



The Funk-Hecke Formula, Harmonic Polynomials, and Derivatives of Radial Distributions

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ABSTRACT: We give a version of the Funk-Hecke formula that holds with minimal assumptions and apply it to obtain formulas for the distributional derivatives of radial distributions in \mathbb{R}^n of the type

$$Y_k(\nabla) \overline{\Delta}^j (f(r)),$$

where Y_k is a harmonic homogeneous polynomial. We show that such derivatives have simpler expressions than those of the form $p(\nabla)(f(r))$ for a general polynomial p .

Key Words: Harmonic polynomials, distributional derivatives, radial distributions.

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1. Introduction

Harmonic polynomials have a long and fruitful history in Mathematics and in Mathematical Physics, but it is noteworthy that there has been interest in several aspects of the relationship between harmonic polynomials and problems in distributional regularization and in the computation of distributional derivatives in recent years. Harmonic polynomials play a fundamental role in the ideas of the late professor Stora on convergent Feynman amplitudes, particularly in his work with Nikolov and Todorov [20]; this can be also seen in the recent article of Várilly and Gracia-Bondía [25]. Parker [21] has pointed out the correct formulas obtained for multipole potentials built from harmonic polynomials, while the author has shown that such multipole potentials have remarkable properties with respect to

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regularization and differentiation [6] and that several product formulas involving n dimensional delta functions simplify only for harmonic polynomials [7].

The aim of this article is to consider the computation of distributional derivatives of the type¹

$$p(\overline{\nabla})(f(r)), \quad (1.1)$$

where f is a radial distribution in \mathbb{R}^n and where p is a polynomial in n variables of the special form $p(\mathbf{x}) = Y(\mathbf{x})|\mathbf{x}|^{2j}$, Y being a harmonic polynomial. Such derivatives, particularly in the case of regularizations of power potentials, are very important in Mathematical Physics [14,19,21], starting with the celebrated Frahm formulas [11] that have become standard material in textbooks [15]. The general derivatives of any order of regularizations of power potentials are available [17], so that, in principle one could evaluate (1.1) for any polynomial p for this type of radial distributions f , but the formulas simplify substantially precisely if p is harmonic.

The main ingredients of our analysis are, first, a minimalistic version of the Funk-Hecke formula that holds for operators that transform polynomials into formal power series, the kernels themselves being formal power series, and which seems to have independent interest. The usual Funk-Hecke formula has proved to be an indispensable aid in the study of multidimensional integral transforms, such as, for example, the Radon transform, from the pioneering work of Ludwig [18] to recent works [22]. Our version of the formula is not only more general than the standard one, but can also be applied in other contexts, as we show in this article. Our second tool is a careful analysis of the spaces of distributions of the type $f(r)Y(\mathbf{x})$, where f is radial and Y is a homogeneous harmonic polynomial, analysis that extends the study of radial distributions of [12] and [5].

2. Notation

We shall write

$$c_{m,n} = \frac{2\Gamma(m+n/2)\pi^{(n-1)/2}}{\Gamma(m+1/2)} = \int_{\mathbb{S}} \omega_j^{2m} d\sigma(\boldsymbol{\omega}), \quad C = c_{0,n}. \quad (2.1)$$

Notice that $c_{0,n} = C = 2\pi^{n/2}/\Gamma(n/2)$, is the surface area of the unit sphere \mathbb{S} of \mathbb{R}^n .

The space of homogeneous polynomials of degree k in n variables will be denoted as \mathcal{P}_k or $\mathcal{P}_k(\mathbb{R}^n)$. The set of all polynomials in n variables will be denoted as \mathcal{P} or as $\mathcal{P}(\mathbb{R}^n)$. In the space $\mathcal{P}(\mathbb{R}^n)$ we consider the inductive limit topology [24, Chp. 14], so that it is an LF space. We denote by $\mathcal{H}_k(\mathbb{R}^n)$ the subspace of $\mathcal{P}_k(\mathbb{R}^n)$ formed by the harmonic homogeneous polynomials of degree k . We may also consider $\mathcal{H}_k(\mathbb{S})$, the set of restrictions to the unit sphere. The elements of $\mathcal{H}_k(\mathbb{S})$ are usually called spherical harmonics, while those of $\mathcal{H}_k(\mathbb{R}^n)$ are referred to as solid harmonics. Notice that the restriction map $\mathcal{H}_k(\mathbb{R}^n) \rightarrow \mathcal{H}_k(\mathbb{S})$ is in fact a bijection because

¹ We follow the Farassat notation of denoting distributional derivatives with an overbar [8]. We will denote by ∇ the gradient: $\nabla = (\partial/\partial x_i)_{i=1}^n$.

of the maximum principle for harmonic functions, and thus one may employ the simpler notation \mathcal{H}_k for this space². The space $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ is the space of all harmonic polynomials, a closed subspace of the topological vector space \mathcal{P} .

The dual space \mathcal{P}' can be identified with the space of formal power series in n variables, and we endow it with the weak topology, which is exactly the topology of simple convergence of the coefficients [24]. Similarly, \mathcal{H}' can be identified with $\prod_{k=0}^{\infty} \mathcal{H}_k$, with the product topology, or alternatively, with the space of formal series of the form $\sum_{k=0}^{\infty} Y_k$ where $Y_k \in \mathcal{H}_k$, with the topology of the simple convergence of each term of the series. When one thinks of the elements of \mathcal{H} and of \mathcal{H}' as objects defined on the sphere \mathbb{S} , then it is many times true that spaces of functions and distributions, X , satisfy $\mathcal{H} \subset X \subset \mathcal{H}'$ and $\mathcal{H} \subset X' \subset \mathcal{H}'$; for example, $\mathcal{H} \subset L^2(\mathbb{S}) \subset \mathcal{H}'$, the elements of $L^2(\mathbb{S})$ being those series $\sum_{k=0}^{\infty} Y_k$ where $Y_k \in \mathcal{H}_k$ with $\sum_{k=0}^{\infty} \|Y_k\|_{L^2}^2 < \infty$, $\mathcal{H} \subset \mathcal{D}(\mathbb{S}) \subset \mathcal{H}'$, the elements of $\mathcal{D}(\mathbb{S})$ being those series with $\|Y_k\|_{L^2} = o(k^{-m})$ as $k \rightarrow \infty$ for all m , while $\mathcal{H} \subset \mathcal{D}'(\mathbb{S}) \subset \mathcal{H}'$, the elements of $\mathcal{D}'(\mathbb{S})$ being those series with $\|Y_k\|_{L^2} = O(k^M)$ as $k \rightarrow \infty$ for some $M \in \mathbb{R}$.

