Bol. Soc. Paran. Mat. (3s.) v. 37 3 (2019): 131–142.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i3.34703

# Certain Results on Generalized $(k,\mu)$-contact Manifolds

Pradip Majhi and Gopal Ghosh

ABSTRACT: The object of the present paper is to study generalized $(k,\mu)$-contact manifolds. At first we consider $\varphi$-semisymmetric generalized $(k,\mu)$-contact manifolds. Beside these we study extended pseudo projectively flat generalized $(k,\mu)$-contact manifolds. Also $(k,\mu)$-contact manifold satisfying $\bar{P}^e \cdot S = 0$ is also considered. As a consequence we obtain several corollaries.

Key Words: Generalized $(k,\mu)$-manifolds, $(k,\mu)$-manifolds, Ricci tensor, Projective curvature tensor, $\varphi$-projective semisymmetric manifold, $\eta$-Einstein manifold.

## Contents



## 1. Introduction

In 1995 Blair, Koufogiorgos and Papantoniou [1] introduced the notion of $(k,\mu)$-contact metric manifolds $M$ of dimension $(2n+1)$ , where $k$ and $\mu$ are real constants, and a full classification of such manifolds was given by E. Boeckx [6]. Actually this class of space was obtained through $D$-homothetic deformation [14] to a contact metric manifold whose curvature tensor satisfying $R(X,Y)\xi = 0$. There exist contact metric manifolds for which $R(X,Y)\xi = 0$. For instance the tangent sphere bundle of flat Riemannian manifold admits such a structure. Further it is well known that [1] the tangent sphere bundle $T_1M$ of a Riemannian manifold of constant curvature $c$ is a $(k,\mu)$-contact metric space where $k = c(2-c)$ and $\mu = -2c$. Thus in one hand there exists examples of $(k,\mu)$-contact manifolds in all dimensions and on the other this class is invariant under $D$-homothetic deformation. It

2010 *Mathematics Subject Classification:* 53C15, 53C15, 53C25.
Submitted December 31, 2016. Published April 20, 2017

is evident that the class of $(k,\mu)$-contact manifolds contains the class of Sasakian manifolds, in which $k=1$. Assuming $k,\mu$ be smooth functions, T. Koufogiorgos and C. Tsichlias introduced the notion of generalized $(k,\mu)$-contact metric manifolds and gave several examples [9]. Also in [9], the authors proved that the generalized $(k,\mu)$-contact metric manifolds exist only for dimension 3 and hence we confined ourselves to the study of 3-dimensional generalized $(k,\mu)$-contact metric manifolds. The $(k,\mu)$-contact metric manifold is of our special interest as it contains both the Sasakian and non-Sasakian cases. Let $M$ be a $(2n+1)$-dimensional Riemannian manifold with metric $g$ and let $\chi(M)$ be the set of all differentiable vector fields on $M$. The Ricci operator $Q$ of $(M,g)$ is defined by $g(QX,Y)=S(X,Y)$ , where $S$ denotes the Ricci tensor of type $(0,2)$ on $M$ and $X,Y\in\chi(M)$. In a Riemannian manifold, if there exist a one-to-one correspondence between each coordinate neighborhood of $M$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n\geq 1$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes. Here $P$ is defined by [13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.1}$$

for all $X,Y,Z\in\chi(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor of type $(0,2)$. In fact, $M$ is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Recently B. Prasad [12] introduced a new type of curvature tensor which is known as pseudo projective curvature tensor and is defined by

$$\begin{aligned}\bar{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{r}{(2n+1)}[\frac{a}{2n}+b][g(Y,Z)X - g(X,Z)Y], \end{aligned}\tag{1.2}$$

where $R$ is the curvature tensor, $S$ is the Ricci tensor of type $(0,2)$, $r$ is the scalar curvature of the manifold and $a,b$ are non-zero constant. For a Riemannian manifold if $a+2b=0$, then $\bar{P}=aP$. Thus in a Riemannian manifold projective curvature tensor is a particular case of pseudo projective curvature tensor.

