{\Lambda}'_{\mathbf{s}}-\widehat{\Lambda}'_{\hat{\mathbf{s}}}\right\|_{\widehat{\mathscr{L}_2}} \leq 2C\ \|\mathbf{s}-\hat{\mathbf{s}}\|_\infty. \tag{3.40}$$

(iv) If $D(\xi) = D_0(\xi)$ for all ξ in $\{a,b\}$, then $h_2(\xi) = 0$ and form the Eq.(3.34) come

$$\langle\left(\widehat{\Upsilon}_{\mathbf{s}}h\right)\widehat{X}(\omega),\widehat{\mathrm{Y}}\rangle = D(a)\frac{\partial \mathrm{w}}{\partial x}(a,\mathbf{s},\omega)\overline{\mathrm{z}}_a - D(b)\frac{\partial \mathrm{w}}{\partial x}(b,\mathbf{s},\omega)\overline{\mathrm{z}}_b,$$

for all $\widehat{\mathrm{Y}} = (\mathrm{v},\mathrm{z}_a,\mathrm{z}_b)\in\widehat{\mathbb{B}}$, where w is the solution of Eq.(3.35)-(3.36).

This concludes the proof. □

Theorem 3.4. *Let ω be a fix frequency, $\hat{\mathbf{s}} = (\nu_0, D_0, \mu_{a_0})$ and $\mathbf{s} = \hat{\mathbf{s}} + h$ be two optical parameters where $h = (h_1, h_2, h_3)$ is the vector of perturbations. We assume the conditions*

$$D(a) = D_0(a) \quad \text{and} \quad D(b) = D_0(b), \tag{3.41}$$

hold. Let $(\widehat{\Lambda}'_{\mathbf{s}} h) : \widehat{\mathrm{L}}_2 \times \mathbb{C} \times \mathbb{C} \to \widehat{\mathbb{B}}$ be the Fréchet derivative of the operator of $\widehat{\Lambda}_{\mathbf{s}}$, then the adjoint operator $(\widehat{\Lambda}'_{\hat{\mathbf{s}}} h)^\star$ of $\widehat{\Lambda}'_{\hat{\mathbf{s}}} h$ is given by the following duality product

$$\begin{aligned}\langle \widehat{X}(\omega), \left(\widehat{\Lambda}'_{\hat{\mathbf{s}}} h\right)^\star Y^\star(\omega) \rangle = \int_a^b \left(\frac{\mathrm{i}\omega}{c_0} h_1(x) - h_3(x) \right) u^\star(x,\omega) \overline{u_0(x,\omega)} dx \\ - \int_a^b h_2(x) \frac{\partial u^\star}{\partial x}(x,\omega) \overline{\frac{\partial u_0}{\partial x}(x,\omega)} dx,\end{aligned} \tag{3.42}$$

where $u^\star$ is the unique solution of the adjoint equation:

$$-\frac{\partial}{\partial x}\left(D(x) \frac{\partial u^\star}{\partial x}(x,\omega) \right) + \left(\mu_a(x) - \frac{\mathrm{i}\omega}{c(x)} \right) u^\star(x,\omega) = 0 \quad \textit{for} \quad x \in]a,b[,$$

with the non-homogeneous boundary conditions

$$\begin{aligned} u^\star(a,\omega) - \frac{\beta_a}{2} D(a) \frac{\partial u^\star(a,\omega)}{\partial x} &= \mathrm{z}_a^\star(\omega), \\ u^\star(b,\omega) + \frac{\beta_b}{2} D(b) \frac{\partial u^\star(b,\omega)}{\partial x} &= \mathrm{z}_b^\star(\omega). \end{aligned} \tag{3.43}$$

where $Y^\star(\omega) = (u^\star(.,\omega), \mathrm{z}_a^\star(\omega), \mathrm{z}_b^\star(\omega)) \in \widehat{\mathbb{B}}$. Furthermore, the adjoint of $\widehat{\Lambda}'_{\mathbf{s}} h$ is given by

$$\left(\widehat{\Lambda}'_{\mathbf{s}} h\right)^\star Y^\star(\omega) = \begin{pmatrix} \frac{\mathrm{i}\omega}{c_0} \; u^\star(.,\omega) \overline{u_0(.,\omega)} \\ -\frac{\partial u^\star}{\partial x}(.,\omega) \overline{\frac{\partial u_0}{\partial x}(.,\omega)} \\ -u^\star(.,\omega) \overline{u_0(.,\omega)} \end{pmatrix}. \tag{3.44}$$

Proof. The adjoint of this linearized equation Eq.(3.35) can be derived using the integration by part

$$\begin{aligned} &\int_a^b \left(\frac{\mathrm{i}\omega}{c(x)} \mathrm{w}(x,\omega) + \mu_a(x) \mathrm{w}(x,\omega) - \frac{\partial}{\partial x}\left(D(x) \frac{\partial \mathrm{w}}{\partial x}(x,\omega) \right) \right) \overline{u^\star(x,\omega)} dx - \\ &\int_a^b \mathrm{w}(x,\omega) \overline{\left(-\frac{\mathrm{i}\omega}{c(x)} u^\star(x,\omega) + \mu_a(x) u^\star(x,\omega) - \frac{\partial}{\partial x}\left(D(x) \frac{\partial u^\star}{\partial x}(x,\omega) \right) \right)} dx = \\ &\qquad D(b) \left(\frac{\partial \mathrm{w}}{\partial x}(b,\omega) \overline{u^\star(b,\omega)} - \mathrm{w}(b,\omega) \overline{\frac{\partial u^\star}{\partial x}(b,\omega)} \right) + \\ &\qquad\qquad D(a) \left(\frac{\partial \mathrm{w}}{\partial x}(a,\omega) \overline{u^\star(a,\omega)} - \mathrm{w}(a,\omega) \overline{\frac{\partial u^\star}{\partial x}(a,\omega)} \right) \end{aligned}$$

Using the assumption (3.41), we have $h_2(a) = h_2(b) = 0$, then the equation (3.34) implies

$$\langle\left(\widehat{\Lambda}'_{\mathbf{s}}h\right)\widehat{X}(\omega), \widehat{Y}^\star(\omega)\rangle = D(a)\frac{\partial \mathrm{w}}{\partial x}(a, \mathbf{s}, \omega)\overline{\mathrm{z}_a^\star(\omega)} - D(b)\frac{\partial \mathrm{w}}{\partial x}(b, \mathbf{s}, \omega)\overline{\mathrm{z}_b^\star(\omega)}. \quad (3.45)$$