The projection from \mathcal{H}' or any of its subspaces to \mathcal{H}_m will be denoted as π_m , that is, $\pi_m(\sum_{k=0}^{\infty} Y_k) = Y_m$. The injection from \mathcal{H}_m to \mathcal{H} or bigger spaces will be denoted as i_m . In fact,

$$\pi_m \{f(\mathbf{u}); \mathbf{v}\} = \langle Z_m(\mathbf{u}, \mathbf{v}), f(\mathbf{u}) \rangle, \tag{2.2}$$

where the kernel $Z_m(\mathbf{u}, \mathbf{v})$ is the zonal harmonic of degree m , the reproducing kernel of the finite dimensional Hilbert space \mathcal{H}_k with structure as a subspace of $L^2(\mathbb{S})$ [2,9,22]. Actually

$$Z_m(\mathbf{u}, \mathbf{v}) = \tilde{P}_m(\mathbf{u} \cdot \mathbf{v}), \tag{2.3}$$

where the \tilde{P}_m are appropriate multiples of the ultraspherical polynomials³ for dimension n [22, (A.6.13)].

3. The Funk-Hecke formula

The Funk-Hecke formula is a very useful tool in harmonic analysis⁴. The work of Funk and Hecke deals with the 3 dimensional case [10,13]; in its n dimensional form it was probably first given by Erdélyi [4]⁵. Here we shall follow the presentation of [22, Appendix A], and to some extent that of [3, Sect. 11.4].

The Funk-Hecke formula is usually written in the following way [22, Thm. A.34]: *In n variables, if $f(t) (1 - t^2)^{(n-3)/2} \in L^1(-1, 1)$ then*

$$\int_{\mathbb{S}} f(\mathbf{u} \cdot \mathbf{v}) Y_k(\mathbf{u}) \, d\sigma(\mathbf{u}) = \lambda_k Y_k(\mathbf{v}), \quad Y_k \in \mathcal{H}_k(\mathbb{S}). \tag{3.1}$$

² See [2,9,22] for the properties of \mathcal{H}_k .

³ The ultraspherical polynomials are also called Gegenbauer polynomials, and they are actually the Chebyshev polynomials in dimension 2 and the Legendre polynomials in dimension 3.

⁴ The author is indebted to Professor Rubin, who shared his deep knowledge of the subject.

⁵ The formula for the constants λ_k of (3.1) given in [4] is not the usual one, but the two expressions can be shown to be equal by applying Parseval's identity.

Notice that the constant λ_k is the *same* for all spherical harmonics of the same degree k . Here we would like to give a general version of the formula that asks minimal regularity of the kernel $f(\mathbf{u} \cdot \mathbf{v})$, that is, by replacing the integral by a suitable evaluation $\langle f(\mathbf{u} \cdot \mathbf{v}), Y_k(\mathbf{u}) \rangle_{\mathbf{u}}$ we shall see that the Funk-Hecke formula continues to hold not only for distributional kernels, but actually for kernels that can be expressed as a formal power series.

Let us start by recalling the following result on invariant functions [22]: A function $g : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ is invariant with respect to the group $O(n)$, $g(\tau\mathbf{u}, \tau\mathbf{v}) = g(\mathbf{u}, \mathbf{v})$ for all $\tau \in O(n)$ and all $(\mathbf{u}, \mathbf{v}) \in \mathbb{S} \times \mathbb{S}$ if and only if $g(\mathbf{u}, \mathbf{v}) = G(\mathbf{u} \cdot \mathbf{v})$ for some function of one variable G ; actually if $n \geq 3$ it is enough to ask invariance with respect to $SO(n)$. From this we obtain the following result on invariant transforms.

Lemma 3.1. *Let $\mathcal{G} : \mathcal{H} \rightarrow C(\mathbb{S})$ be a linear transform given by the formula*

$$\mathcal{G}\{f\} = \mathcal{G}\{f(\mathbf{u}); \mathbf{v}\} = \int_{\mathbb{S}} g(\mathbf{u}, \mathbf{v}) f(\mathbf{u}) \, d\sigma(\mathbf{u}), \quad (3.2)$$

where $g \in C(\mathbb{S} \times \mathbb{S})$. Then \mathcal{G} is invariant with respect to $O(n)$,

$$\mathcal{G}\{f(\tau\mathbf{u}); \mathbf{v}\} = \mathcal{G}\{f(\mathbf{u}); \tau\mathbf{v}\}, \quad \tau \in O(n), \quad (3.3)$$

if and only if

$$g(\mathbf{u}, \mathbf{v}) = G(\mathbf{u} \cdot \mathbf{v}), \quad (3.4)$$

for some $G \in C(\mathbb{S})$. If $n \geq 3$ it is enough to ask invariance with respect to $SO(n)$.

Proof: If (3.4) holds, then (3.3) is obtained by a simple change of variables. Conversely, if (3.3) is satisfied, then for any $\tau \in O(n)$ the same change of variables gives

$$\int_{\mathbb{S}} (g(\mathbf{u}, \mathbf{v}) - g(\tau\mathbf{u}, \tau\mathbf{v})) f(\mathbf{u}) \, d\sigma(\mathbf{u}) = 0, \quad (3.5)$$

for all $f \in \mathcal{H}$, and the density of \mathcal{H} in $(C(\mathbb{S}))'$ thus yields $g(\tau\mathbf{u}, \tau\mathbf{v}) = g(\mathbf{u}, \mathbf{v})$; (3.4) follows. \square

We shall improve this result to more general kernels g , but this weaker form will be useful in our analysis.