Now we define extended pseudo projective curvature tensor $\bar{P}^e$ of type $(1,3)$ as follows:

$$\begin{aligned}\bar{P}^e(X,Y)Z &= \bar{P}(X,Y)Z - \eta(X)\bar{P}(\xi,Y)Z \\ &\quad -\eta(Y)\bar{P}(X,\xi)Z - \eta(Z)\bar{P}(X,Y)\xi. \end{aligned}\tag{1.3}$$

From the definition of extended pseudo projective curvature tensor it is clear that if the manifold is pseudo projectively flat then it is also extended pseudo projectively flat, but converse is not true, in general.

In 2006, Venkatasha and C. S. Bagewadi [16] studied pseudo projective $\varphi$-recurrent Kenmotsu manifold. Later C. S. Bagewadi et. al. [5] studied pseudo projective

curvature tensor of a contact metric manifold. In ([10], [11]), the authors have extended this notion to Kenmotsu manifolds, Sasakian manifolds, LP-Sasakian manifolds and $(k,\mu)$-contact manifolds and obtain the condition for these manifolds to be of Einstein, $\eta$-Einstein and pseudo projectively flat.
Motivated by the above studies, in this paper we study extended pseudo projectively flat and the curvature condition $\bar{P}^e \cdot S = 0$ in generalized $(k,\mu)$-contact manifold. Besides this in this paper we study $\varphi$-projectively semisymmetric generalized $(k,\mu)$-contact manifolds.
The paper is organized as follows:
After brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. In section 3, we consider $\varphi$-semisymmetric generalized $(k,\mu)$-contact manifolds and prove that a $\varphi$-projectively semisymmetric non-Sasakian generalized $(k,\mu)$-contact manifold is an $N(k)$-contact manifold. Section 4 is devoted to study extended pseudo projectively flat generalized $(k,\mu)$-contact manifolds. Finally, we consider $(k,\mu)$-contact manifold satisfying $\bar{P}^e \cdot S = 0$. As a consequence we obtain several corollaries.

## 2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $M$ is said to admit an almost contact metric structure if it admits a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying ([2], [3])

$$(a)\ \ \varphi^2 = -I + \eta \otimes \xi, \ \ (b)\ \ \eta(\xi) = 1, \ \ (c)\ \ \varphi\xi = 0, \ \ (d)\ \ \eta \circ \varphi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the almost complex structure $J$ on the product manifold is defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is the smooth function on $M \times \mathbb{R}$. Let $g$ be a compatible Riemannian metric with almost contact structure $(\varphi, \eta, \xi)$, that is,

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then $M$ becomes an almost contact metric structure $(\varphi, \xi, \eta, g)$. From (2.1) it can be easily seen that

$$(a) \quad g(X, \varphi Y) = -g(\varphi X, Y), \qquad (b) \quad g(X,\xi) = \eta(X). \tag{2.3}$$

for all vector fields $X, Y$. An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X,Y), \tag{2.4}$$

for all vectors fields $X, Y$. The 1-form $\eta$ is called a contact form and $\xi$ is its characteristic vector field. We define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2}\pounds_\xi \varphi$, where

$\pounds$ denote the Lie derivative. Then $h$ is symmetric and satisfies the conditions $h\varphi = -\varphi h$, $Tr.h = Tr.\varphi h = 0$ and $h\xi = 0$. Also

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{2.5}$$

holds in a contact metric manifold. A normal contact manifold is a Sasakian manifold. An almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.6}$$

where $X, Y \in \chi(M)$ and $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ for which $\xi$ is a Killing vector is said to be a $K$-contact metric manifold. A Sasakian manifold is $K$-contact but not conversely. However a 3-dimensional $K$-contact manifold is Sasakian. It is known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$. On the other hand, on a Sasakian manifold the following relation holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case: D.E. Blair, T. Koufogiorgos and B. J. Papantoniou [1] introduced the $(k,\mu)$-nullity distribution on a contact metric manifold and gave several reasons for studying it.