An immediate consequence of the theorem 3.3 is as follows

$$\begin{aligned}
\langle\left(\widehat{\Lambda}'_{\mathbf{s}}h\right)\widehat{X}(\omega), Y^\star(\omega)\rangle &= -\frac{\mathrm{i}\omega}{c_0}\int_a^b h_1(x)u_0(x,\omega)\overline{u^\star(x,\omega)}dx \\
&\quad -\int_a^b h_2(x)\frac{\partial u_0}{\partial x}(x,\omega)\overline{\frac{\partial u^\star}{\partial x}(x,\omega)}dx \\
&\quad -\int_a^b h_3(x)u_0(x,\omega)\overline{u^\star(x,\omega)}dx \\
&= \overline{\langle\widehat{X}(\omega), (\Lambda'_{\mathbf{s}}h)^\star Y^\star(\omega)\rangle}.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle\widehat{X}(\omega), (\Lambda'_{\mathbf{s}}h)^\star Y^\star(\omega)\rangle &= \int_a^b \left(\frac{\mathrm{i}\omega}{c_0}h_1(x) - h_3(x)\right)u^\star(x,\omega)\overline{u_0(x,\omega)}dx \\
&\quad -\int_a^b h_2(x)\frac{\partial u^\star}{\partial x}(x,\omega)\overline{\frac{\partial u_0}{\partial x}(x,\omega)}dx,
\end{aligned}$$

which achieves the proof. □

4. Computed solution for the inverse problem

4.1. Discretized inverse problem, Tikhonov regularization and Newton method

The inverse problem treated here, consists of an estimation the parameters $\mathbf{s}^* = (\nu^*, D^*, \mu_a^*)$ around an observed optical parameter $\hat{\mathbf{s}} = (\nu_0, D_0, \mu_{a0})$. The nominal valued vector function $\hat{\mathbf{s}}$ is composed by the observed optical parameters ν_0, D_0 and μ_{a0}. We assume that N_s source terms $X_j(.,\omega) = (f_j(.,\omega), g_{aj}(\omega), g_{bj}(\omega))$, for $1 \le j \le N_s$ are available. Consider the cost functional $\mathcal{J}$ given by

$$\mathcal{J} : (\mathrm{L}^\infty([a,b],\mathbb{R}))^3 \longrightarrow [0,+\infty[, \quad \mathcal{J}(\mathbf{s}) = \sum_{j=1}^{N_s}\left\|\widehat{\Lambda}_{\mathbf{s}}(\widehat{X}_j(.,\omega)) - \widehat{\Lambda}_{\hat{\mathbf{s}}}(\widehat{X}_j(.,\omega))\right\|_{\widehat{\mathbb{B}}}^2, \quad (4.1)$$

for all $\mathbf{s} = (\nu, D, \mu_a) \in (\mathrm{L}^\infty([a,b],\mathbb{R}))^3$. For every $1 \le j \le N_s$, we have

$$\widehat{\Lambda}_{\mathbf{s}}(\widehat{X}_j(.,\omega)) - \widehat{\Lambda}_{\hat{\mathbf{s}}}(\widehat{X}_j(.,\omega)) = (\Gamma_{2j-1}(\mathbf{s}), \Gamma_{2j}(\mathbf{s}))^T,$$

where

$$\begin{aligned}
\Gamma_{2j-1}(\mathbf{s}) &= D(a)\frac{\partial u_j}{\partial x}(a,\mathbf{s},\omega) - D_0(a)\frac{\partial u_j}{\partial x}(a,\hat{\mathbf{s}},\omega) \\
&= 2\zeta_a(u_j(a,\mathbf{s},\omega) - u_j(a,\hat{\mathbf{s}},\omega)), \\
\Gamma_{2j}(\mathbf{s}) &= D(b)\frac{\partial u_j}{\partial x}(b,\mathbf{s},\omega) - D_0(b)\frac{\partial u_j}{\partial x}(b,\hat{\mathbf{s}},\omega) = 2\zeta_b(u_j(b,\mathbf{s},\omega) - u_j(b,\hat{\mathbf{s}},\omega)).
\end{aligned}$$

Let $u_j(.,\mathbf{s},.)$ and $u_j(.,\hat{\mathbf{s}},.)$ denote the exact solutions of the problem (2.8)-(2.9) with respect to the sources term $\widehat{X}_j(.,\omega)$ and the parameters $\mathbf{s}$ and $\hat{\mathbf{s}}$, respectively. For $j = 1,\ldots,N_s$, we have

$$\begin{aligned} u_j(a,\mathbf{s},\omega) &= u_j(a,\hat{\mathbf{s}},\omega) + \varepsilon_{aj} \\ u_j(b,\mathbf{s},\omega) &= u_j(b,\hat{\mathbf{s}},\omega) + \varepsilon_{bj} \end{aligned}$$

where ε_{aj} and ε_{bj} are the noise perturbations due to the beam measured at the boundary a and b for a given source term $\widehat{X}_j(.,\omega)$. In general, measurement of $u_j(x,\mathbf{s},\omega)$ may be not possible, only some observable part $\Gamma_{2j-1}(\mathbf{s})$ and $\Gamma_{2j}(\mathbf{s})$ of the actual state $u_j(x,\mathbf{s},\omega)$ may be measured. Also, the cost functional $\mathcal{J}$ is given by

$$\mathcal{J}(\mathbf{s}) = \sum_{l=1}^{2N_s} |\Gamma_l(\mathbf{s})|^2 . \tag{4.2}$$

The objective of the inverse problem is to find the vector parameter $\mathbf{s}_*$ in the domain $[a,b]$, that minimizes the cost functional (4.2) over all possible $\mathbf{s}$ in $[a,b]$ subject to $u_j(x,\mathbf{s},\omega)$ satisfying the diffusion problem (2.8)-(2.9) with the source terms $\widehat{X}_j(.,\omega)$ for all $j = 1,\ldots,N_s$.

So, we consider $\mathbf{s}_*$ as the solution of the following minimization problem

$$\mathbf{s}_* = \arg\left(\min_{\mathbf{s}\in(\mathrm{L}^\infty([a,b],\mathbb{R}))^3} \mathcal{J}(\mathbf{s})\right). \tag{4.3}$$

Now, we consider the new discretization of the interval $[a,b]$ using equally spaced knots $x_i = a + ih'$, for $i = 0,1,2\ldots,n'$ with $x_0 = a, x_{n'} = b$ and $h' = (b-a)/n'$. For $i = -3,\ldots,n'+1$, let us consider a partition of $(n'+5)$ nodes of the interval $[a-3h', b+h']$

$$x_{-3} < x_{-2} < x_{-1} < a = x_0 < x_1 < x_2 < \ldots < x_{n'-1} < b = x_{n'} < x_{n'+1}. \tag{4.4}$$

Consider the following space $\mathbb{S}_{h'}$ given by

$$\mathbb{S}_{h'} = \{\mathbf{s}_{h'} \in \mathcal{C}^2([a,b]):\ \mathbf{s}_{h'}|_{[x_i,x_{i+1}[} \in \mathbb{P}_3,\ \text{ for } 0 \le i \le n'-1\}, \tag{4.5}$$

where $\mathbb{P}_3$ is the space of polynomials of degree ≤ 3. It is well known that the dimension of the subspace $\mathbb{S}_h$ is $\mathrm{m} = n'+3$. We use similar notations by considering the classical cubic Spline as a basis $B_1,\ldots,B_{\mathrm{m}}$ of the subspace $\mathbb{S}_{h'}$. Here, $(B_i)_{1\le i\le \mathrm{m}}$ denote the Splines associated with the given partition. We have

$$B_i(x) = \mathcal{B}_3\left(\frac{x - x_{i-4}}{h'}\right),\ i = 1,\ldots,\mathrm{m}, \tag{4.6}$$

where $x_{i-4} = a + (i-4)h'$, for $i = 1,\ldots,\mathrm{m}$. We use the piecewise approximations for $\mathbf{s} = (\nu, D, \mu_a)$ in the cubic spline basis $(\mathrm{B}_\ell)_{1\le\ell\le\mathrm{m}}$ as follows

$$\nu_{h'}(x) = \sum_{\ell=1}^{\mathrm{m}} \nu_\ell \mathrm{B}_\ell(x),\quad D_{h'}(x) = \sum_{\ell=1}^{\mathrm{m}} d_\ell \mathrm{B}_\ell(x) \ \text{ and }\ \mu_{ah'}(x) = \sum_{\ell=1}^{\mathrm{m}} \mu_{a\ell}\mathrm{B}_\ell(x). \tag{4.7}$$