Lemma 3.2. *Let $\mathcal{G}_k : \mathcal{H}_k \rightarrow \mathcal{H}'$ be a linear transform that is invariant with respect to $O(n)$, $n = 2$, or $SO(n)$, $n \geq 3$. Then there exists a constant λ_k such that $\mathcal{G}_k = \lambda_k i_k$, $i_k : \mathcal{H}_k \rightarrow \mathcal{H}'$ being the canonical injection.*

Proof: The operator $\pi_m \circ \mathcal{G}_k \circ \pi_k$ is invariant for any m , and thus the Lemma 3.1 gives that it comes from a continuous kernel $g_m(\mathbf{u}, \mathbf{v}) = G_m(\mathbf{u} \cdot \mathbf{v})$. If we now apply the Funk-Hecke formula for integrable kernels, we obtain that for each l , $(\pi_m \circ \mathcal{G}_k \circ \pi_k) \circ i_l = \lambda_{k,m,l} i_l$ for some constants $\lambda_{k,m,l}$; clearly $\lambda_{k,m,l} = 0$ unless

$m = l = k$, so that, in particular, if we put $\lambda_k = \lambda_{k,k,k}$ we obtain $\pi_m \circ \mathcal{G}_k = (\pi_m \circ \mathcal{G}_k \circ \pi_k) \circ i_k = \lambda_k \delta_{k,m} i_k$ for all m , and this naturally gives $\mathcal{G}_k = \lambda_k i_k$. \square

We are now ready to give our version of the Funk-Hecke formula.

Theorem 3.3. *Let $\mathcal{G} : \mathcal{H} \rightarrow \mathcal{H}'$ be a linear transform that is invariant with respect to $O(n)$, $n = 2$, or $SO(n)$, $n \geq 3$. Then there exist constants λ_k such that*

$$\mathcal{G}(Y_k) = \lambda_k Y_k, \quad Y_k \in \mathcal{H}_k. \tag{3.6}$$

Furthermore⁶

$$\mathcal{G} = \sum_{k=0}^{\infty} \lambda_k \pi_k, \tag{3.7}$$

and for any $Y \in \mathcal{H}$

$$\mathcal{G}\{Y(\mathbf{u}); \mathbf{v}\} = \langle G(\mathbf{u} \cdot \mathbf{v}), Y(\mathbf{u}) \rangle, \tag{3.8}$$

where G is the formal series⁷

$$G(t) = \sum_{k=0}^{\infty} \lambda_k \tilde{P}_k(t), \tag{3.9}$$

the \tilde{P}_k being the normalized ultraspherical polynomials (2.3) for dimension n .

Proof: Indeed, for any operator $\mathcal{G} : \mathcal{H} \rightarrow \mathcal{H}'$ we have $\mathcal{G} = \sum_{k=0}^{\infty} \mathcal{G}_k \circ \pi_k$, where $\mathcal{G}_k = \mathcal{G} \circ i_k$. The Lemma 3.2 gives $\mathcal{G}_k = \lambda_k i_k$, for some constants λ_k , and this yields (3.7). The expansion (3.9) for the kernel G is obtained from (2.2). \square

Notice that when the invariant operator \mathcal{G} can be considered as acting from function or distribution spaces X and Y , with $\mathcal{H} \hookrightarrow X \hookrightarrow \mathcal{H}'$ and $\mathcal{H} \hookrightarrow Y \hookrightarrow \mathcal{H}'$ then the formal series G corresponds to a function or distribution, and the expansion (3.9) becomes convergent in a stronger sense. For instance, if \mathcal{G} is an operator from $\mathcal{D}(\mathbb{S})$ to $\mathcal{D}'(\mathbb{S})$ then G converges distributionally. The operators that send given spaces X to Y can be characterized by studying the properties of the sequence $\{\lambda_k\}_{k=0}^{\infty}$ [22, Sct. A.12]; in particular \mathcal{G} is an operator from $\mathcal{D}(\mathbb{S})$ to $\mathcal{D}'(\mathbb{S})$ if and only if $\lambda_k = O(k^M)$ as $k \rightarrow \infty$ for some $M \in \mathbb{R}$.

We may now employ the Theorem 3.3 to obtain the form of several operators acting on polynomials in n variables.

Proposition 3.4. *Let $\mathcal{T} : \mathcal{P}_k(\mathbb{R}^n) \rightarrow \mathcal{P}_k(\mathbb{R}^n)$ be a linear operator, invariant with respect to $O(n)$, $n = 2$, or $SO(n)$, $n \geq 3$. Then there are constants λ_{k-2j} , $0 \leq 2j \leq k$, such that*

$$\mathcal{T}\left\{|\mathbf{x}|^{2j} Y_{k-2j}(\mathbf{x}); \mathbf{z}\right\} = \lambda_{k-2j} |\mathbf{z}|^{2j} Y_{k-2j}(\mathbf{z}), \quad Y_{k-2j} \in \mathcal{H}_{k-2j}(\mathbb{R}^n). \tag{3.10}$$

⁶ The series $\mathcal{G}(Y) = \sum_{k=0}^{\infty} \lambda_k \pi_k(Y)$ converges in the topology of \mathcal{H}' , since in \mathcal{H}' the weak and strong topologies coincide [24].

⁷ Since the P_k are polynomials, this is actually a formal power series.

Proof: Apply the Funk-Hecke formula to the operator \mathcal{G} defined on \mathcal{H} as $\mathcal{G} = R \circ \mathcal{T} \circ R^{-1} \circ \left(\sum_{j=0}^{\lfloor k/2 \rfloor} \pi_{k-2j} \right)$, where R is the restriction operator from $\mathcal{P}_k(\mathbb{R}^n)$ to \mathcal{H} , $R\{p(\mathbf{x}); \mathbf{u}\} = p(\mathbf{u})$, $\mathbf{u} \in \mathbb{S}$. There are constants λ_{k-2j} , $0 \leq 2j \leq k$, such that $\mathcal{G}(Y_{k-2j}) = \lambda_{k-2j} Y_{k-2j}$ for $Y_{k-2j} \in \mathcal{H}_{k-2j}$. Hence,

$$\begin{aligned} \mathcal{T}\left\{|\mathbf{x}|^{2j} Y_{k-2j}(\mathbf{x}); \mathbf{z}\right\} &= (R^{-1} \circ \mathcal{G} \circ R)\left\{|\mathbf{x}|^{2j} Y_{k-2j}(\mathbf{x}); \mathbf{z}\right\} \\ &= R^{-1}\left\{\mathcal{G}\{Y_{k-2j}(\mathbf{u}); \mathbf{v}\}; \mathbf{z}\right\} \\ &= R^{-1}\left\{\lambda_{k-2j} Y_{k-2j}(\mathbf{v}); \mathbf{z}\right\} = \lambda_{k-2j} |\mathbf{z}|^{2j} p_{k-2j}(\mathbf{z}), \end{aligned}$$

as required. \square

We can also rewrite these results in the ensuing way.