The $(k,\mu)$-nullity distribution $N(k,\mu)$ [1] of a contact metric manifold $M$ is defined by

$$\begin{aligned} N(k,\mu) \quad : \quad & p \longrightarrow N_p(k,\mu) \\ & = \{W \in T_pM : R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. Thus we have

$$R(X,Y)\xi = (kI + \mu h)R_0(X,Y)\xi, \tag{2.8}$$

where $R_0(X,Y)\xi = \eta(Y)X - \eta(X)Y$ .

A contact metric manifold $M$ with $\xi \in N(k,\mu)$ is called a $(k,\mu)$-contact metric manifold. If $\mu = 0$, the $(k,\mu)$-nullity distribution reduces to $k$-nullity distribution [15]. The $k$-nullity distribution $N(k)$ of a Riemannian manifold is defined by [15]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

$k$ being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold a $N(k)$-contact metric manifold. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.

Moreover, in a $(k,\mu)$ -contact manifold the following relation holds :

$$h^2 = (k-1)\varphi^2, k \leq 1. \tag{2.9}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.10}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ &\quad +[2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), n \geq 1. \end{aligned} \tag{2.11}$$

$$\begin{aligned} QX &= [2(n-1) - n\mu]X + [2(n-1)+\mu]hX \\ &\quad +[2(1-n) + n(2k+\mu)]\eta(X)\xi, n \geq 1. \end{aligned} \tag{2.12}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.13}$$

$$Q\xi = 2nk\xi, \tag{2.14}$$

$$(\nabla_X \eta)Y = g(X + hX, \varphi Y), \tag{2.15}$$

$$\begin{aligned} (\nabla_X h)Y &= (1-k)g(X,\varphi Y)\xi + g(X, h\varphi Y)\xi + \eta(Y)(h(\varphi X + \varphi hX)) \\ &\quad -\mu\eta(X)\varphi hY. \end{aligned} \tag{2.16}$$

A generalized $(k,\mu)$-contract manifold $M^3$ $(\varphi,\xi,\eta,g)$ is a $(k,\mu)$-contract manifold in which $k,\mu$ are smooth functions $M^3$. A generalized $(k,\mu)$ contract metric manifold does not exist for dimension greater then three [9]. In a generalized $(k,\mu)$-contact metric manifold $M^3(\varphi,\xi,\eta,g)$, beside the relations (2.1) − (2.15) the following relations also hold [9]

$$\xi k = 0, \tag{2.17}$$

$$h \; grad \; \mu = grad \; k. \tag{2.18}$$

Generalized $(k,\mu)$-contact manifolds have been studied by several authors such as Gouli-Andreou et al. [8], Yildiz et. al. [18], De et al. [7] and many others.

**Definition 2.1.** *A generalized $(k,\mu)$-contact manifold is said to be $\varphi$-projectively semisymmetric if*

$$(P(X,Y)\cdot\varphi)Z = 0,$$

*for all smooth vector fields $X, Y, Z$.*

Now we state the following Lemma and Proposition which will be used latter.

**Lemma 2.2.** *[4] Let $M^3$ be a contact metric manifold with contact metric structure $(\varphi, \xi, \eta, g)$. Then the following conditions are equivalent:*
*(a) $M^3$ is $\eta$-Einstein manifold,*
*(b) $Q\varphi = \varphi Q$,*
*(c) $\xi$ belongs to the $k$-nullity distribution.*

**Proposition 2.3.** *[17] In a non-Sasakian $(k, \mu)$-contact metric manifold, the following are equivalent :*
*(a) $\eta$-Einstein manifold,*
*(b) $Q\varphi = \varphi Q$.*