The functions $\nu_{h'}$, $D_{h'}$ and $\mu_{ah'}$ belong to the space $\mathbb{S}_{h'}$.
Let $\mathbf{s}^{\mathrm{m}} = (\nu_1, \ldots, \nu_{\mathrm{m}}, d_1, \ldots, d_{\mathrm{m}}, \mu_{a1}, \ldots, \mu_{a\mathrm{m}})^T$ be a vector in $\mathbb{R}^{3\mathrm{m}}$ with the components are the coefficients given in the discrete compositions (4.7). Thus, we get a relationship between the continuous and the discrete cases

$$\begin{aligned} \Theta : (\mathrm{L}^{\infty}([a,b], \mathbb{R}))^3 &\longrightarrow \mathbb{S}_{h'} \times \mathbb{S}_{h'} \times \mathbb{S}_{h'} \longrightarrow \mathbb{R}^{3\mathrm{m}} \\ \mathbf{s} &\longmapsto \mathbf{s}_{h'} \longmapsto \Theta(\mathbf{s}) := \mathbf{s}^{\mathrm{m}}. \end{aligned} \tag{4.8}$$

Now, we consider the discrete cost functional $\mathbf{s}^{\mathrm{m}} \to \mathrm{J}(\mathbf{s}^{\mathrm{m}})$ given as follows

$$\mathrm{J} : \mathbb{R}^{3\mathrm{m}} \longrightarrow [0, +\infty[, \quad \mathbf{s}^{\mathrm{m}} \longmapsto \mathrm{J}(\mathbf{s}^{\mathrm{m}}) := \|\mathrm{R}(\mathbf{s}^{\mathrm{m}})\|_2^2 = \sum_{l=1}^{2N_s} |\mathrm{R}_l(\mathbf{s}^{\mathrm{m}})|^2 , \tag{4.9}$$

where $\mathrm{R}(\mathbf{s}^{\mathrm{m}}) = (\mathrm{R}_1(\mathbf{s}^{\mathrm{m}}), \mathrm{R}_2(\mathbf{s}^{\mathrm{m}}), \ldots, \mathrm{R}_{2N_s}(\mathbf{s}^{\mathrm{m}}))^T \in \mathbb{C}^{2N_s}$ is the vector of noises given by

$$\begin{aligned} \mathrm{R}_{2j-1}(\mathbf{s}^{\mathrm{m}}) &:= 2\zeta_a \left(u_{hj}(a, \mathbf{s}^{\mathrm{m}}, \omega) - u_j(a, \hat{\mathbf{s}}^{\mathrm{m}}, \omega)\right), \\ \mathrm{R}_{2j}(\mathbf{s}^{\mathrm{m}}) &:= 2\zeta_b \left(u_{hj}(b, \mathbf{s}^{\mathrm{m}}, \omega) - u_j(b, \hat{\mathbf{s}}^{\mathrm{m}}, \omega)\right), \end{aligned} \tag{4.10}$$

for $j = 1, \ldots, N_s$ and $u_{hj}(., \mathbf{s}^{\mathrm{m}}, .)$ is a solution of the direct problem with respect to the sources term $\widehat{X}_j(., \omega)$, and $u_j(a, \hat{\mathbf{s}}^{\mathrm{m}}, \omega)$ and $u_j(b, \hat{\mathbf{s}}^{\mathrm{m}}, \omega)$ are the data measurements at the boundary a and b for given source terms $\widehat{X}_j(., \omega)$. The norm $\|\mathrm{R}(\mathbf{s}^{\mathrm{m}})\|_2$ is the residual at $\mathbf{s}^{\mathrm{m}}$. The discrete minimization problem associated with the problem (4.3) is

$$\mathbf{s}_*^{\mathrm{m}} = \arg \min_{\mathbf{s}^{\mathrm{m}} \in \mathbb{R}^{3\mathrm{m}}} \mathrm{J}(\mathbf{s}^{\mathrm{m}}). \tag{4.11}$$

The problems (4.3) and (4.11) are ill-posed problems. In order to diminish the effects of the noise in the data, we replace the original problem (4.11) by a better conditioned one. One of the most popular regularization methods is due to Tikhonov [18]. The method replaces the problem (4.11) by the following unconstrained optimization problem

$$\mathbf{s}_*^{\mathrm{m}} = \arg \min_{\mathbf{s}^{\mathrm{m}} \in \mathbb{R}^{3\mathrm{m}}} \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}), \tag{4.12}$$

where the typical cost functional J_λ is

$$\mathbf{s}^{\mathrm{m}} \in \mathbb{R}^{3\mathrm{m}} \longrightarrow \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) := \|\mathrm{R}(\mathbf{s}^{\mathrm{m}})\|_2^2 + \lambda \|\mathbf{s}^{\mathrm{m}} - \hat{\mathbf{s}}^{\mathrm{m}}\|_2^2 . \tag{4.13}$$

and the discrete parameter $\hat{\mathbf{s}}^{\mathrm{m}}$ is a nominal vector. The coefficient $\lambda > 0$ is the regularization parameter to be chosen and $\hat{\mathbf{s}}^{\mathrm{m}}$ is the coefficient of an observed nominal parameter $\hat{\mathbf{s}}$ in the Spline basis functions.

Since the function J_λ is Gâteaux-differentiable and convex it follows that $\mathrm{s}_*^{\mathrm{m}}$ is a solution of the unconstrained optimization problem (4.12) if and only if

$$\nabla \mathrm{J}_\lambda(\mathbf{s}_*^{\mathrm{m}}) = 0. \tag{4.14}$$

The Hessian $\nabla^2 \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}})$ is positive semi-definite for all $\mathbf{s}^{\mathrm{m}} \in \mathbb{R}^{3\mathrm{m}}$. The Newton iterations consist of generating a vector sequence $(\mathbf{s}_k^{\mathrm{m}})_k$ from a guess $\mathbf{s}_0^{\mathrm{m}}$ in a neighborhood of $\mathrm{s}_*^{\mathrm{m}}$ and the next iterations are computed from the scheme

$$\mathbf{s}_{k+1}^{\mathrm{m}} = \mathbf{s}_k^{\mathrm{m}} - \eta_k \left[\nabla^2 \mathrm{J}_{\lambda_k}(\mathbf{s}_k^{\mathrm{m}})\right]^{-1} \nabla \mathrm{J}_{\lambda_k}(\mathbf{s}_k^{\mathrm{m}}), \quad k = 0, 1, 2 \ldots, k_{max},$$

where η_k is backtracking strategy coefficient. Let $\widetilde{d}_k$ be the decent direction, $\widetilde{d}_k$ is a solution of the system

$$\left[\nabla^2 \mathrm{J}_{\lambda_k}(\mathbf{s}_k^{\mathrm{m}})\right] \widetilde{d}_k = -\nabla \mathrm{J}_{\lambda_k}(\mathbf{s}_k^{\mathrm{m}}). \tag{4.15}$$

We get the following iterations scheme

$$\mathbf{s}_{k+1}^{\mathrm{m}} = \mathbf{s}_k^{\mathrm{m}} + \eta_k \widetilde{d}_k, \quad k = 0, 1, 2 \ldots, k_{max}.$$