Proposition 3.5. *Let $\mathcal{T} : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n)$ be a linear operator that satisfies*

$$\mathcal{T}\{p(a\tau\mathbf{x}); \mathbf{z}\} = \mathcal{T}\{p(\mathbf{x}); a\tau\mathbf{z}\}, \quad (3.11)$$

for $a \in \mathbb{R} \setminus \{0\}$ and for $\tau \in SO(n)$. Then there are constants $\lambda_{k,j}$ such that

$$\mathcal{T}\left\{|\mathbf{x}|^{2j} Y_k(\mathbf{x}); \mathbf{z}\right\} = \lambda_{k,j} |\mathbf{z}|^{2j} Y_k(\mathbf{z}), \quad Y_k \in \mathcal{H}_k(\mathbb{R}^n). \quad (3.12)$$

Proof: It follows from the Proposition 3.4 since (3.11) implies that \mathcal{T} sends $\mathcal{P}_k(\mathbb{R}^n)$ to $\mathcal{P}_k(\mathbb{R}^n)$. \square

Notice that in a natural fashion one can consider $\mathcal{P}(\mathbb{R}^n)$ as a subspace of $\mathcal{P}(\mathbb{R}^{n'})$ if $n \leq n'$; denote the corresponding injection as $i_{n,n'}$. Suppose now that $\{\mathcal{T}_n\}_{n=1}^{\infty}$ is a family of operators $\mathcal{T}_n : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n)$. If $i_{n,n'} \circ \mathcal{T}_n = \mathcal{T}_{n'} \circ i_{n,n'}$ whenever $n \leq n'$, then one may consider the family as a *single* operator $\mathcal{T} = \{\mathcal{T}_n\}_{n=1}^{\infty}$, and one can write $\mathcal{T}\{p\}$ instead of $\mathcal{T}_n\{p\}$ if $p \in \mathcal{P}(\mathbb{R}^n)$. In general the constants given in (3.12) will *depend* on n , that is, if $\mathbf{x}_n \in \mathbb{R}^n$,

$$\mathcal{T}\left\{|\mathbf{x}_n|^{2j} Y_k(\mathbf{x}_n); \mathbf{z}_n\right\} = \lambda_{k,j}^{\{n\}} |\mathbf{z}_n|^{2j} Y_k(\mathbf{z}_n), \quad Y_k \in \mathcal{H}_k(\mathbb{R}^n). \quad (3.13)$$

However, the $\lambda_{k,0}^{\{n\}}$ are actually *independent* of n , and can be found rather easily.

Proposition 3.6. *Let $\mathcal{T} = \{\mathcal{T}_n\}_{n=1}^{\infty}$ be a family of linear operators that sends $\mathcal{P}(\mathbb{R}^n)$ to itself for each n , which satisfies $i_{n,n'} \circ \mathcal{T}_n = \mathcal{T}_{n'} \circ i_{n,n'}$ whenever $n \leq n'$, and which is also invariant in the sense of (3.11). Then $\lambda_{k,0}^{\{n\}} = \lambda_{k,0}$ for all n , and*

$$\mathcal{T}\{x_{i_1} \cdots x_{i_k}; \mathbf{z}\} = \lambda_{k,0} z_{i_1} \cdots z_{i_k}, \quad (3.14)$$

whenever the indices i_1, \dots, i_k are all different.

Proof: Let $Y \in \mathcal{H}_k(\mathbb{R}^n)$ and let $n' \geq n$. Then $Y \in \mathcal{H}_k(\mathbb{R}^{n'})$ also, so that (3.13) with $j = 0$ holds for both $\lambda_{k,0}^{\{n\}}$ and $\lambda_{k,0}^{\{n'\}}$, and hence we obtain $\lambda_{k,0}^{\{n\}} = \lambda_{k,0}^{\{n'\}}$. Let us thus denote the $\lambda_{k,0}^{\{n\}}$ as $\lambda_{k,0}$. We have that $x_{i_1} \cdots x_{i_k} \in \mathcal{H}_k(\mathbb{R}^n)$, therefore relation (3.14) follows. \square

We now present an example where all the computations are basically trivial, but — because of this — will allow the reader to appreciate the main ideas of our approach.

Example 3.1. Let $p \in \mathcal{P}_k(\mathbb{R}^n)$ and let f be a smooth function defined in $(0, \infty)$. Consider the Laplacian $\Delta(f(r)p(\mathbf{x}))$, $r = |\mathbf{x}|$: it can be written, in several ways, as $q(r, \mathbf{x})$ where $q(\rho, \mathbf{x})$ is a polynomial in \mathbf{x} whose coefficients are functions of ρ , but there is a unique expression of this form where for each ρ the polynomial $q = q_k$ belongs to $\mathcal{P}_k(\mathbb{R}^n)$. Write

$$\mathcal{T}_n\{p\} = \mathcal{T}_{f,r}\{p\} = q_k(r, \mathbf{x}) = \Delta(f(r)p(\mathbf{x})). \tag{3.15}$$

The operator \mathcal{T}_n can be extended to $\mathcal{P}(\mathbb{R}^n)$ by linearity. Notice, however, that the \mathcal{T}_n depend on n . Clearly \mathcal{T}_n is invariant in the sense of (3.11). Hence

$$\Delta\left(f(r)|\mathbf{x}_n|^{2j}Y_k(\mathbf{x}_n)\right) = \Lambda_{k,j}^{\{n\}}(f)|\mathbf{x}_n|^{2j}Y_k(\mathbf{x}_n), \quad Y_k \in \mathcal{H}_k(\mathbb{R}^n), \tag{3.16}$$

for some operators $\Lambda_{k,j}^{\{n\}}$ that send smooth functions in $(0, \infty)$ to smooth functions in $(0, \infty)$. Moreover, $\Lambda_{k,j}^{\{n\}}(f(r)) = r^{-2j}\Lambda_{k,0}^{\{n\}}(r^{2j}f(r))$. On the other hand, the $\Lambda_{k,0}^{\{n\}}$ can be obtained by observing that $\mathcal{T}_n\{p\} - \Delta_n(f(r))p(\mathbf{x})$ is independent of n , so that we may take $Y_k(\mathbf{x}) = x_1 \cdots x_k$, $\Delta(f(r)x_1 \cdots x_k) = \Lambda_{k,0}^{\{n\}}(f(r))x_1 \cdots x_k$, so that

$$\Lambda_{k,0}^{\{n\}}(f) = (\Delta_n + 2k)(f) = (r^2L^2 + nL + 2k)(f), \quad , \quad L(f) = \frac{1}{r}\frac{df}{dr}. \tag{3.17}$$