## 3. $\varphi$-projectively Semisymmetric Generalized $(k, \mu)$-Contact Manifolds

In this section we deal with $\varphi$-projectively semisymmetric generalized $(k, \mu)$-contact manifolds. Suppose

$$(P(X, Y) \cdot \varphi)W = 0.$$

Then

$$P(X, Y)\varphi W - \varphi(P(X, Y)W) = 0. \tag{3.1}$$

Using (1.1), (2.8) and (2.11) in (3.1) yields

$$\begin{aligned} 0 \;=\;& \{(1-k)[g(\varphi Y, W)\eta(X) - g(\varphi X, W)\eta(Y)] - (1-\mu)[g(\varphi hY, W)\eta(X) \\ & -g(\varphi hX, W)\eta(Y)]\}\xi - g(Y + hY, W)(\varphi X, \varphi hX) \\ & +g(X + hX, W)(\varphi Y, \varphi hY) - g(\varphi Y + \varphi hY, W)(X + hX) \\ & +g(\varphi X + \varphi hX, W)(Y + hY) - \eta(W)\{(1-k)[\eta(X)\varphi Y - \eta(Y)\varphi X] \\ & +(1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]\} - \frac{1}{2}[S(Y, \varphi W)X - S(X, \varphi W)Y \\ & -S(Y, W)\varphi X + S(X, W)\varphi Y]. \end{aligned} \tag{3.2}$$

Putting $Y = W = \xi$ in the above equation yields

$$\mu(\varphi hX) = 0. \tag{3.3}$$

Taking inner product with $Y$ in (3.3) implies

$$\mu g(\varphi hX, Y) = 0.$$

Therefore $\mu = 0$, since $g(\varphi hX, Y) \neq 0$ for non-Sasakian generalized $(k, \mu)$-contact manifold . Therefore we can state the following:

**Theorem 3.1.** *A $\varphi$-projectively semisymmetric non-Sasakian generalized $(k, \mu)$-contact manifold is an $N(k)$-contact manifold.*

Now taking inner product with $Z$ in (3.2) yields

$$\begin{aligned}0 &= \{(1-k)[g(\varphi Y,W)\eta(X)-g(\varphi X,W)\eta(Y)]-(1-\mu)[g(\varphi hY,W)\eta(X)\\&-g(\varphi hX,W)\eta(Y)]\}\eta(Z)-g(Y+hY,W)[g(\varphi X,Z)+g(\varphi hX,Z)]\\&+g(X+hX,W)[g(\varphi Y,Z)+g(\varphi hY,Z)]-g(\varphi Y+\varphi hY,W)[g(X,Z)\\&+g(hX,Z)]+g(\varphi X+\varphi hX,W)[g(Y,Z)+g(hY,Z)]\\&-\eta(W)\{(1-k)[\eta(X)g(\varphi Y,Z)-\eta(Y)g(\varphi X,Z)]+(1-\mu)[\eta(X)g(\varphi hY,Z)\\&-\eta(Y)g(\varphi hX,Z)]\}-\frac{1}{2}[S(Y,\varphi W)g(X,Z)-S(X,\varphi W)g(Y,Z)\\&-S(Y,W)g(\varphi X,Z)+S(X,W)g(\varphi Y,Z)].\end{aligned}\tag{3.4}$$

Contracting $Y$ and $Z$ in the above equation yields

$$\begin{aligned}0 &= (k-3)g(\varphi X,W)+(2+\mu)g(\varphi hX,W)-2g(h\varphi W,hX)+S(X,\varphi W)\\&+\frac{1}{2}S(\varphi X,W).\end{aligned}\tag{3.5}$$

Replacing $W$ by $\varphi W$ in the above equation we have

$$\begin{aligned}0 &= (k-3)g(\varphi X,\varphi W)+(2+\mu)g(\varphi hX,\varphi W)+2g(\varphi hW,\varphi hX)+S(\varphi X,\varphi W)\\&+\frac{1}{2}S(\varphi X,\varphi W).\end{aligned}\tag{3.6}$$

Using (2.2) and (2.11) in (3.6) implies,

$$\begin{aligned}0 = &\ (k-3-2\mu)[g(X,W)-\eta(X)\eta(W)]+(3\mu+1)g(hX,W)\\&+g(h^2W,X).\end{aligned}\tag{3.7}$$

Using (2.9) in (3.7) yields

$$\begin{aligned}0 &= (k-3-2\mu)[g(X,W)-\eta(X)\eta(W)]+(3\mu+1)g(hX,W)\\&+(k-1)g(\varphi^2W,X).\end{aligned}\tag{3.8}$$