4.2. Computation of the Jacobian and the Hessian

Taking into account the expressions in (4.7), then $\forall\, (i,j) \in \{1, \ldots, N\}^2$, the coefficients $A_{ij}(\omega)$ of the matrix $A(\omega)$ may be written in the following form

$$\begin{aligned} A_{ij}(\omega) \;=\; & \frac{1}{h}\sum_{\ell=1}^{\mathrm{m}} d_\ell \mathrm{D}_{ij}^{(\ell)} + h \sum_{\ell=1}^{\mathrm{m}} \mu_{a\ell} \mathrm{U}_{ij}^{(\ell)} \\ & + \mathrm{i}\frac{\omega h}{c_0}\sum_{\ell=1}^{\mathrm{m}} \nu_\ell \mathrm{U}_{ij}^{(\ell)} + 2\zeta_a \mathcal{B}_3(-i+4)\,\mathcal{B}_3(-j+4) && (4.16) \\ & + 2\zeta_b \mathcal{B}_3(n-i+4)\,\mathcal{B}_3(n-j+4), && (4.17) \end{aligned}$$

where $\mathrm{D}_{ij}^{(\ell)}$ and $\mathrm{U}_{ij}^{(\ell)}$ are given as

$$\begin{aligned} \mathrm{D}_{ij}^{(\ell)} &= \int_0^n \mathcal{B}_3\left(\frac{h'}{h}(\tau - \ell + 4)\right) \mathcal{B}_3'(\tau - i + 4)\mathcal{B}_3'(\tau - j + 4)d\tau, \\ \mathrm{U}_{ij}^{(\ell)} &= \int_0^n \mathcal{B}_3\left(\frac{h'}{h}(\tau - \ell + 4)\right) \mathcal{B}_3(\tau - i + 4)\mathcal{B}_3(\tau - j + 4)d\tau, \end{aligned} \tag{4.18}$$

respectively. Let D_ℓ and U_ℓ be the matrices of size $\mathrm{m} \times \mathrm{m}$ with coefficients $\mathrm{D}_{ij}^{(\ell)}$ and $\mathrm{U}_{ij}^{(\ell)}$ respectively. Thus,

$$\mathrm{A}(\omega) = \frac{1}{h}\sum_{\ell=1}^{\mathrm{m}} d_\ell \mathrm{D}_\ell + h\sum_{\ell=1}^{\mathrm{m}} \mu_{a_\ell}\mathrm{U}_\ell + \mathrm{i}\frac{\omega h}{c_0}\sum_{\ell=1}^{\mathrm{m}} \nu_\ell \mathrm{U}_\ell + 2\mathrm{Q}, \tag{4.19}$$

where Q is a matrix with coefficient

$$\mathrm{Q}_{ij} = \zeta_a \mathcal{B}_3(-i+4)\,\mathcal{B}_3(-j+4) + \zeta_b \mathcal{B}_3(n-i+4)\,\mathcal{B}_3(n-j+4).$$

Let $\mathbf{s}^{\mathrm{m}} = (s_1, \ldots, \ldots, s_{3\mathrm{m}})^T$ be a generic vector in $\mathbb{R}^{3\mathrm{m}}$. The first derivatives of (2.21) with respect to the coefficients s_ℓ are given as follows

$$\mathrm{A}(\omega)\frac{\partial \mathrm{z}_h(\omega)}{\partial s_\ell} + \frac{\partial \mathrm{A}(\omega)}{\partial s_\ell}\mathrm{z}_h(\omega) = \frac{\partial \mathrm{b}_h(\omega)}{\partial s_\ell}, \quad \ell = 1, \ldots, 3\mathrm{m},$$

and, the second derivatives are given for all $\ell, \ell' = 1, \ldots, 3\mathrm{m}$ as

$$\frac{\partial \mathrm{A}(\omega)}{\partial s_{\ell'}} \frac{\partial \mathrm{z}_h(\omega)}{\partial s_\ell} + \mathrm{A}(\omega) \frac{\partial^2 \mathrm{z}_h(\omega)}{\partial s_{\ell'} \partial s_\ell} + \frac{\partial \mathrm{A}(\omega)}{\partial s_\ell} \frac{\partial \mathrm{z}_h(\omega)}{\partial s_{\ell'}} + \frac{\partial^2 \mathrm{A}(\omega)}{\partial s_{\ell'} \partial s_\ell} \mathrm{z}_h(\omega) = \frac{\partial^2 \mathrm{b}_h(\omega)}{\partial s_{\ell'} \partial s_\ell}.$$

Since $\mathrm{b}_h(\omega)$ is independent of $\mathbf{s}$, then we have $\dfrac{\partial \mathrm{b}_h(\omega)}{\partial s_\ell} = 0$ and $\dfrac{\partial^2 \mathrm{b}_h(\omega)}{\partial s_{\ell'} \partial s_\ell} = 0$. It follows that

$$\frac{\partial \mathrm{z}_h(\omega)}{\partial s_\ell} = -\mathrm{A}^{-1}(\omega) \frac{\partial \mathrm{A}(\omega)}{\partial s_\ell} \mathrm{z}_h(\omega), \tag{4.20}$$

and

$$\frac{\partial^2 \mathrm{z}_h(\omega)}{\partial s_{\ell'} \partial s_\ell} = -\mathrm{A}^{-1}(\omega) \left[\frac{\partial \mathrm{A}(\omega)}{\partial s_{\ell'}} \mathrm{A}^{-1}(\omega) \frac{\partial \mathrm{A}(\omega)}{\partial s_\ell} + \frac{\partial \mathrm{A}(\omega)}{\partial s_\ell} \mathrm{A}^{-1}(\omega) \frac{\partial \mathrm{A}(\omega)}{\partial s_{\ell'}} \right] \mathrm{z}_h(\omega). \tag{4.21}$$

By using the relations (4.19), we compute the derivatives $\dfrac{\partial \mathrm{A}(\omega)}{\partial s_\ell}$ for all $1 \leq \ell \leq 3\mathrm{m}$, from the following expressions

$$\frac{\partial \mathrm{A}(\omega)}{\partial s_\ell} = \begin{cases} \mathrm{i} \dfrac{\omega h}{c_0} \mathrm{U}_\ell \ , & 1 \leq \ell \leq \mathrm{m}, \\ \dfrac{1}{h} \mathrm{D}_{\ell - \mathrm{m}} \ , & \mathrm{m} + 1 \leq \ell \leq 2\mathrm{m}, \\ h \mathrm{U}_{\ell - 2\mathrm{m}} \ , & 2\mathrm{m} + 1 \leq \ell \leq 3\mathrm{m}, \end{cases} \tag{4.22}$$

By using the (4.22) in (4.20), we obtain

$$\frac{\partial \mathrm{z}_h(\omega)}{\partial s_\ell} = \begin{cases} -\mathrm{i} \dfrac{h\omega}{c_0} \mathrm{A}^{-1}(\omega) \mathrm{U}_\ell \mathrm{z}_h(\omega) \ , & 1 \leq \ell \leq \mathrm{m}, \\ -\dfrac{1}{h} \mathrm{A}^{-1}(\omega) \mathrm{D}_{\ell - \mathrm{m}} \mathrm{z}_h(\omega) \ , & \mathrm{m} + 1 \leq \ell \leq 2\mathrm{m}, \\ -h \mathrm{A}^{-1}(\omega) \mathrm{U}_{\ell - 2\mathrm{m}} \mathrm{z}_h(\omega) \ , & 2\mathrm{m} + 1 \leq \ell \leq 3\mathrm{m}. \end{cases} \tag{4.23}$$

For $j = 1, \ldots, N_s$, let $X_j(., t) = (f_j(., t), g_{aj}(t), g_{bj}(t))$ be the j^{th} sources term, then the corresponding solution $u_{hj}(., \mathbf{s}^{\mathrm{m}}, \omega)$ is given by

$$u_{hj}(x, \mathbf{s}^{\mathrm{m}}, \omega) = \sum_{k=1}^{N} \mathrm{z}_{h,k}^{(j)}(\omega) B_k(x), \quad \forall x \in [a, b], \ j = 1, \ldots, N_s.$$

