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Lower Bounds of Forwarding Indices of Graph Products

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ABSTRACT: In this paper, we give lower bounds of vertex and edge forwarding indices for cartesian product, join, composition, disjunction and symmetric difference of graphs. Moreover, we derive further lower bounds for several operators on connected graphs, such as subdivision graph and total graphs.

Key Words: Forwarding indices, Graph product, Subdivision graphs.

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1. Introduction

A routing R of a connected graph $G = (V, E)$ of order n consists in a set of $n(n-1)$ elementary paths $R(u, v)$ specified for all ordered pairs u, v of vertices of G.

To measure the efficiency of a routing deterministically, Chung, Coffman, Reiman and Simon [3] introduced the concept of forwarding index of a routing.

The load of a vertex v (resp. an edge e) in a given routing R of $G = (V, E)$, denoted by $\xi(G, R, v)$ (resp. $\pi(G, R, e)$), is the number of paths of R going through v (resp. e), where v is not an end vertex. The parameters

$$\xi(G, R) = \max_{v \in V(G)} \xi(G, R, v)$$

and

$$\pi(G, R) = \max_{e \in E(G)} \pi(G, R, e)$$

are defined as the vertex forwarding (resp. the edge forwarding) index of G with respect to R, and

$$\xi(G) = \min_{R} \xi(G, R) \text{ and } \pi(G) = \min_{R} \pi(G, R)$$

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are defined as the vertex forwarding (resp. the edge forwarding) index of G.

The original research of the forwarding indices is motivated by the fact that maximizing network capacity can be reduced to minimizing vertex-forwarding index or edge-forwarding index of a routing. Thus, the forwarding index problem has been studied widely by many researchers (see, for example, [1,12]).

Although, determining the forwarding index problem has been shown to be NP-complete by Saad [11]. The exact values of the forwarding index of many important classes of graphs have been computed [2,4,5,6,8]. For more complete results on forwarding indices, we can refer to the survey of Xu et Al. [15].

For a given connected graph $G = (V, E)$ of order n and number of edges m, set

$$A(G) = \frac{1}{n}\sum_{u\in V}\left(\sum_{v\in V\setminus\{u\}}(d_G(u,v)-1)\right)$$

$$B(G) = \frac{1}{m}\sum_{(u,v)\in V\times V} d_G(u,v).$$

where $d_G(u, v)$ denotes the distance from the vertex u to the vertex v in G. Let $d(G, k)$ be the number of pairs of vertices of a graph G that are at distance k.

The following bounds of $\xi(G)$ and $\pi(G)$ were first established by Chung et al. [3] and Heydemann et al. [7], respectively.

$$A(G) \leq \xi(G) \leq (n-1)(n-2)$$

$$B(G) \leq \pi(G) \leq \lfloor \frac{1}{2}n^2 \rfloor.$$

The paper is organized as follows, in section 2, we present the graph products. Our main results on graph products are presented in section 3, we compute lower bounds of forwarding indices for join, composition, disjunction and symmetric difference of graphs. In section 4, lower bounds of several operators on connected graphs, such as subdivision graph and total graphs are computed.

2. Graph Products

The Cartesian product $G \times H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$, and any two vertices (a, b) and (u, v) are adjacent in $G \times H$ if and only if either $a = u$ and b is adjacent with v, or $b = v$ and a is adjacent with u, see [10] for details.

The join $G = G + H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$.

The composition $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2), [10].

The disjunction $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent with (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$.

The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G) or\ u_2v_2 \in E(H)\ but\ not\ both\}$.

The following lemma is related to distance properties of some graph products.