What happens if f is now a distribution of one variable, with support in $[0, \infty)$? It is not clear if the operators $\Lambda_{k,j}^{\{n\}}$ can be defined for such distributions, since $L(f)$ cannot be defined as an element of $\mathcal{D}'(\mathbb{R})$ for all $f \in \mathcal{D}'(\mathbb{R})$. However, there should be a way to extend the $\Lambda_{k,j}^{\{n\}}$ to distributions since $\Delta(f(r)p(\mathbf{x}))$ is a well defined distribution of $\mathcal{D}'(\mathbb{R}^n)$ for any radial distribution $f(r)$ of $\mathcal{D}'(\mathbb{R}^n)$; we explain how this is achieved in Section 6.

4. Derivatives of smooth radial functions

In this section we shall apply the Propositions 3.5 and 3.6 to find the formulas for the computation of the action of certain differential operators on radial functions in \mathbb{R}^n .

We now assume that the radial functions are smooth, and then extend our analysis to distributional derivatives in the Section 6. Thus f will be a smooth

function defined in some open subinterval of $(0, \infty)$, so that $f(r)$ will be smooth in some annular region in \mathbb{R}^n .

Theorem 4.1. *Let L be the differential operator*

$$L(f)(r) = \frac{1}{r} \frac{df}{dr}. \quad (4.1)$$

Then in \mathbb{R}^n ,

$$Y_k(\nabla) \Delta^j (f(r)) = \Lambda_{k,j}^{\{n\}}(f) |\mathbf{x}|^{2j} Y_k(\mathbf{x}), \quad Y_k \in \mathcal{H}_k(\mathbb{R}^n), \quad (4.2)$$

where

$$\Lambda_{k,j}^{\{n\}}(f) = \frac{1}{r^{2j}} L^k \Delta^j (f) = \frac{1}{r^{2j}} L^k (r^2 L^2 + nL)^j (f). \quad (4.3)$$

Proof: Let r be a fixed number in the domain of f . Consider the operator $\mathcal{T} = \mathcal{T}_n = \mathcal{T}_{f,r,n}$ from $\mathcal{P}(\mathbb{R}^n)$ to $\mathcal{P}(\mathbb{R}^n)$ defined by $q = \mathcal{T}(p)$, where $p(\nabla)(f(r)) = q(\mathbf{x})$. Clearly $\mathcal{T} = \{\mathcal{T}_n\}_{n=1}^\infty$ does not depend on n . Furthermore, \mathcal{T} is invariant in the sense of (3.11). Hence (4.2) follows for some operators $\Lambda_{k,j}^{\{n\}}$. But $\nabla_i f(r) = L(f)(r) x_i$, so that if i_1, \dots, i_k are all different,

$$\nabla_{i_1} \cdots \nabla_{i_k} f(r) = L^k(f)(r) x_{i_1} \cdots x_{i_k}. \quad (4.4)$$

Therefore, by taking $Y_k(\mathbf{x}) = x_1 \cdots x_k$ we obtain

$$\Lambda_{k,0}(f) = L^k(f). \quad (4.5)$$

Since $\Lambda_{k,j}^{\{n\}}(f) = r^{-2j} \Lambda_{k,0} \Delta^j (f)$, the relation (4.3) follows. \square

It is interesting to notice that the first expression for the operator $\Lambda_{k,j}^{\{n\}}$ in (4.3) is, in a way, independent of n . Naturally, of course, $\Lambda_{k,j}^{\{n\}}(f)$ does depend on n if $j > 0$.

In order to appreciate the Theorem 4.1, it is instructive to consider a particular case, the derivatives of the power potentials $f(r) = r^\lambda$. The distributional derivatives of any order of r^λ were obtained by the author and are studied in the textbooks [17]; they generalize the important formulas of Frahm [11]. Here we just consider the ordinary part of the formulas, that is, for $r > 0$; the delta part will be considered in Section 6. For example for derivatives of the fourth order we have,

$$\begin{aligned} \nabla^4 (r^\lambda) &= \lambda(\lambda-2)(\lambda-4)(\lambda-6) \mathbf{x}^4 r^{\lambda-8} \\ &+ 6\lambda(\lambda-2)(\lambda-4) \mathbf{x}^2 \delta r^{\lambda-6} + 3\lambda(\lambda-2) \delta^2 r^{\lambda-4}. \end{aligned} \quad (4.6)$$

Here and in similar formulas, ∇^N denotes the symmetric tensor of order N with components $\nabla_{i_1} \cdots \nabla_{i_N}$, \mathbf{x}^N is the tensor with components $x_{i_1} \cdots x_{i_N}$, while δ is the tensor of the second order with components $\delta_{i_1 i_2}$. If \mathbf{S} and \mathbf{T} are symmetric tensors, then \mathbf{ST} is their *symmetric product*, that is, the symmetrization of their

tensor product $\mathbf{S} \otimes \mathbf{T}$; the notation \mathbf{S}^Q will be used for the symmetric product of \mathbf{S} with itself Q times. For example, $\mathbf{x}^2 \boldsymbol{\delta}$ is a fourth order tensor with components

$$\frac{1}{4!} \sum_{\sigma \in S_4} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} \delta_{\sigma(3)\sigma(4)} = \frac{1}{6} \sum_{\{i,j\} \cup \{k,l\} = \{1,2,3,4\}} x_i x_j \delta_{kl}. \quad (4.7)$$

The Theorem 4.1 then yields

$$Y(\nabla)(r^\lambda) = \lambda(\lambda - 2)(\lambda - 4)(\lambda - 6) Y(\mathbf{x}) r^{\lambda-8}, \quad Y \in \mathcal{H}_4(\mathbb{R}^n). \quad (4.8)$$

In the case of order N , $\nabla^N(r^\lambda)$ consists of a sum of $\lfloor N/2 \rfloor + 1$ terms, and so does, in general, $p(\nabla)(r^\lambda)$ if $p \in \mathcal{P}_N(\mathbb{R}^n)$. However, if Y is a harmonic polynomial of degree N then $Y(\nabla)(r^\lambda)$ reduces to just the first term,

$$Y(\nabla)(r^\lambda) = \lambda(\lambda - 2) \cdots (\lambda - N + 2) Y(\mathbf{x}) r^{\lambda-2N}, \quad Y \in \mathcal{H}_N(\mathbb{R}^n).$$

Since [2,9] any polynomial p in \mathbb{R}^n can be expressed, uniquely, as a sum of terms of the form $|\mathbf{x}|^{2j} Y_{k,j}(\mathbf{x})$, $Y_{k,j} \in \mathcal{H}_k(\mathbb{R}^n)$, one can, *in principle*, employ this theorem to find $p(\nabla)(f(r))$ for any polynomial p .