By using the Fréchet derivative, we compute the Jacobian matrix $\mathrm{K}(\mathbf{s}^{\mathrm{m}})$ of the vector valued function $\mathrm{R}(\mathbf{s}^{\mathrm{m}})$ as the $2N_s \times 3\mathrm{m}$ matrix with the entries are given by

$$\mathrm{K}_{j,\ell}(\mathbf{s}^{\mathrm{m}}) \quad = \quad \frac{\partial \mathrm{R}_j(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell}, \quad \ell = 1, ..., 3\mathrm{m} \quad \text{and} \quad j = 1, \ldots, 2N_s. \tag{4.24}$$

Denote the vector $\mu(x)$ by

$$\sigma(x) = (B_1(x), B_2(x), \ldots, B_N(x))^T \ , \quad \forall x \in [a, b].$$

Using the relations given in (3.12) and (3.13), we obtain

$$\left.\begin{array}{l} \mathrm{R}_{2j-1}(\mathbf{s}^{\mathrm{m}}) = 2\zeta_a\left(u_{hj}(a,\mathbf{s}^{\mathrm{m}},\omega) - u_j(a,\hat{\mathbf{s}}^{\mathrm{m}},\omega)\right) \\ \mathrm{R}_{2j}(\mathbf{s}^{\mathrm{m}}) = 2\zeta_b\left(u_{hj}(b,\mathbf{s}^{\mathrm{m}},\omega) - u_j(b,\hat{\mathbf{s}}^{\mathrm{m}},\omega)\right) \end{array}\right., \quad j = 1,\ldots,N_s \tag{4.25}$$

Since $u_{hj}(a,\mathbf{s}^{\mathrm{m}},\omega) = \sum_{k=1}^{N} \mathrm{z}_{h,k}^{(j)}(\omega)B_k(a)$ and $u_{hj}(b,\mathbf{s}^{\mathrm{m}},\omega) = \sum_{k=1}^{N} \mathrm{z}_{h,k}^{(j)}(\omega)B_k(b)$ then we have

$$\begin{array}{l} \frac{\partial \mathrm{R}_{2j-1}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = 2\zeta_a \frac{\partial u_{hj}}{\partial s_\ell}(a,\mathbf{s}^{\mathrm{m}},\omega) = 2\zeta_a \sum_{k=1}^{N} \frac{\partial \mathrm{z}_{h,k}^{(j)}(\omega)}{\partial s_\ell} B_k(a), \\ \frac{\partial \mathrm{R}_{2j}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = 2\zeta_b \frac{\partial u_{hj}}{\partial s_\ell}(b,\mathbf{s}^{\mathrm{m}},\omega) = 2\zeta_b \sum_{k=1}^{N} \frac{\partial \mathrm{z}_{h,k}^{(j)}(\omega)}{\partial s_\ell} B_k(b), \end{array} \quad j = 1,\ldots,N_s. \tag{4.26}$$

Now, using the explicit expression (2.19) of $u_{hj}(a,\mathbf{s}^{\mathrm{m}},\omega)$ and $u_{hj}(b,\mathbf{s}^{\mathrm{m}},\omega)$ in the equations (4.26), we get

$$\left.\begin{array}{l} \frac{\partial \mathrm{R}_{2j-1}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = 2\zeta_a \sigma(a) \frac{\partial \mathrm{z}_h^{(j)}(\omega)}{\partial s_\ell} \\ \frac{\partial \mathrm{R}_{2j}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = 2\zeta_b \sigma(b) \frac{\partial \mathrm{z}_h^{(j)}(\omega)}{\partial s_\ell} \end{array}\right., \quad j = 1,\ldots,N_s. \tag{4.27}$$

Therefore, using the equations (4.23) for the expression $\frac{\partial \mathrm{z}_h^{(j)}(\omega)}{\partial s_\ell}$ in (4.27) we compute the Jacobian matrix $\mathrm{K}(\mathbf{s}^{\mathrm{m}})$ for our calculations. And, for $j = 1,\ldots,N_s$ we have

$$\frac{\partial \mathrm{R}_{2j-1}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = \begin{cases} -2\zeta_a \mathrm{i}\frac{h\omega}{c_0}\sigma(a)\mathrm{A}^{-1}(\omega)\mathrm{U}_\ell \mathrm{z}_h^{(j)}(\omega), & 1 \le \ell \le \mathrm{m}, \\ -2\zeta_a \frac{1}{h}\sigma(a)\mathrm{A}^{-1}(\omega)\mathrm{D}_{\ell-\mathrm{m}}\mathrm{z}_h^{(j)}(\omega), & \mathrm{m}+1 \le \ell \le 2\mathrm{m}, \\ -2\zeta_a h\sigma(a)\mathrm{A}^{-1}(\omega)\mathrm{U}_{\ell-2\mathrm{m}}\mathrm{z}_h^{(j)}(\omega), & 2\mathrm{m}+1 \le \ell \le 3\mathrm{m} \end{cases} \tag{4.28}$$

and

$$\frac{\partial \mathrm{R}_{2j}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell} = \begin{cases} -2\zeta_b \mathrm{i}\frac{h\omega}{c_0}\sigma(b)\mathrm{A}^{-1}(\omega)\mathrm{U}_\ell \mathrm{z}_h^{(j)}(\omega), & 1 \le \ell \le \mathrm{m}, \\ -2\zeta_b \frac{1}{h}\sigma(b)\mathrm{A}^{-1}(\omega)\mathrm{D}_{\ell-\mathrm{m}}\mathrm{z}_h^{(j)}(\omega), & \mathrm{m}+1 \le \ell \le 2\mathrm{m}, \\ -2\zeta_b h\sigma(b)\mathrm{A}^{-1}(\omega)\mathrm{U}_{\ell-2\mathrm{m}}\mathrm{z}_h^{(j)}(\omega), & 2\mathrm{m}+1 \le \ell \le 3\mathrm{m}. \end{cases} \tag{4.29}$$

The gradients $\nabla \mathrm{J}(\mathbf{s}^{\mathrm{m}}) = \left(\frac{\partial \mathrm{J}(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell}\right)^T_{1\le\ell\le 3\mathrm{m}}$ and $\nabla \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) = \left(\frac{\partial \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell}\right)^T_{1\le\ell\le 3\mathrm{m}}$ of the cost functions J and J_λ, are given for all $\mathbf{s}^{\mathrm{m}}$ by

$$\nabla \mathrm{J}(\mathbf{s}^{\mathrm{m}}) = 2\mathrm{K}^*(\mathbf{s}^{\mathrm{m}})\mathrm{R}(\mathbf{s}^{\mathrm{m}}) \quad \text{and} \quad \nabla \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) = 2\mathrm{K}^*(\mathbf{s}^{\mathrm{m}})\mathrm{R}(\mathbf{s}^{\mathrm{m}}) + 2\lambda(\mathbf{s}^{\mathrm{m}} - \hat{\mathbf{s}}^{\mathrm{m}}), \tag{4.30}$$

respectively.