Lemma 2.1. *(Hossein-Zadeh et Al. [9])*
Let G and H be graphs. Then we have:

1. $|V(G \times H)| = |V(G \vee H)| = |V(G[H])| = |V(G \oplus H)| = |V(G)| \times |V(H)|$,
$|E(G \times H)| = |E(G)| \times |V(H)| + |V(G)| \times |E(H)|$,
$|E(G + H)| = |E(G)| + |E(H)| + |V(G)|.|V(H)|$,
$|E(G[H])| = |E(G)|.|V(H)|^2 + |E(H)|.|V(G)|$,
$|E(G \vee H)| = |E(G)|.|V(H)|^2 + |E(H)|.|V(G)|^2 - 2|E(G)|.|E(H)|$,
$|E(G \oplus H)| = |E(G)|.|V(H)|^2 + |E(H)|.|V(G)|^2 - 4|E(G)|.|E(H)|$.

2. *$G \times H$ is connected if and only if G and H are connected.*

3. *If (a, c) and (b, d) are vertices of $G \times H$ then $d_{G\times H}((a, c), (b, d)) = d_G(a, b) + d_H(c, d)$.*

4. *The Cartesian product, join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except from composition.*

5. $d_{G+H}(u, v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G)\ or\ uv \in E(H)\ or\ (u \in V(G), v \in V(H)) \\ 2 & otherwise \end{cases}$

6. $d_{G[H]}((a, b), (c, d)) = \begin{cases} d_G(a, c) & a \neq c \\ 0 & a = c\ \&\ b = d \\ 1 & a = c\ \&\ bd \in E(H) \\ 2 & a = c\ \&\ bd \notin E(H) \end{cases}$

7. $d_{G\vee H}((a, b), (c, d)) = \begin{cases} 0 & a = c\ \&\ b = d \\ 1 & ac \in E(G)\ or\ bd \in E(H) \\ 2 & otherwise \end{cases}$

8. $d_{G\oplus H}((a, b), (c, d)) = \begin{cases} 0 & a = c\ \&\ b = d \\ 1 & ac \in E(G)\ or\ bd \in E(H)\ but\ not\ both \\ 2 & otherwise \end{cases}$

3. Main results

In this section, some exact formulae for expressions $A(G)$ and $B(G)$ of the Cartesian product, composition, join, disjunction and symmetric difference of graphs are presented.

The Forwarding indices of the Cartesian product graphs were studied in [14]. In the following propositions, we compute the lower bounds of the forwarding indices $A(G)$ and $B(G)$ for known product graphs.

Theorem 3.1. *Let G and H be connected graphs. Then*
$A(G+H) = |E(\overline{G})| + |E(\overline{H})|$,
$$B(G+H) = \frac{2.(|E(\overline{G})| + |E(\overline{H})|)}{|E(G)| + |E(H)| + |V(G)|.|V(H)|} + 1.$$

Proof. By Lemma 2.1, we have:
$$A(G+H) = \frac{1}{|V(G+H)|}\sum_{u\in V}\left(\sum_{v\in V\setminus\{u\}} d_{G+H}(u,v) - 1\right),$$
$$A(G+H) = \frac{1}{|V(G+H)|}\sum_{u\in V}(d(G+H,1)(1-1) + d(G+H,2)(2-1)),$$
$$A(G+H) = \frac{1}{|V(G+H)|}\sum_{u\in V}|E(\overline{H})| + |E(\overline{G})|,$$ and hence the result.

Further, we have that
$$B(G+H) = \frac{1}{|E(G+H)|}\sum_{(u,v)\in V\times V} d_{G+H}(u,v),$$
$$B(G+H) = \frac{1}{|E(G+H)|}(d(G+H,1) + 2.d(G+H,2)),$$
$$B(G+H) = \frac{1}{|E(G+H)|}(|E(G)|+|E(H)|+|V(G)|.|V(H)|+2.|E(\overline{H})|+2.|E(\overline{G})|),$$
proving the result. □

Corollary 3.2. *Let $nG = G_1 + G_2 + \cdots + G_n$ denotes the join of n copies of G, then*
$$A(nG) = \sum_{i=1}^{n}|E(\overline{G_i})|, \text{ and } B(nG) = \frac{2.\sum_{i=1}^{n}|E(\overline{G_i})|}{|E(nG)|} + 1.$$