Using (2.1) in (3.8) we get

$$-2(\mu+1)g(X,W)+2(\mu+1)\eta(X)\eta(W)+(3\mu+1)g(hX,W)=0.\tag{3.9}$$

Putting the value of $g(hX,W)$ from (2.11) in the above equation yields

$$S(X,W)=\left(\frac{\mu(5\mu+3)}{3\mu+1}\right)g(X,W)+[2k+\frac{\mu(\mu-1)}{3\mu+1}]\eta(X)\eta(W).$$

Thus we can state the following:

**Theorem 3.2.** *A $\varphi$-projectively semisymmetric generalized $(k,\mu)$-contact manifold reduces to an $\eta$-Einstein manifold.*

Thus in virtue of Prop 2.1 and Theorem 3.2 we can state the following:

**Corollary 3.3.** *Let $M$ be a non-Sasakian $\varphi$-projectively semisymmetric generalized $(k,\mu)$-contact manifold. Then the Ricci operator commutes with $\varphi$. That is, $Q\varphi = \varphi Q$.*

Suppose $Q\varphi = \varphi Q$. Then from (2.12)

$$\mu h\varphi = 0. \tag{3.10}$$

Therefore either $\mu = 0$ or $h\varphi = 0$. Suppose $h\varphi = 0$, operating $h$ both sides of this equation and using (2.9) we have $(k-1)\varphi X = 0$, that is, $k = 1$ and consequently the manifold becomes Sasakian. Thus for a non-Sasakian generalized $(k,\mu)$-contact manifold, applying Lemma 2.1 we can state the following:

**Corollary 3.4.** *Let $M^3(\varphi,\xi,\eta,g)$ be a $\varphi$-projective semisymmetric non-Sasakian generalized $(k,\mu)$ contact metric manifold. Then the following conditions are equivalent:*
*$(a)$ $M^3$ is $\eta$-Einstein manifold,*
*$(b)$ $Q\varphi = \varphi Q$,*
*$(c)$ $\xi$ belongs to the $k$-nullity distribution.*

## 4. Extended Pseudo Projectively Flat Generalized $(k,\mu)$-Contact Manifolds

In this section we study extended pseudo projectively flat generalized $(k,\mu)$-contact manifolds. From (2.8) we have

$$\begin{aligned} R(\xi,X)\xi &= R_0(\xi,(kI+\mu h)X)\xi \\ &= \eta((kI+\mu h)X)\xi - \eta(\xi)(kI+\mu h)X \\ &= k\eta(X)\xi - kX - \mu hX. \end{aligned} \tag{4.1}$$

Putting $Z = \xi$ in (1.2) we have

$$\begin{aligned} \bar{P}(X,Y)\xi &= aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] \\ &\quad -\frac{r}{(2n+1)}[\frac{a}{2n} + b][g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{4.2}$$

Using (2.11) in (4.2) we obtain

$$\begin{aligned} \bar{P}(X,Y)\xi &= a(kI+\mu h) - 2kb - \frac{r}{(2n+1)}[\frac{a}{2n} + b])R_0(X,Y)\xi. \\ &= [(a+2nb)(k - \frac{r}{(2n+1)})I + a\mu h]R_0(X,Y)\xi. \end{aligned} \tag{4.3}$$

Also from the above equation,

$$\bar{P}(\xi,X) = [(a+2nb)(k - \frac{r}{2n(2n+1)})]R_0(X,\xi) + a\mu R_0(\xi,hX). \tag{4.4}$$

Consequently, we have

$$\bar{P}(\xi, X)\xi = [(a+2nb)(k - \frac{r}{2n(2n+1)})](\eta(X)\xi - X) - a\mu hX. \quad (4.5)$$

Now putting $Y = Z = \xi$ in (1.3) we obtain

$$\begin{aligned} 0 = P^e(X,\xi)\xi &= \bar{P}(X,\xi)\xi - \eta(X)\bar{P}(\xi,\xi)\xi \\ &\quad -\eta(\xi)\bar{P}(X,\xi)\xi - \eta(\xi)\bar{P}(X,\xi)\xi. \end{aligned} \quad (4.6)$$

