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On New Applications of Fractional Calculus

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abstract: In the present paper author derive a number of integrals concerning various special functions which are applications of the one of Osler result. Osler provided extensions to the familiar Leibniz rule for the nth derivative of product of two functions.

Key Words: Fractional calculus, generalized Leibniz rule, Generalized hypergeometric series.

Contents

1. Introduction

One of the most frequently encountered tools in the theory of fractional calculus (that is, differentiation and integration of an arbitrary real or complex order) is furnished by the familiar differentiable operator ${}_aD_x^q$ defined and represented by Oldham and Spanier [18]:

$$ {}_aD_x^q = \frac{1}{\Gamma(-q)} \int_a^x (x-y)^{-q-1} f(y)dy \qquad (a \in \mathbb{C}, \Re(q) < 0)\,, \tag{1.1}$$

and

$$\begin{aligned} {}_aD_x^q &= D_x^n \left[\frac{1}{\Gamma(n-q)} \int_a^x (x-y)^{n-q-1} f(y)dy \right] \\ &= D_x^n \left[{}_aD_x^{q-n} f(x) \right], \end{aligned} \tag{1.2}$$

where n is the least positive integer such that $n > q$.

The operator ${}_aD_x^q$ provides a generalization of the familiar differential and integral operator, viz., $D \left(\equiv \frac{d}{dx}\right)$ and D^{-1}. For $a = 0$ the operator D_x^q is given by:

$$D_x^q =_a D_x^q \qquad (q \in \mathbb{C})\,, \tag{1.3}$$

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corresponding essentially to the classical Riemann-Liouville fractional derivative (or integral) of order q (or $-q$). Moreover, when $a \to \infty$, Equation (1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order q (or $-q$).

In recent years there has appeared a great deal of literature discussing the application of the aforementioned fractional calculus operators in a number of areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, summation of series, et cetera) and now stands on fairly firm footing through the research contribution of various authors (cf., e.g., [1,2,3,4,5,6, 7,8,9,10,11,12,14,16,17]). Here, we aim derive some summations series concerning generalized hypergeometric function which are applications of the one of Samko result.

The integral analog of the generalized Leibnitz rule in the theory of fractional calculus was given by Osler [19] in the following form:

$$D_z^{\alpha}\left[u(z)\,v(z)\right] = \int_{-\infty}^{\infty} \binom{\alpha}{\eta} D_z^{\alpha-\eta}\left[u(z)\right] D_z^{\eta}\left[v(z)\right] \quad (\alpha, \eta \in \mathbb{C}) \tag{1.4}$$

The generalized hypergeometric function of one variable viz., ${}_pF_q[.;.;z]$ defined as (see, for details, Srivastava and Karisson 1985, [20, p.19]):

$$\begin{aligned} {}_pF_q[z] &= {}_pF_q\left[\begin{array}{c} (a_p); \\ (b_q); \end{array} z\right] \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.5}$$

provided $p \leqq$, $p = q + 1$ and $|z| < 1$.

2. Main Results

Here, we establish certain summations of series formulas for the generalized hypergeometric function.

1. For $\Re(\lambda + \mu) > 1$

$$\begin{aligned} &\int_{-\infty}^{\infty} \binom{\alpha}{\eta} \frac{\left[{}_1F_1(\mu; \mu - \alpha + \eta; iaz) + {}_1F_1(\mu; \mu - \alpha + \eta; -iaz)\right]}{\Gamma(\lambda - \eta)\,\Gamma(\mu - \alpha + \eta)} \\ &\qquad \left[{}_1F_1(\lambda; \lambda - \eta; ibz) + {}_1F_1(\lambda; \lambda - \eta; -ibz)\right] \\ &= \frac{\Gamma(\mu + \lambda - 1)}{\Gamma(\mu)\,\Gamma(\lambda)\,\Gamma(\mu + \lambda - \alpha - 1)} \left[{}_1F_1(\mu + \lambda - 1; \mu + \lambda - \alpha - 1; i(a + b)z)\right. \\ &+ {}_1F_1(\mu + \lambda - 1; \mu + \lambda - \alpha - 1; -i(a + b)z) \\ &+ {}_1F_1(\mu + \lambda - 1; \mu + \lambda - \alpha - 1; i(a - b)z) \\ &\left.+ {}_1F_1(\mu + \lambda - 1; \mu + \lambda - \alpha - 1; -i(a - b)z)\right] \end{aligned} \tag{2.1}$$