Theorem 3.3. *Let G and H be connected graphs. Then*
$A(G\vee H) = |V(H)||E(\overline{G})| + |V(G)||E(\overline{H})| + 2.|E(\overline{G})|.|E(\overline{H})|$,
$$B(G\vee H) = 1 + 2.\frac{A(G\vee H)}{|E(G\vee H|}.$$

Proof. By Lemma 2.1, we have:
$$A(G\vee H) = \frac{1}{|V(G\vee H)|}\sum_{u\in V}\left(\sum_{v\in V\setminus\{u\}} d_{G\vee H}(u,v) - 1\right),$$
$$A(G\vee H) = \frac{1}{|V(G\vee H)|}(d(G\vee H,1)(1-1) + d(G\vee H,2)(2-1)),$$

$$A(G \vee H) = \frac{1}{|V(G \vee H)|}(|V(H)||E(\overline{G})|$$
$$+|V(G)||E(\overline{H})| + 2.|E(\overline{G})|.|E(\overline{H})|).$$

Now, for $B(G \vee H)$, we have,

$$B(G \vee H) = \frac{1}{|E(G \vee H)|} \sum_{(u,v) \in V \times V} d_{G \vee H}(u, v),$$

$$B(G \vee H) = \frac{1}{|E(G \vee H)|}(d(G \vee H, 1) + 2.d(G \vee H, 2)),$$

$$\begin{aligned} B(G \vee H) &= \frac{1}{|E(G \vee H)|}\big(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2.|E(G)||E(H)| \\ &\quad +2.(|V(G)||E(\overline{H})| + |V(H)||E(\overline{G})| + 2.|E(\overline{G})|.|E(\overline{H})|\big), \end{aligned}$$

proving the result. □

Theorem 3.4. *Let G and H be connected graphs. Then*
$A(G \oplus H) = 2.|E(G)||E(H)| + |V(H)||E(\overline{G})| + |V(G)|.|E(\overline{H})| + 2.|E(\overline{G})||E(\overline{H})|$,
$B(G \oplus H) = 1 + 2.\dfrac{|E(\overline{G \oplus H})|}{|E(G \oplus H|}$.

Proof. By Lemma 2.1, we have:

$$A(G \oplus H) = \frac{1}{|V(G \oplus H)|} \sum_{(a,b) \in V} \left(\sum_{(c,d) \in V \setminus \{(a,b)\}} d_{G \oplus H}((a, b), (c, d)) - 1 \right),$$

$$A(G \oplus H) = \frac{1}{|V(G \oplus H)|}(d(G \oplus H, 1)(1 - 1) + d(G \oplus H, 2)(2 - 1)),$$

$$\begin{aligned} A(G \oplus H) &= \frac{1}{|V(G \oplus H)|} \sum_{(a,b) \in V} (2.|E(G)||E(H)| + |V(H)||E(\overline{G})| \\ &\quad +|V(G)|.|E(\overline{H})| + 2.|E(\overline{G})||E(\overline{H})|), \end{aligned}$$

Now, for $B(G \oplus H)$, we have,

$$B(G \oplus H) = \frac{1}{|E(G \oplus H)|} \sum_{(u,v) \in V \times V} d_{G \oplus H}(u, v),$$

$$B(G \oplus H) = \frac{1}{|E(G \oplus H)|}(d(G \oplus H, 1) + 2.d(G \oplus H, 2)),$$

$$\begin{aligned} B(G \oplus H) &= \frac{1}{|E(G \oplus H)|}(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 + 4.|E(\overline{G})|.|E(\overline{H})| \\ &\quad +2.|V(G)||E(\overline{H})| + 2.|V(H)||E(\overline{G})|), \end{aligned}$$

proving the result. □

Theorem 3.5. *Let G and H be connected graphs. Then*

$$A(G[H]) = A(G) + \frac{|E(\overline{H})|}{|V(H)|},$$

$$B(G[H]) = B(G) + \frac{1}{2|E(G[H])|}\left(|V(G)||E(H)| + 2.|V(G)||E(\overline{H})|\right).$$