Let $\mathrm{H}(\mathbf{s}^{\mathrm{m}}) = \mathrm{K}^*(\mathbf{s}^{\mathrm{m}})\mathrm{K}(\mathbf{s}^{\mathrm{m}}) + \mathrm{E}(\mathbf{s}^{\mathrm{m}})$ where $\mathrm{E}_{\ell\ell'}(\mathbf{s}^{\mathrm{m}}) = \sum_{i=1}^{2N_s} \mathrm{R}_i(\mathbf{s}^{\mathrm{m}}) \frac{\partial^2 \mathrm{R}_i(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell \partial s_{\ell'}}$, $1 \le \ell, \ell' \le 3\mathrm{m}$. It is obvious that there exists a constant $C > 0$ such that for all $\mathbf{s}^{\mathrm{m}} \in \mathbb{R}^{3\mathrm{m}}$ and for all $\widetilde{h} \in \mathbb{R}^{3\mathrm{m}}$ such that

$$\left\| \nabla \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}} + \widetilde{h}) - \nabla \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) - 2\left(\mathrm{H}(\mathbf{s}^{\mathrm{m}}) + \lambda \mathrm{I}_{3\mathrm{m}}\right)\widetilde{h} \right\| \le C \|\widetilde{h}\|^2_{\mathbb{R}^{3\mathrm{m}}}.$$

Then, the Hessian matrix $\nabla^2 \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) = \left(\frac{\partial^2 \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}})}{\partial s_\ell \partial s_{\ell'}} \right)_{1 \le \ell, \ell' \le 3\mathrm{m}}$ is given for all $\mathbf{s}^{\mathrm{m}}$ by

$$\nabla^2 \mathrm{J}_\lambda(\mathbf{s}^{\mathrm{m}}) = 2\left(\mathrm{H}(\mathbf{s}^{\mathrm{m}}) + \lambda \mathrm{I}_{3\mathrm{m}}\right) = 2\mathrm{K}^*(\mathbf{s}^{\mathrm{m}})\mathrm{K}(\mathbf{s}^{\mathrm{m}}) + 2\mathrm{E}(\mathbf{s}^{\mathrm{m}}) + 2\lambda \mathrm{I}_{3\mathrm{m}}. \tag{4.31}$$

Since $\mathrm{E}(\mathbf{s}^{\mathrm{m}}_k) \simeq 0$, then the Hessian $\nabla^2 \mathrm{J}_{\lambda_k}(\mathbf{s}^{\mathrm{m}}_k)$ is approximated by

$$\nabla^2 \mathrm{J}_{\lambda_k}(\mathbf{s}^{\mathrm{m}}_k) \simeq 2\mathrm{K}^*(\mathbf{s}^{\mathrm{m}}_k)\mathrm{K}(\mathbf{s}^{\mathrm{m}}_k) + 2\lambda_k \mathrm{I}_{3\mathrm{m}}.$$

Hence, the linear systems (4.15) may be written in the following form

$$\left(\mathrm{H}(\mathbf{s}^{\mathrm{m}}_k) + \lambda_k \mathrm{I}_{3\mathrm{m}}\right)\widetilde{d}_k = -\left(\mathrm{K}^*(\mathbf{s}^{\mathrm{m}}_k)\mathrm{R}(\mathbf{s}^{\mathrm{m}}_k) + \lambda_k(\mathbf{s}^{\mathrm{m}}_k - \hat{\mathbf{s}}^{\mathrm{m}})\right), \ k = 1, 2, \ldots, k_{max}.$$

And come the following iterations

$$\mathbf{s}^{\mathrm{m}}_{k+1} = \mathbf{s}^{\mathrm{m}}_k - \eta_k \left(\mathrm{H}(\mathbf{s}^{\mathrm{m}}_k) + \lambda_k \mathrm{I}_{3\mathrm{m}}\right)^{-1} \left(\mathrm{K}^*(\mathbf{s}^{\mathrm{m}}_k)\mathrm{R}(\mathbf{s}^{\mathrm{m}}_k) + \lambda_k(\mathbf{s}^{\mathrm{m}}_k - \hat{\mathbf{s}}^{\mathrm{m}})\right), \tag{4.32}$$

for $k = 0, 1, 2 \ldots, k_{max}$, where η_k is a backtracking small positif number.

5. Numerical results and applications

In this section, we give numerical examples to illustrate the effectiveness of our proposed method. All computations were carried out using Matlab 6.5 on an Intel Pentium workstation with about 16 significant decimal digits. Let $[a, b]$ be the computational domain and $\omega = \frac{2\pi}{\upsilon}$ be a modulation frequency parameter where υ is the wavelength number of laser light in the wavelength range between 650 and 1000 nm. Let $\delta_{-t_s} : t \mapsto \delta(t - t_s)$ be a Dirac measure function defined for all time variable t in $[0, T]$ where t_s is a given time center in $[0, T]$. The Fourier transform of δ_{-t_s} considered as a time forcing distribution source is given by (for more detail see [16]):

$$\widehat{\delta}_\omega = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i} t_s \omega}. \tag{5.1}$$

5.1. Verification tests for the forward solver

To test our forward solver, we first computed the approximate solutions in two cases:

Example 5.1. *Let $c = 1$, $\mu_a = 1$, $\mu_s = 2$ and $\vartheta \in [0,1]$, then $D = \frac{1}{3-2\vartheta}$. For a given time center t_s and the source terms $F(x,\omega)$, $G_a(\omega)$ and $G_b(\omega)$, the analytic solution $u_e(x,\omega)$ of the diffusion problem in frequency domain is*

$$u_e(x,\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\mathrm{i}t_s\omega - \frac{\omega^2}{2}\right) x^2 e^{-x}, \quad \forall x \in]0,6[.$$

We consider the relative L^p-norm error function defined as

$$\mathrm{L}^p - error = \frac{\|u_e(.,\omega) - u_h(.,\omega)\|_{\mathrm{L}^p}}{\|u_e(.,\omega)\|_{\mathrm{L}^p}}, \quad for \quad p = 1,2,\infty$$

where $\|\cdot\|_{\mathrm{L}^p}$ is the L^p-norm, u_h and u_e are respectively, the computed and analytic solutions. First we check the accuracy of the direct solver with respect to the number of gridpoints used in the computational domain. To this end we summarize in Table 1 the L^∞-error, L^1-error and L^2-error norms at time $t_s = 0.5$ using different values of N. It is evident that increasing the number of gridpoints in the computational domain results in a decrease in all error norms along with an increase in the computational cost. It should be stressed that the main part of the CPU times listed in Table 1 are used by the direct solver for solving the associated linear systems Eq.(2.21).

Table 1: Error norms for Example 1 using different numbers of gridpoints and the wavelength number $\upsilon = 750$, the anisotropic coefficient $\vartheta = 0.9$ and $t_s = 0.5$.

N	L^∞-error	L^1-error	L^2-error	CPU (in second)
5	1.89E-02	6.89E-03	9.93E-03	4.93E-03
10	1.61E-03	2.90E-04	5.87E-04	1.24E-02
20	1.09E-04	1.89E-05	3.72E-05	1.31E-02
40	1.01E-05	5.88E-06	5.49E-06	1.43E-02
80	3.91E-06	5.76E-06	5.15E-06	1.67E-02

Example 5.2. *Let $\nu = 1.5$, $\mu_a = 0.05$, $\mu_s = 3.75$ and $\vartheta = 0.8$, then $D = 5$. For a given time center t_s and the source terms*

$$F(x,\omega) = 0, \quad \forall x \in]a,b[\quad and \quad G_\xi(\omega) = \widehat{\delta}_\omega\, \delta(\xi - x_s), \quad \forall \xi \in \{a,b\},$$

where $\xi \mapsto \delta(\xi - x_s)$ is a Dirac's measure function at the boundary $\xi \in \{a,b\}$, $x_s \in]a,b[$ is a center position of the source and $\widehat{\delta}_\omega$ is the Fourier transform of Dirac function $t \mapsto \delta(t - t_s)$ given in Eq.(5.1).

Table 2: Error norms for Example 2 using different numbers of gridpoints and the wavelength number $\upsilon = 650$, $\vartheta = 0.8$, $t_s = 0.5$, $\zeta_a = \zeta_b = 0.5$, $[a,b] = [0,20]$, $x_s = (b+a)/2$, $t_s = 0.5$ and $c_0 = 3000$.