2. For $\Re(\lambda+\mu)>1$

$$\int_{-\infty}^{\infty}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\frac{[{}_1F_1(\mu;\mu-\alpha+\eta;iaz)-{}_1F_1(\mu;\mu-\alpha+\eta;-iaz)]}{\Gamma(\lambda-\eta)\,\Gamma(\mu-\alpha+\eta)}$$
$$[{}_1F_1(\lambda;\lambda-\eta;ibz)+{}_1F_1(\lambda;\lambda-\eta;-ibz)]$$
$$=2\,\frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu)\,\Gamma(\lambda)\,\Gamma(\mu+\lambda-\alpha-1)}\,[{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;i(a+b)z)$$
$$-\,{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;-i(a+b)z)$$
$$+\,{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;i(a-b)z)$$
$$-{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;-i(a-b)z)] \tag{2.2}$$

3. For $\Re(\lambda+\mu)>1$

$$\int_{-\infty}^{\infty}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\frac{{}_2F_2(a+m+1,\mu;a+1,\mu-\alpha+\eta;-k_1z)}{\Gamma(\lambda-\eta)\,\Gamma(\mu-\alpha+\eta)}{}_1F_1(\lambda;\lambda-\eta;k_1z)$$
$$=\frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu)\,\Gamma(\lambda)\,\Gamma(\mu+\lambda-\alpha-1)}{}_2F_2(-m,\mu+\lambda-1;a+1,\mu+\lambda-\alpha-1;k_1z) \tag{2.3}$$

4. For $\Re(\lambda+\mu)>1$

$$\int_{-\infty}^{\infty}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\frac{{}_1F_1\left(\mu;\frac{1}{2},\mu-\alpha+\eta+n;\frac{-a^2z}{4}\right)}{\Gamma(\lambda-\eta-n)\,\Gamma(\mu-\alpha+\eta+n)}{}_1F_1\left(\lambda;\frac{1}{2},\lambda-\eta-n;\frac{-b^2z}{4}\right)$$
$$=\frac{\Gamma(\mu+\lambda-1)}{2\Gamma(\mu+\lambda-\alpha-1)}\left[{}_1F_1\left(\mu+\lambda-1;\frac{1}{2},\mu+\lambda-\alpha-1;-\frac{(a+b)^2z}{4}\right)\right.$$
$$\left.+{}_1F_1\left(\mu+\lambda-1;\frac{1}{2},\mu+\lambda-\alpha-1;-\frac{(a-b)^2z}{4}\right)\right] \tag{2.4}$$

5. For $\Re(\lambda+\mu)>1$

$$\int_{-\infty}^{\infty}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[f(\alpha,\mu,\eta,z)g(\lambda,\eta,z)]$$
$$=\left[C_1F_1\left(\mu+\lambda-1;\frac{1}{2},\mu+\lambda-\alpha-1;\frac{(a+b)^2z}{4}\right)z^{-1}\right. \tag{2.5}$$
$$\left.+D_1F_2\left(\mu+\lambda-\frac{1}{2};\frac{3}{2},\mu+\lambda-\alpha-\frac{1}{2};\frac{(a+b)^2z}{4}\right)z^{-1/2}\right],$$