Example 4.1. Let $p(\mathbf{x}) = \mathbf{x} \cdot A \mathbf{x} = a_{ij} x_i x_j$, where $A = (a_{ij})$. Then p is harmonic if and only if $\text{tr}(A) = 0$, so that we write $a_{ij} = (a_{ij} - (\text{tr}(A)/n) \delta_{ij}) + (\text{tr}(A)/n) \delta_{ij}$ to obtain $p(\mathbf{x}) = Y_2(\mathbf{x}) + (\text{tr}(A)/n) |\mathbf{x}|^2$, $Y_2(\mathbf{x}) = (a_{ij} - (\text{tr}(A)/n) \delta_{ij}) x_i x_j$ and thus

$$p(\nabla)(f(r)) = Y_2(\mathbf{x}) L^2(f) + \frac{\text{tr}(A)}{n} (r^2 L^2 + nL)(f) = p(\mathbf{x}) L^2(f) + \text{tr}(A) L(f). \quad (4.9)$$

5. Radial and related distributions

We shall now consider radial distributions and distributions that are radial multiples of a harmonic homogeneous polynomial. In order to fix the notation, we shall give the details in the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, but naturally the same considerations apply in the spaces $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$, the spaces $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$, or other dual pairs, without much change. A test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is called *radial* if it is a function of r , $\phi(\mathbf{x}) = \varphi(r)$, for some even function $\varphi \in \mathcal{S}(\mathbb{R})$; the space of all radial test functions of $\mathcal{S}(\mathbb{R}^n)$ is denoted as $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$. Similarly, we denote as $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ the space of all radial tempered distributions; a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is radial if $f(\tau \mathbf{x}) = f(\mathbf{x})$ for any $\tau \in O(n)$, and this actually means that $f(\mathbf{x}) = f_1(r)$ for some distribution of one variable f_1 . Notice, however, that while φ is uniquely determined by ϕ , for a given $f \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ there are several possible distributions $f_1 \in \mathcal{S}'(\mathbb{R})$. When $n = 1$ then $\mathcal{S}_{\text{rad}}(\mathbb{R})$ and $\mathcal{S}'_{\text{rad}}(\mathbb{R})$ become the spaces of even rapidly decreasing test functions and tempered distributions, respectively, and are also denoted as $\mathcal{S}_{\text{even}}(\mathbb{R})$ and $\mathcal{S}'_{\text{even}}(\mathbb{R})$.

Observe that the space $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ is naturally isomorphic to the dual space $(\mathcal{S}_{\text{rad}}(\mathbb{R}^n))'$, that is to say, if the action of a radial distribution is known in all

radial test functions, then it can be obtained for arbitrary test functions. Indeed, if $f \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\langle f(\mathbf{x}), \phi(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \tilde{\phi}(\mathbf{x}) \rangle, \tag{5.1}$$

where $\tilde{\phi} \in \mathcal{S}_{\text{rad}}(\mathbb{R})$ is given as $\tilde{\phi}(\mathbf{x}) = \phi^o(|\mathbf{x}|)$, $\phi^o \in \mathcal{S}_{\text{even}}(\mathbb{R})$ being defined as

$$\phi^o(r) = \frac{1}{C} \int_{\mathbb{S}} \phi(r\boldsymbol{\theta}) \, d\sigma(\boldsymbol{\theta}). \tag{5.2}$$

Following [12], we shall denote by $\mathcal{R}_n = r^{n-1}\mathcal{S}_{\text{even}}(\mathbb{R})$. Also [5], if \mathcal{A} is a subspace of $\mathcal{S}(\mathbb{R})$ we shall denote by $\mathcal{A}[0, \infty)$ the space of restrictions of elements of \mathcal{A} to $[0, \infty)$. The operator $\mathfrak{J} : \mathcal{S}'_{\text{rad}}(\mathbb{R}^n) \rightarrow \mathcal{R}'_n[0, \infty)$, given by $F = \mathfrak{J}(f)$,

$$\langle F(r), \phi^o(r)r^{n-1} \rangle_{\mathcal{R}'_n[0, \infty) \times \mathcal{R}_n[0, \infty)} = \frac{1}{C} \langle f(\mathbf{x}), \phi(\mathbf{x}) \rangle_{\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)}. \tag{5.3}$$

is an isomorphism [5]. What this means is that when considering a radial distribution of $\mathcal{S}'(\mathbb{R}^n)$, we can express it, in a unique fashion, as $f = F(r)$, where $F \in \mathcal{R}'_n[0, \infty)$.

It is interesting to observe that a radial distribution with support $\{0\}$ should have the form $\sum_{j=0}^N \alpha_j \nabla^{2j} \delta(\mathbf{x})$, for some N and some constants α_j , $0 \leq j \leq N$. Notice also the formulas

$$\mathfrak{J}(\nabla^{2j} \delta(\mathbf{x})) = \frac{(-1)^{n-1} \delta^{(n+2j-1)}(r)}{(n+2j-1)!c_{j,m}}, \tag{5.4}$$

where the $c_{j,m}$ are given in (2.1).

Let now $Y \in \mathcal{H}_k(\mathbb{R}^n)$ be a solid harmonic and consider the multiplication map $M_Y : \mathcal{S}'_{\text{rad}}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ given by $M_Y(f) = fY$. A distribution of $\mathcal{S}'(\mathbb{R}^n)$ will be called a *radial multiple* of Y if it belongs to the image of the map, $M_Y(\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)) = \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)Y$, that is, if it is of the form $F(r)Y(\mathbf{x})$ for some distribution of one variable F . As we explained, we can take $F \in \mathcal{R}'_n[0, \infty)$, but actually we shall now show that a better choice will be $F \in \mathcal{R}'_{n+2k}[0, \infty)$ since the operator M_Y is not injective, but rather has a non trivial kernel.