N	L^{∞}-error	L^1-error	L^2-error	CPU (in second)
10	1.82E-06	1.14E-06	1.32E-06	1.12E-02
20	1.29E-07	7.29E-08	8.52E-08	1.21E-02
40	7.63E-09	4.78E-09	5.47E-09	1.32E-02
80	4.70E-10	2.98E-10	3.43E-10	1.56E-02

In the example 2, we check the accuracy of the direct solver with respect to the number of gridpoints used in the computational domain. To this end, we summarize in Table 2 the L^{∞}*-error,* L^1*-error and* L^2*-error norms using different values of* N*. It is evident that increasing the number of gridpoints in the computational domain results in a decrease in all error norms along with an increase in the computational cost. It should be stressed that the main part of the CPU times listed in Table 2 are used by the direct solver for solving the associated linear systems Eq.(2.21).*

5.2. Numerical experiment for the inverse problem solver

To test our inverse problem solver, we compute the approximate solutions in two cases:

Example 5.3. *We consider the diffusion problem in frequency domain (2.8)-(2.9) with the source terms given, for* $j = 1, \dots, N_s$, $x_s^{(j)} = a + \frac{j}{N_s}(b-a)$ *and* $t_s^{(j)} = \frac{j}{N_s}T$*, by*

$$F_j(x,\omega) = 0, \quad \forall x \in]a,b[\quad \text{and} \quad G_{\xi j}(\omega) = \frac{A_o}{\sqrt{2\pi}}\, e^{-\mathrm{i}t_s^{(j)}\omega}\delta(\xi - x_s^{(j)}), \quad \forall \xi \in \{a,b\},$$

where $\xi \mapsto \delta(\xi - x_s^{(j)})$ *is a Dirac's measure function at the boundary* $\xi \in \{a,b\}$, $x_s^{(j)} \in]a,b[$ *is a center position of the* j^{th} *source and* A_o *is an amplitude of the sources terms. The flow measurements at the boundary of computational domain are computed for* $j = 1, \dots, N_s$ *as follows*

$$\Gamma_{2j-1}(\omega) = D(a)\frac{\partial u_j}{\partial x}(a,\mathbf{s},\omega) = 2\zeta_a\left(u_j(a,\mathbf{s},\omega) - G_{aj}(\omega)\right),$$
$$\Gamma_{2j}(\omega) = -D(b)\frac{\partial u_j}{\partial x}(b,\mathbf{s},\omega) = 2\zeta_b\left(u_j(b,\mathbf{s},\omega) - G_{bj}(\omega)\right).$$

In this example, we fix a source functions and detectors at the boundary position. In the following computations, we use a Newton scheme (4.32) with the initial estimation $\mathbf{s}_0 = (1,\dots,1) \in \mathbb{R}^{3(n'+3)\times 1}$*, a tolerance (*$Tol = 10^{-8}$*), a number of iterations* $k_{max} = 20$*, a backtracking number* $\eta = 0.5$ *and Tikhonov regularization parameter* $\lambda = 10^{-6}$*. In the left-hand side of Figure 1, we present respectively,* *the simulation plots of the diffusion coefficient* $x \mapsto D(x)$*, the absorption coefficient* $x \mapsto \mu_a(x)$ *and the refractive index coefficient* $x \mapsto \nu(x)$ *for* $n = n' = 20$*. And, in the right-hand side of Figure 1, we give respectively, the simulations plots of the diffusion coefficient* $x \mapsto D(x)$*, the absorption coefficient* $x \mapsto \mu_a(x)$ *and the refractive index coefficient* $x \mapsto \nu(x)$ *for* $n = n' = 160$*. The line corresponds to truth optical parameters* (D, μ_a, ν) *and the star corresponds to the constructed optical parameters* (D, μ_a, ν)*. The comparison of the left-hand side of Figure 1 and the right-hand side of Figure 1 demonstrates the accurate convergence of the presented method, along with increasing the number of gridpoints in the computational domain. Therefore, we have provided and investigated the theoretical convergence and implementation of our algorithm with accurate reconstruction of the optical parameters.*

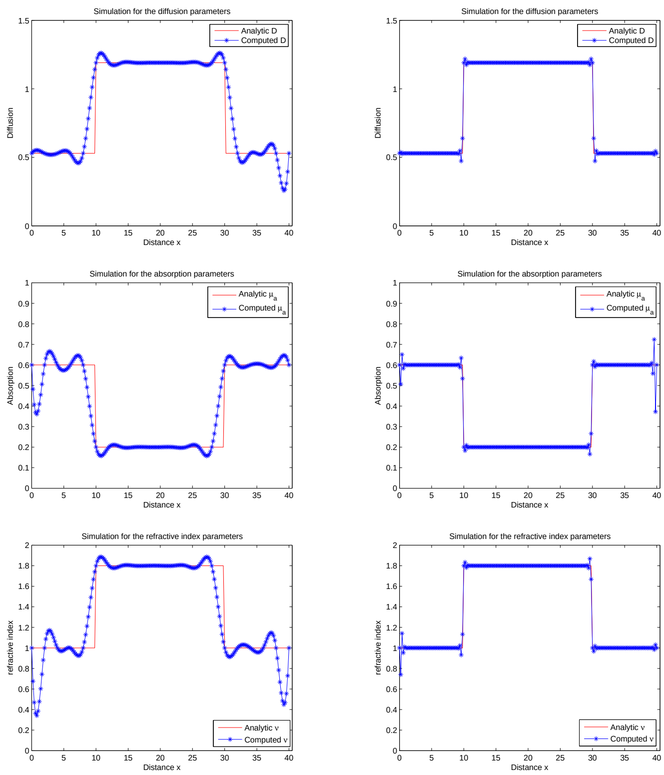

Figure 1: Example with the parameters: $a = 0$, $b = 40$, $\lambda = 10^{-6}$, $\eta = 0.5$, $\omega = 100$, $\zeta_a = \zeta_b = 1$, $c_0 = 3000$, $\vartheta = 0.9$, $A_o = 1$, $t_s = T/2$, $T = 1$, $Tol = 10^{-8}$, $k_{max} = 20$, $N_s = 2$.

Example 5.4. *In this example, we consider the diffusion problem in frequency domain (2.8)-(2.9) with the source terms given for* $j = 1, \ldots, N_s$ *by* $G_{aj}(\omega) = 0$,

$G_{bj}(\omega) = 0$ *and*

$$F_j(x,\omega) = \frac{A_o}{\sqrt{2\pi}}\, e^{-\mathrm{i}\, t_s^{(j)}\omega}\delta(x - x_s^{(j)}), \quad \forall x \in]a,b[,$$

where A_o *is an amplitude of the source terms. We modeled the sources with* $(x_s, t_s) = \left(\frac{15a+b}{16}, \frac{T}{16}\right)$, $\left(\frac{7a+b}{8}, \frac{T}{8}\right)$, $\left(\frac{3a+b}{4}, \frac{T}{4}\right)$, $\left(\frac{b+a}{2}, \frac{T}{2}\right)$, $\left(\frac{3b+a}{4}, \frac{3T}{4}\right)$, $\left(\frac{7b+a}{8}, \frac{7T}{8}\right)$ *and* $\left(\frac{15b+a}{16}, \frac{15T}{16}\right)$. *The flow measurements at the boundary of our computational domain are computed for* $j = 1, \ldots, N_s - 1$ *as follows*

$$\Gamma_{2j-1}(\mathbf{s}) = D(a)\frac{\partial u_j}{\partial x}(a,\mathbf{s},\omega) = 2\zeta_a u_j(a,\mathbf{s},\omega),$$
$$\Gamma_{2j}(\mathbf{s}) = -D(b)\frac{\partial u_j}{\partial x}(b,\mathbf{s},\omega) = 2\zeta_b u_j(b,\mathbf{s},\omega)$$