where

$$
\begin{aligned}
f(\alpha,\mu,\eta,z) =& \frac{\Gamma(\mu)}{(\mu-\alpha+\eta)}\,{}_1F_2\left(\mu;\frac{1}{2},\mu-\alpha+\eta;\frac{a^2z}{4}\right)z^{-1/2}\\
&+\frac{a\Gamma\left(\mu+\frac{1}{2}\right)}{\left(\mu-\alpha+\eta+\frac{1}{2}\right)}\,{}_1F_2\left(\mu+\frac{1}{2};\frac{3}{2},\mu-\alpha+\eta+\frac{1}{2};\frac{a^2z}{4}\right)\\
g(\lambda,\eta,z) =& \frac{\Gamma(\lambda)}{(\lambda-\eta)}\,{}_1F_2\left(\lambda;\frac{1}{2},\lambda-\eta;\frac{b^2z}{4}\right)z^{-1/2}\\
&+\frac{b\Gamma\left(\lambda+\frac{1}{2}\right)}{\left(\lambda-\eta+\frac{1}{2}\right)}\,{}_1F_2\left(\lambda+\frac{1}{2};\frac{3}{2},\lambda-\eta+\frac{1}{2};\frac{b^2z}{4}\right)\\
C =& \frac{\Gamma(\mu+\lambda-1)}{(\mu+\lambda-\alpha-1)}\\
D =& \frac{(a+b)\Gamma\left(\mu+\lambda-\frac{1}{2}\right)}{\left(\mu+\lambda-\alpha-\frac{1}{2}\right)}
\end{aligned}
$$

3. Method of derivation

Let $\mathbb{L}$ and $\mathbb{R}$ are the left-hand and right-hand sides of the result (1.4). By choosing specific following values of $u(z)$ and $v(z)$

$$u(z) = z^{\mu-1}\cos az \tag{3.1}$$

$$v(z) = z^{\lambda-1}\cos bz \tag{3.2}$$

Then LHS of (1.4), becomes

$$\mathbb{L} = D_z^{\alpha}\left[u(z)v(z)\right] \tag{3.3}$$

$$D_z^{\alpha}\left[\frac{1}{2}z^{\mu+\lambda-2}\{\cos(a+b)z+\cos(a-b)z\}\right] \tag{3.4}$$

and using known results [15, p.189], we obtain

$$
\begin{aligned}
= \quad & \frac{1}{4}\frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu+\lambda-\alpha-1)}\left[{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;i(a+b)z)\right. && (3.5)\\
&+{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;-i(a+b)z) && (3.6)\\
&+{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;i(a-b)z) && (3.7)\\
&\left.+{}_1F_1(\mu+\lambda-1;\mu+\lambda-\alpha-1;-i(a-b)z)\right]z^{\mu+\lambda-\alpha-2} && (3.8)
\end{aligned}
$$

For R. H. S., we similarly have

$$\mathbb{R} = \int_{-\infty}^{\infty}\binom{\alpha}{\eta}D_z^{\alpha-\eta}\left[z^{\mu-1}\cos az\right]D_z^{\eta}\left[z^{\lambda-1}\cos bz\right], \quad \text{where} \tag{3.9}$$

$$D_z^{\alpha-\eta}\left[z^{\mu-1}\cos az\right] = \frac{\Gamma(\mu)}{2\Gamma(\mu-\alpha+\eta)}\left[{}_1F_1(\mu;\mu-\alpha+\eta;iaz)\right. \tag{3.10}$$

$$\left.+{}_1F_1(\mu;\mu-\alpha+\eta;-iaz)\right]z^{\mu-\alpha+\eta-1}. \tag{3.11}$$

$$D_z^{\eta}\left[z^{\lambda-1}\cos bz\right] = \frac{\Gamma(\lambda)}{2\Gamma(\lambda-\eta)}\left[{}_1F_1(\lambda;\lambda-\eta;ibz)+{}_1F_1(\lambda;\lambda-\eta;-ibz)\right]z^{\lambda-\eta-1} \tag{3.12}$$

By using (1.4), we get the desired result (2.1).
Summation series from (2.2) to (3.1) are similarly established by choosing;

$$\begin{cases} u(z) = z^{\mu-1}\sin az \\ v(z) = z^{\lambda-1}\cos bz \end{cases} \tag{3.13}$$

$$\begin{cases} u(z) = z^{\mu-1}L_m^{\alpha}(k_1 z)e^{-k_1 z} \\ v(z) = z^{\lambda-1}e^{k_1 z} \end{cases} \tag{3.14}$$

$$\begin{cases} u(z) = z^{\mu-1}\cos\left(az^{1/2}\right) \\ v(z) = z^{\lambda-1}\cos\left(bz^{1/2}\right) \end{cases} \tag{3.15}$$

in (1.3) and using known results [15, p.189,Eq.(26) and (32); p.193, Eq.(51), p.192, Eq.(50) and p.187, Eq. (14); p. 190,Eq. (35); p.188, Eq.(21)], respectively.