Therefore,

$$0 = -\bar{P}(X,\xi)\xi. \quad (4.7)$$

Using (4.3) in (4.7) yields

$$0 = (a+2b)(k - \frac{r}{6})(\eta(X)\xi - X) - a\mu hX. \quad (4.8)$$

Using (2.1) in (4.8) implies

$$(a+2b)(\frac{2k+\mu}{3})\varphi^2 X = a\mu hX. \quad (4.9)$$

Using (2.9) in (4.9) we have

$$(a+2b)(2k+\mu)h^2 X = 3a\mu(k-1)hX. \quad (4.10)$$

Therefore,

$$h^2 = \frac{3a\mu(k-1)}{(a+2b)(2k+\mu)}h.$$

Taking trace on both sides of the above equation we have

$$Tr.h^2 = 0. \quad (4.11)$$

Using (2.9) in (4.11) yields

$$k = 1.$$

Thus we have:

**Theorem 4.1.** *An extended pseudo projectively flat generalized $(k,\mu)$-contact manifold is a Sasakian manifold.*

As a corollary we obtain the following:

**Corollary 4.2.** *A pseudo projectively flat generalized $(k,\mu)$-contact manifold is a Sasakian manifold.*

## 5. Generalized $(k,\mu)$-Contact Manifold Satisfying $\bar{P^e}\cdot S=0$

In this section we characterize a generalized $(k,\mu)$-contact manifold satisfying the curvature condition $(\bar{P^e}(X,Y)\cdot S)(U,V)=0$.

Now using (2.3), (2.11) and (2.12) we have,

$$\begin{aligned}\bar{P^e}(\xi,Y)U &= (a+2b)[(k-\frac{r}{6})\eta(U)Y-(k+\frac{r}{3})\eta(X)\eta(Y)\xi \\ &\quad +a\mu\eta(U)hY]. \end{aligned}\tag{5.1}$$

Putting $U=\xi$ in the above equation we obtain,

$$\begin{aligned}\bar{P}(\xi,Y)\xi &= (a+2b)[(k-\frac{r}{6})Y-(k+\frac{r}{3})\eta(Y)\xi \\ &\quad +a\mu hY]. \end{aligned}\tag{5.2}$$

Suppose

$$(\bar{P^e}(X,Y)\cdot S)(U,V)=0.$$

Then

$$S(\bar{P^e}(X,Y)U,V))+S(U,\bar{P^e}(X,Y)V)=0. \tag{5.3}$$

Putting $X=V=\xi$ in (5.3) we have

$$S(\bar{P^e}(\xi,Y)U,\xi))+S(U,\bar{P^e}(\xi,Y)\xi)=0. \tag{5.4}$$

Using (5.1) and (5.2), in (5.4), we get

$$\begin{aligned}0 &= -(a+2b)(\frac{rk}{3}-2k^2)\eta(Y)\eta U+(a+2b)(k+\frac{r}{6})S(Y,U)] \\ &\quad +a\mu S(hX,Y). \end{aligned}\tag{5.5}$$

From (2.11) in (5.5) yields

$$\begin{aligned}0 &= -(a+2b)(\frac{rk}{3}-2k^2)\eta(Y)\eta(U)+(a+2b)(k+\frac{r}{6})S(Y,U) \\ &\quad +a\mu[-\mu g(hY,U)+\mu g(h^2Y,U)]. \end{aligned}\tag{5.6}$$

Using (2.1) and (2.11) in (5.6) yields

$$S(Y,U)=\alpha g(Y,U)+\beta\eta(X)\eta(Y),$$

where

$$\alpha=\frac{3ak}{(a+2b)(4k-\mu)-3a\mu}$$

and

$$\beta=\frac{-k(a+2b)(3a-2\mu-4k)}{(a+2b)(4k-\mu)-3a\mu}.$$

Thus we can state the following:

**Theorem 5.1.** *A generalized $(k,\mu)$-contact manifold satisfying the curvature condition $\bar{P^e}\cdot S=0$ is an $\eta$-Einstein manifold, if $a+2b\neq 0$ and Einstein manifold if $a+2b=0$ .*

## Acknowledgments

The authors are thankful to the referee for his/her valuable suggestions towards improvement of the paper.