Proposition 5.1. *Let f be a radial distribution and let $Y \in \mathcal{H}_k(\mathbb{R}^n)$. Then $fY = M_Y(f) = 0$ if and only if*

$$f(\mathbf{x}) = \sum_{j=0}^{k-1} \alpha_j \nabla^{2j} \delta(\mathbf{x}), \tag{5.5}$$

for some constants α_j , $0 \leq j < k$.

Proof: Indeed, if $fY = 0$ then $\text{supp } f = \{0\}$, so that, because f is radial, $f(\mathbf{x}) = \sum_{j=0}^N \alpha_j \nabla^{2j} \delta(\mathbf{x})$ for some N and some constants α_j . However [7] $Y(\mathbf{x}) \nabla^{2m} \delta(\mathbf{x}) = 0$, if $m < k$, while if $m \geq k$ then

$$Y(\mathbf{x}) \nabla^{2m} \delta(\mathbf{x}) = \frac{(-1)^k 2^k m!}{(m-k)!} Y(\nabla) \nabla^{2m-2k} \delta(\mathbf{x}). \tag{5.6}$$

The result follows. □

Collecting ideas we can give the ensuing representation of radial multiples of a solid harmonic.

Proposition 5.2. *Let $Y \in \mathcal{H}_k(\mathbb{R}^n)$. There is an isomorphism $\mathfrak{J}_k : \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)Y \rightarrow \mathcal{R}'_{n+2k}[0, \infty)$, so that any radial multiple of Y can be written as $F(r)Y(\mathbf{x})$ for some $F \in \mathcal{R}'_{n+2k}[0, \infty)$.*

Proof: Let $g \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)Y$. Then there exists $f \in \mathcal{R}'_n[0, \infty)$ such that $g(\mathbf{x}) = f(r)Y(\mathbf{x})$. Define $\mathfrak{J}_k(g) = f|_{\mathcal{R}_{n+2k}[0, \infty)}$. Then $\mathfrak{J}_k(g)$ is well defined, since if $g(\mathbf{x}) = f_1(r)Y(\mathbf{x}) = f_2(r)Y(\mathbf{x})$ then $(f_1(r) - f_2(r))Y(\mathbf{x}) = 0$, hence for some constants α_j , $0 \leq j < k$, $f_1(r) - f_2(r) = \mathfrak{J}^{-1}(\sum_{j=0}^{k-1} \alpha_j \nabla^{2j} \delta(\mathbf{x}))$, that is $f_1(r) - f_2(r)$ equals $\sum_{j=0}^{k-1} \{(-1)^{n-1} \alpha_j / (n+2j-1)! c_{j,m}\} \delta^{(n+2j-1)}(r)$, and consequently this gives $(f_1 - f_2)|_{\mathcal{R}_{n+2k}[0, \infty)} = 0$. □

It also important to point out that $(\mathcal{S}_{\text{rad}}(\mathbb{R}^n)Y)'$ is naturally isomorphic to $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)Y$. If $g \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)Y$ and $\phi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)Y$, $\phi(\mathbf{x}) = \varphi(|\mathbf{x}|)Y(\mathbf{x})$, then

$$\langle g(\mathbf{x}), \phi(\mathbf{x}) \rangle_{\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} = A \langle \mathfrak{J}_k(g)(r), r^{2k+n-1} \varphi(r) \rangle_{\mathcal{R}'_{n+2k}[0, \infty) \times \mathcal{R}_{n+2k}[0, \infty)}, \tag{5.7}$$

where $A = \int_{\mathbb{S}} Y^2(\boldsymbol{\theta}) d\sigma(\boldsymbol{\theta})$.

6. Derivatives of radial distributions

Our next task is to show that the Theorem 4.1 holds in the distributional sense for the derivatives of radial distributions. This will follow from a careful examination of the operator L .

Proposition 6.1. *The derivative operator \bar{d}/dr , $f \rightarrow \bar{d}f/dr$ is a well defined operator from $\mathcal{R}'_n[0, \infty)$ to $\mathcal{R}'_{n+1}[0, \infty)$. If $q \in \mathbb{Z}$, then the operator \bar{M}_{r^q} , $f(r) \rightarrow r^q f(r)$, is a well defined operator from $\mathcal{R}'_n[0, \infty)$ to $\mathcal{R}'_{n-q}[0, \infty)$.*

Proof: The results follow immediately by considering the transpose operators. Indeed, we have $(\bar{d}/dr)^T = -d/dr$ which sends $\mathcal{R}_{n+1}[0, \infty)$ to $\mathcal{R}_n[0, \infty)$ and $(\bar{M}_{r^q})^T = M_{r^{-q}}$ sends $\mathcal{R}_{n-q}[0, \infty)$ to $\mathcal{R}_n[0, \infty)$. □

Therefore we obtain the following for the operator \bar{L} .

Proposition 6.2. *The operator \bar{L} given by*

$$\bar{L}(f)(r) = \frac{1}{r} f'(r), \tag{6.1}$$

sends $\mathcal{R}'_n[0, \infty)$ to $\mathcal{R}'_{n+2}[0, \infty)$.

Applying the Proposition 6.1 again, we obtain that, in general if a is a constant, then the operator $r^2\bar{L}^2 + a\bar{L}$ sends $\mathcal{R}'_n[0, \infty)$ to $\mathcal{R}'_{n+2}[0, \infty)$. Nevertheless, in the special case when $a = n$, then more is true.

Proposition 6.3. *The operator*

$$\bar{D}_n = r^2\bar{L}^2 + n\bar{L}, \tag{6.2}$$

sends $\mathcal{R}'_n[0, \infty)$ to itself.

Proof: We just need to verify that $(\bar{D}_n)^T$ sends $\mathcal{R}_n[0, \infty)$ to itself, but this is clear since $(\bar{D}_n)^T \varphi(r) = (r^{-1}(r\varphi)')' - n(r^{-1}\varphi)'$. □

We can now state the formula for the application of harmonic polynomial derivative operators to radial distributions.