Proof. By Lemma 2.1, we have:

$$A(G[H]) = \frac{1}{|V(G[H])|}\sum_{(a,b)\in V}\left(\sum_{(c,d)\in V\setminus\{(a,b)\}} d_{G[H]}((a,b),(c,d)) - 1\right),$$

$A(G[H]) = A(G) + \dfrac{1}{|V(G[H]|}|V(G)||E(\overline{H})|$, proving the result.

Moreover we have,

$$\begin{aligned}
B(G[H]) &= \frac{1}{|E(G[H])|}\sum_{((a,b),(c,d))\in V\times V} d_{G[H]}((a,b),(c,d)),\\
B(G[H]) &= \frac{1}{2.|E(G[H])|}\sum_{(a,b)\in V(G[H)}\ \sum_{(c,d)\in V(G[H)} d_{G[H]}((a,b),(c,d)),\\
B(G[H]) &= \frac{1}{2.|E(G[H])|}\sum_{(a,c)\subseteq V(G),a\neq c} d_G(a,c) + |V(G)||E(H)|\\
&\quad +2.|V(G)||E(\overline{H})|.
\end{aligned}$$

□

4. Graph operations

For a connected graph G, define four related graphs as follows

Definition 4.1. *Let G be a graph, then*

1. *$S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.*
2. *$R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.*
3. *$Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.*
4. *$T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.*

The graphs $S(G)$ and $T(G)$ are called the subdivision and total graph of G, respectively. For more details on these operations we refer the reader to [13,16].

In this section, we compute lower bounds of forwarding indices $A(G)$ and $B(G)$ for some graph operations. We use the following lemma in [16].

Lemma 4.2. *(Yan et Al 2007 [16])*
Let G be a graph, then

1. *For any $v, v' \in V(G)$, then*
$d_{S(G)}(v, v') = d_{T(G)}(v, v') = d_{R(G)}(v, v') = d_{Q(G)}(v, v') - 1 = d_G(v, v')$.
2. *For any $e, e' \in E(G)$, then*
$d_{S(G)}(e, e') = d_{T(G)}(e, e') = d_{R(G)}(e, e') - 1 = d_{Q(G)}(e, e') = d_{L(G)}(e, e')$.
3. *For any $e \in E(G)$, $v \in V(G)$, then*
$\frac{1}{2}.d_{S(G)}(e, v) + 1 = d_{T(G)}(e, v) = d_{R(G)}(e, v) = d_{Q(G)}(e, v)$.

Theorem 4.3. *If G is the connected graph with m edges and n vertices, then*

$$\begin{aligned}
&A(S(G)) = 2.A(T(G)) - \frac{mn}{m+n}, && B(S(G)) = B(T(G)) - \frac{n}{2}.\\
&A(R(G)) = A(T(G)) + \frac{m(m-1)}{2.m+n}, && B(R(G)) = B(T(G)) + \frac{m(m-1)}{2.|E(R(G))|}.\ .\\
&A(Q(G)) = A(T(G)) + \frac{n(n-1)}{2.m+n}, && B(Q(G)) = A(T(G)) + \frac{n(n-1)}{2.|E(Q(G))|}
\end{aligned}$$

Proof. By Lemma 4.2 and splitting the sum in $A(G)$ and $B(G)$ over 2-subsets $\{x, x'\}$ of $E(G) \cup V(G)$ in three ways into sums over $\{v, v'\}$ and $\{e, e'\}$, and $(e \in E(G), v \in V(G))$. □

Corollary 4.4. *Let G be a connected graph, then*

$$A(S(G)) = A(R(G)) + A(Q(G)) - \binom{m+n-1}{2},$$

$$B(S(G)) = B(R(G)) - \frac{3n+m-1}{6}.$$

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