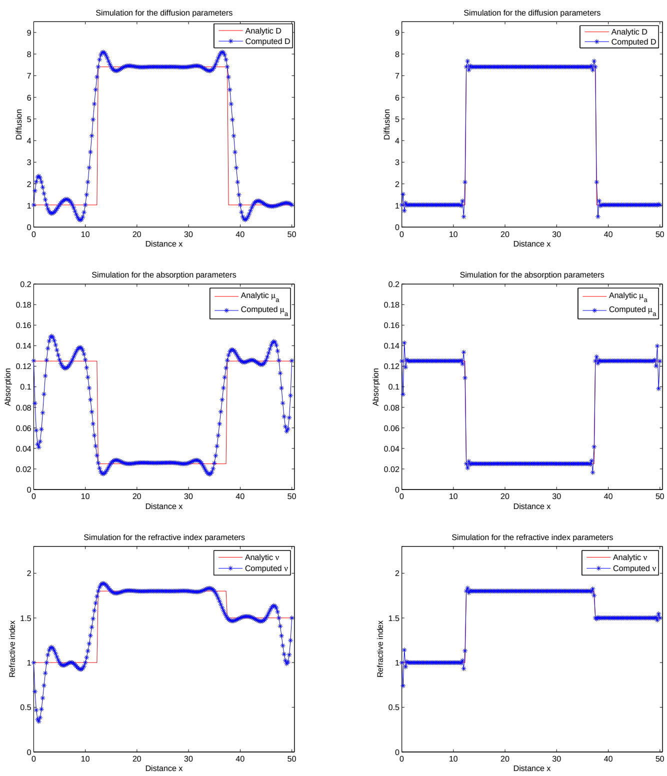

Figure 2: Example with the parameters: $a = 0$, $b = 50$, $c_0 = 3000$, $\zeta_1 = \zeta_2 = 1$, $\vartheta = 0.95$, $\omega = 100$, $N_s = 7$, $T = 1$, $Tol = 10^{-10}$, $k_{max} = 20$, $\lambda = 10^{-2}$ and $A_o = 1$.

In the following computations, we use a Newton scheme (4.32) with the initial estimation $\mathbf{s}_0 = (1, \ldots, 1) \in \mathbb{R}^{3(n'+3)\times 1}$, *a tolerance* ($Tol = 10^{-10}$), *a number of* *iterations* $k_{max} = 20$*, a backtracking number* $\eta = 0.5$ *and Tikhonov regularization parameter* $\lambda = 10^{-2}$*. In the left-hand side of Figure 2, we present respectively, the simulation plots of the diffusion coefficient* $x \mapsto D(x)$*, the absorption coefficient* $x \mapsto \mu_a(x)$ *and the refractive index coefficient* $x \mapsto \nu(x)$*, for* $n = n' = 20$*. And, in the right-hand side of Figure 2, we give respectively, the simulation plots of the diffusion coefficient* $x \mapsto D(x)$*, the absorption coefficient* $x \mapsto \mu_a(x)$ *and the refractive index coefficient* $x \mapsto \nu(x)$*, for* $n = n' = 160$*. The line corresponds to truth optical parameters* (D, μ_a, ν) *and the star corresponds to the constructed optical parameters* (D, μ_a, ν)*. The comparison of the left-hand side of Figure 2 and the right-hand side of Figure 2 demonstrates the accurate convergence of the presented method, along with an increasing the number of gridpoints in the computational domain. Therefore, we have provided and investigated the theoretical convergence and implementation of our algorithm with accurate reconstruction of the optical parameters.*

6. Conclusions

In this paper, we presented an non-linear inverse problem for one-dimensional diffusion transport equation with Robin boundary conditions. For the forward diffusion problem, we gave results such as existence, uniqueness and smoothness solutions of the diffusion problem in the frequency domain. And, for the inverse problem, we gave the same theoretical results such as the continuity, the stability and Fréchet derivatives of the Dirichlet-to-Neumann non-linear map. Also, we discretized the inverse problem using the cubic spline basis functions. The Tikhonov regularization and Newton solver are used to compute the solution of the considered inverse problem. Consequently, an approximate of the optical parameters are obtained in the cubic spline basis functions. Some numerical examples are presented to prove the efficacious of the proposed method for the forward solver and backward solver. From a modeling point of view, the problem considered in this paper, has some limitations such as the applications in mammography, breast cancer treatment, and optical tomography. The $2D$-inverse problem or Higher dimensional problem for more realistic applications in optical tomography, needs more investigation and future work.

References

1. R. Adams, *Sobolev Spaces*, Academic Press, NewYork, (1975).

2. M. Addam, *Approximation du Problème de Diffusion en Tomographie Optique et Problème Inverse*, Dissertation, LMPA, Université Lille-Nord de France, (2010).

3. M. Addam, A. Bouhamidi, K. Jbilou, *A numerical method for one dimensional diffusion problem using Fourier transform and the B-spline Galerkin method.* Appl. Math. Comput. 215 4067-4079, (2010).

4. M. Addam, A. Bouhamidi, K. Jbilou, *Signal reconstruction for the diffusion transport equation using tensorial spline Galerkin approximation.* Appl. Num. Math. 62 1089-1108, (2012).

5. C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York, (1978).

6. R. Chandrasekhar,*Radiation Transfer*, Oxford Clarendon, (1950).

7. J. E. Dennis, R. B. Schnabel,*Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, (1983).

8. C. Evans Lawrence,*Partial Differential Equations*, Graduate Studies in Mathematics, Vol.19, AMS, (1998).

9. T. Hohage,*Fast numerical solution of the electromagnetic medium scattering problem and applications to the inverse problem.* J. Comput. Phys. 214 224-38 (2006).

10. V. Isakov,*Inverse Problems for Partial Differential Equations*, Springer, New York, (1998).

11. A. Ishimaru,*Single Scattering and Transport Theory: Wave Propagation and Scattering in Random Media I and II*, Academic Press, New York, (1978).

12. A. D. Klose, E. W. Larsen,*Light transport in biological tissue based on the simplified spherical harmonics equations.* J. Comput. Phys. 220 441-470 (2006).

13. E. W. Larsen,*Solution of Neutron Transport Problems in* L_1. Comm. Pure Appl. Math., XXVIII, pp. 729-746 (1975).

14. F. Natterer, H. Sielschott, O. Dorn, T. Dierkes, V. Palamodov,*Fréchet derivatives for some bilinear inverse problems.* SIAM J. Appl. Math. 62 6 2092-2113 (2002).

15. R.G. Novikov,$\overline{\partial}$*-Method with Nonzero Background Potential Application to Inverse Scattering for the Two-dimensional Acoustic Equation.* Comm. Part. Diff. Eq., 21 (3& 4) pp. 597-618 (1996).

16. R. Samuelides, L. Touzillier,*Analyse Harmonique*, Cepadues-Editions, Toulouse, (1990).

17. I. Teleaga, M. Seaïd, I. Gasser, A. Klar, J. Strukmeier,*Radiation models for thermal flows at low Mach number.* J. Comput. Phys. 205 506-525 (2006).

18. A. N. Tikhonov, V. Y. Arsenin,*Solutions of Ill-Posed Problems*, Winston Wiley, New York, (1977).

Mohamed ADDAM,
Department of Mathematics and Computer Sciences,
ENSAH, Université Mohammed Premier,
Oujda.
E-mail address: m.addam@ump.ma