4. Special Cases

In view of the large number of parameters involved in the results established above, these integrals are capable of yielding a number of known results and new interesting results including integrals due to Arora and Koul [13, p.932, Eqs. (2.1) and (2.2)]. These are not record here for lack of space.

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References

1. Agarwal, P., *Some inequalities involving Hadmard type k-fractional integral operators*, Math. Meth. Appl. Sci., (2017). doi: 10.1002/mma.4270.

2. Agarwal, P., *Fractional integration of the product of two multivariables H-function and a general class of polynomials*, In: Adva. Appl. Math. Approx. Theo., (Springer Proc. in Mathematics and Statistics), 41, 359-374, (2013).

3. Agarwal, P., *Further results on fractional calculus of Saigo operators*, Appl. Appl. Math., 7(2), 585-594, (2012).

4. Agarwal, P., *Generalized fractional integration of the $\overline{H}$-function*, *Le Matematiche*, LXVII, 107-118, (2012).

5. Agarwal, R. P. and Agarwal, P., *Extended Caputo fractional derivative operator*, Adv. Studies Contemp. Math., 25 (3), (2015).

6. Agarwal, P. and Jain, S., *Further results on fractional calculus of Srivastava polynomials*, Bull. Math. Anal. Appl., 3(2), 167-174, (2011).

7. Agarwal, P. and Choi, J., *Fractional calculus operators and their image formulas*, J. Korean Math. Soc. 53(5), 1183-1210 (2016).

8. Agarwal, P. and Choi, J., *Certain fractional integral inequalities associated with Pathway fractional integral operators*, Bull. Korean Math. Soc, (2016).

9. Agarwal, P., Chand, M. and Singh, G., *Certain fractional kinetic equations involving the product of generalized k-Bessel function*, Alexandria Engrg. J. **55**(4), 3053–3059, (2016).

10. Agarwal, P., Dragomir, S. S., Park, J., Jain, S. *q-Integral inequalities associated with some fractional q-integral operators*, J. Inequal. Appl., (2015).

11. Agarwal, P. and Choi, J. *Certain fractional integral inequalities associated with Pathway fractional integral operators*, Bull. Korean Math. Soc., 53(1), 181-–193, (2016).

12. Agarwal, P., Tariboon, J., Ntouyas, S. K., *Some generalized Riemann-Liouville k-fractional integral inequalities*, J. Inequal. Appl., 2016:122, (2016). DOI 10.1186/s13660-016-1067-3.

13. Arora, A. K. and Koul, C. L., *Applications of fractional calculus*, Indian J. Pure Appl.Math. 18 (10), 931-937, (1987).

14. Choi, J. and Agarwal, P., *A Note on Fractional Integral Operator Associated with Multiindex Mittag-Leffler Functions*, Filomat, 30(7), 2016, 1931-1939

15. Erdelyi, A., *Tables of Integral Transforms*, Vol.2, McGraw Hill Book Co., New York, (1954).

16. Kiryakova, V, *Generalized Fractional calculus and Applications*, Pitman Research Notes in Mathematics Series 301, Longman Scientific and Technical, Harlow, Essex (Johan Wiley and Sons, New York),(1994).

17. Nishimoto, K., *Fractional Calculus*, Vols. I, II, III and IV, Descartes Press, Koriyama, (1984, 1987, 1989 and 1991)..

18. Oldham, K. B. and Spanier, J., *The Fractional Calculus*, Academic Press, New York and London, (1974).

19. Osler, T.J., *The integral analog of Leibniz rule,* Math.Comp. 26, 903-15, (1972).

20. Srivastava, H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, John Wiley and Sons, New York, (1985).

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