## References

1. Blair, D. E., Koufogiorgos, T., Papantoniou, B. J., *Contact metric manifolds satisfying a nullity condition,* Israel J. Math. 91, 189-214, (1995).

2. Blair, D. E., *Contact manifolds in Riemannian geomatry,* Lecture notes in math., springer-verlag, 509, (1976).

3. Blair, D. E., *Riemannian geomatry of contact and Sympletic Manifolds ,* Progress in Math., Birkhuser, Boston, 203, (2002).

4. Blair, D. E.,Koufogiorgos, T., Sharma, R., *A classification of* 3*-dimensional contact metric manifolds with* $Q\varphi = \varphi Q$*,* Kodai Math. J., 13, 391-401, (1990).

5. Bagewadi, C. S., Prakasha, D. G., Venkatasha., *On pseudo projective curvature tensor of a contact metric manifold,* Sut. J. Math., 43, 115-126, (2007).

6. Boeckx, E., *A full clasification of contact metric* $(k, \mu)$*- spaces,* Illinois J. Math., 44, 212-219, (2000).

7. De, U. C., Samui. S., *Quasi-conformal curvature tensor on generalized* $(k, \mu)$*-contact metric manifolds,* Acta Univ. Apulensis Math. Inform., 40, 291-303, (2014).

8. Gouli-Andreou., Xenos, P. J., *A class of contact metric 3-manifolds with* $\xi \in (k, \mu)$ *and* $k, \mu$ *functions,* Algebras, Groups and Geom., 17, 401-407, (2000).

9. Koufogiorgos, T., Tsichlias, C., *On the existance of new class of Contact metric manifolds,* Canad. Math. Bull., XX(Y), 1-8, (2000).

10. Maralabhavi, Y. B., Shivaprasanna, G. S., *On pseudo projective curvature tensor in LP-Sasakian manifolds,* International Mathematical Forum., 7, 1121-1128, (2012).

11. Nagaraja, H. G., Somashekara, G., *On pseudo projective curvature tensor in Sasakian manifolds,* Int. J. Contemp. Math. Sciences., 6, 1319-1328, (2011).

12. Prasad, B., *A pseudo projective curvature tensor on a Riemannian manifold,* Bull. Cal. Math. Soc., 94, 163-166, (2002).

13. Soos, G., *Uber die geodatischen Abbildungen von Riemannaschen Raumen auf projektiv symmetrische Riemannasche Rauume,* Acta. Math. Acad. Sci. Hunger., 9, 359-361, (1958).

14. Tanno, T., *The topology of contact Riemannian-manifolds*, Illinois J. Math., 12, 700-717, (1968).

15. Tanno, S., *Ricci curvature of contact Riemannian manifolds*, Tohoku Math. J., 40, 441-448, (1988).

16. Venkatasha and Bagewadi, C. S., *On pseudo projective* $\varphi$*-recurrent Kenmotsu manifolds,* Soochow Journal of Mathematics., 32, 433-439, (2006).

17. Yildiz, A., De, U. C., *A classification of* $(k, \mu)$*-contact metric manifols,* Comm. Korean. Math. Soc., 27, 327-339, (2012).

18. Yildiz, A., De, U. C. and Cetinkaya, A., *On some classes of* 3*-dimensional generalized* $(k, \mu)$*-contact metric manifols,* Turkish J. Math., 39, 356-368, (2015).

*Pradip Majhi,*
*Department of Pure Mathematics*
*University of Calcutta*
*35, Ballygunge Circular Road*
*Kol- 700019, West Bengal, India.*
*E-mail address:* mpradipmajhi@gmail.com, pmpm@caluniv.ac.in

*and*

*Gopal Ghosh,*
*Department of Pure Mathematics*
*University of Calcutta*
*35, Ballygunge Circular Road*
*Kol- 700019, West Bengal, India.*
*E-mail address:* ghoshgopal.pmath@gmail.com