Theorem 6.4. *Let f_0 be a radial tempered distribution in n variables, $f_0(\mathbf{x}) = f(r)$ for $f \in \mathcal{R}'_n[0, \infty)$. Then*

$$Y_k(\bar{\nabla})\bar{\Delta}^j(f(r)) = \bar{\Lambda}_{k,j}^{\{n\}}(f)|\mathbf{x}|^{2j}Y_k(\mathbf{x}), \quad Y_k \in \mathcal{H}_k(\mathbb{R}^n), \tag{6.3}$$

where $\bar{\Lambda}_{k,j}^{\{n\}}$ is the operator from $\mathcal{R}'_n[0, \infty)$ to $\mathcal{R}'_{n+2k}[0, \infty)$ given by

$$\bar{\Lambda}_{k,j}^{\{n\}}(f) = \frac{1}{r^{2j}}\bar{L}^k\bar{D}^j(f). \tag{6.4}$$

Proof: The Theorem 4.1 tells us that (6.3) holds if the distribution f is a regular distribution corresponding to a smooth function. Since such regular distributions are dense in the space of all distributions, the result is obtained. □

Let us now consider the distributional derivatives of the inverse power potentials $r^{-\alpha}$. Since $r^{-\alpha}$ is not integrable at the origin if $\alpha \geq n$, it does not give a distribution directly, so that we need to employ a regularization⁸. We shall employ the Hadamard regularization or pseudofunction [23] $\mathcal{P}f(r^{-\alpha})$, $\langle \mathcal{P}f(r^{-\alpha}), \phi(\mathbf{x}) \rangle = \text{F.p.} \int_{\mathbb{R}^n} |\mathbf{x}|^{-\alpha} \phi(\mathbf{x}) \, d\mathbf{x}$, the Hadamard finite part of the divergent integral. It is proved in [17] that the distributional derivatives $\bar{\nabla}^N \mathcal{P}f(r^{-\alpha})$ are also pseudofunctions unless α is an integer q and $q - n$ is even, say $q - n = 2m$; in that case we have that $\bar{\nabla}^N \mathcal{P}f(r^{-\alpha})$ has the expression

$$\sum_{j=0}^{\lfloor N/2 \rfloor} \frac{(-1)^{N-j} 2^{N-2j} \Gamma(\alpha/2 + N - j) N!}{\Gamma(\alpha/2) (N - 2j)! j!} \delta^j \mathbf{x}^{N-2j} \mathcal{P}f(r^{-\alpha-2N+2j}). \tag{6.5}$$

⁸ Regularization methods are considered in the texts on distributions [16,17,23].

Hence, $Y_k (\overline{\nabla}) \mathcal{P}f (r^{-\alpha})$ is given as

$$\frac{2^N \Gamma (\alpha/2 + k)}{\Gamma (\alpha/2)} \mathcal{P}f (r^{-\alpha-2k}) Y_k (\mathbf{x}), \quad Y_k \in \mathcal{H}_k (\mathbb{R}^n), \quad (6.6)$$

and, more generally, since

$$\overline{\Delta}^j \mathcal{P}f (r^{-\alpha}) = B_{\alpha,j} \mathcal{P}f (r^{-\alpha-2j}), \quad (6.7)$$

where

$$B_{\alpha,j} = \alpha \cdots (\alpha + 2j - 2) (\alpha + 2 - n) \cdots (\alpha + 2j - n), \quad (6.8)$$

we obtain that $Y_k (\overline{\nabla}) \overline{\Delta}^j \mathcal{P}f (r^{-\alpha})$ equals

$$= \frac{2^N \Gamma (\alpha/2 + k) B_{\alpha,j}}{\Gamma (\alpha/2)} \mathcal{P}f (r^{-\alpha-2k-2j}) Y_k (\mathbf{x}), \quad Y_k \in \mathcal{H}_k (\mathbb{R}^n). \quad (6.9)$$

On the other hand, when we replace α by q , with $q = n + 2m$, we obtain [17]

$$\begin{aligned} \overline{\nabla}^N \mathcal{P}f (r^{-q}) &= \sum_{j=0}^{\lfloor N/2 \rfloor} \frac{(-1)^{N-j} 2^{N-2j} \Gamma (q/2 + N - j) N!}{\Gamma (q/2) (N - 2j)! j!} \delta^j \mathbf{x}^{N-2j} \mathcal{P}f (r^{-q-2N+2j}) \\ &\quad - \sum_{j=(|m|-m)/2}^{\lfloor N/2 \rfloor} \frac{N! \Gamma (q/2 + j) c_{m+j,n} \beta_{N,j}}{(N - 2j)! \Gamma (q/2) j! (2m + 2j)!} \delta^j \overline{\nabla}^N \overline{\Delta}^{m+j} \delta (\mathbf{x}). \end{aligned} \quad (6.10)$$

Here

$$\beta_{N,0} = \frac{1}{q} + \frac{1}{q+2} + \cdots + \frac{1}{q+2N-2} = \frac{1}{2} \left(\psi \left(\frac{q}{2} + N \right) - \psi \left(\frac{q}{2} \right) \right), \quad (6.11)$$

where $\psi (s) = \Gamma' (s) / \Gamma (s)$ is the digamma function, and if $l \geq 1$,

$$\beta_{N,l} = \frac{1}{2} \sum_{j=0}^l \binom{l}{j} (-1)^j \psi \left(\frac{q}{2} + N - j \right). \quad (6.12)$$

We thus obtain that if $Y_k \in \mathcal{H}_k (\mathbb{R}^n)$, then

$$\begin{aligned} Y_k (\overline{\nabla}) \mathcal{P}f (r^{-q}) &= \frac{2^N \Gamma (q/2 + k)}{\Gamma (q/2)} \mathcal{P}f (r^{-q-2k}) Y_k (\mathbf{x}) \\ &\quad + \frac{c_{m,n} \beta_{N,0}}{(2m)!} Y_k (\overline{\nabla}) \overline{\Delta}^m \delta (\mathbf{x}). \end{aligned} \quad (6.13)$$

We would like to finish by pointing out that the structure of the Fourier transform in n variables can be studied by employing our ideas. However, the results obtained are hardly new. In fact Watson in the Section 11.5 of his treatise [26]

gives the most relevant formulas — several addition theorems for Bessel functions — citing the work of Bauer in 1859 for the 3 dimensional case and the work of Gegenbauer of 1874 in the general case. Interestingly, Erdélyi [4] actually employs these addition theorems in his study of the n dimensional Funk-Hecke formula. Such developments of the Fourier kernel are called Rayleigh expansions in the Physics literature, where they are still employed in the study of Fourier transforms [1].

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