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Dispersive Semiflows on Fiber Bundles *

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Abstract: This paper studies dispersiveness of semiflows on fiber bundles. The main result says that a right invariant semiflow on a fiber bundle is dispersive on the base space if and only if there is no almost periodic point and the semiflow is dispersive on the total space. A special result states that linear semiflows on vector bundles are not dispersive.

Key Words: Semiflows, recursiveness, dispersiveness, Poisson stability, nonwandering point.

Contents

1. Introduction

A right invariant semiflow μ_t on the total space Q of a principal bundle $Q \to B$ induces a semiflow on the base space B and a semiflow on the associated bundle $E = Q \times_G F$. Moreover, the semiflow μ_t generates a semigroup action on the typical fiber F of the associated bundle E. The relationship between these dynamics has been extensively studied in the literature. Morse decompositions and chain recurrence for semiflows on fiber bundles were investigated in [4], [5], [6], [7], [9], [10], and [13]. Lyapunov stability and attraction for semiflows on fiber bundles were studied in [11]. A generalization of Lyapunov stability and attraction for semigroup actions on fiber bundles was recently published ([3]). The present paper contributes to the studies of recursiveness and dispersiveness for semiflows on fiber bundles.

A topological method of studying dynamical concepts on nonmetric spaces was formulated by Patrão and San Martin [8]. They introduced the notion of admissible family of open coverings to study chain recurrence and Morse decomposition of semiflows on topological spaces. The concept of admissible family of open coverings was reformulated in [12] by adding the direct set property. A topological space that admits an admissible family of open covering is called an admissible space. Recently, one discoveries that a topological space is uniformizable if and only if it is admissible ([1]). In the case of a locally trivial bundle, an admissible family of

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open coverings of the total space can be constructed by means of the admissible family of open coverings of the base space, in such a way that the projection become uniformly continuous ([9]).

If $\pi : Q \to B$ is a principal bundle with structure group G and μ is a right invariant semiflow on Q, the collection of the limit sets and the prolongational limit sets is G-invariant (Proposition 3.3). It follows that every point in the fiber qG is forward Poisson stable (nonwandering) whenever q is forward Poisson stable (nonwandering). Therefore, if Q is a homogeneous space of G, all the points in Q are forward Poisson stable (nonwandering) if there is a forward Poisson stable point (nonwandering point). In particular, if Q is a compact homogeneous space of G, then every point in Q is Poisson stable. On the other hand, the semiflow is dispersive if $J(q) = \emptyset$ for some $q \in Q$, where $J(q)$ means the prolongational limit set of q. By speaking on the associated bundle $E = Q \times_G F$, a forward Poisson stable point v in the fiber F induces forward Poisson stable points $[q, v]$ in E and $\pi(q)$ in B (Theorem 3.8). This fact implies that linear semiflows on vector bundles are not dispersive. Similarly, a nonwandering point v in F induces nonwandering points $[q, v]$ in E and $\pi(q)$ in B. If the semiflow induced on the base space B is dispersive, then all the semiflows on Q, E, and F are dispersive. On the other hand, if there is no almost periodic point in B and the semiflow is dispersive on E then the semiflow on B is dispersive (Theorem 3.18).

2. Recursiveness and dispersiveness

This section contains the basic definitions and properties of recursiveness and dispersiveness for semiflows on admissible spaces.

Let $\mu : \mathbb{R}^+ \times M \to M$ be a continuous semiflow on the Hausdorff space M. We denote by $\mu_t : M \to M$ the map defined by $\mu_t(x) = \mu(t, x)$. For $t \in \mathbb{R}^+$ and $X \subset M$ we define the sets

$$\mu_t^+(X) = \bigcup_{s \geq t} \mu_s(X) \qquad \text{and} \qquad \mu_t^-(X) = \bigcup_{s \geq t} \mu_s^{-1}(X).$$

The forward and backward orbits of X are the sets $\mu^+(X) = \mu_0^+(X)$ and $\mu^-(X) = \mu_0^-(X)$, respectively. We say that X is forward invariant if $\mu^+(X) \subset X$; it is backward invariant if $\mu^-(X) \subset X$; and it is invariant if it is forward and backward invariant. The ω-*limit set* of X is defined as

$$\omega(X) = \bigcap_{t>0} \mathrm{cls}(\mu_t^+(X)),$$

and the ω^*-*limit set* of X is defined by

$$\omega^*(X) = \bigcap_{t>0} \mathrm{cls}(\mu_t^-(X)).$$

In general, $\omega(X)$ is forward invariant and $\mathrm{cls}(\mu^+(x)) = \mu^+(x) \cup \omega(x)$ for every $x \in M$.

A set $N \subset M$ is a *(forward) minimal set* if N is nonempty, closed, forward invariant, and N has no proper subset with these properties. It is well-known that a set N is a minimal set if and only if $\mathrm{cls}\left(\mu_t^+(x)\right) = N$ for all $x \in N$ and $t \in \mathbb{R}^+$. Moreover, if $X \subset M$ is a compact forward invariant set then there is a compact minimal set in X. A point $x \in M$ is said to be *almost periodic* if $\mathrm{cls}\left(\mu^+(x)\right)$ is a minimal set. In particular, singular points and periodic points are almost periodic.

Note that the singular points, the periodic trajectories, and the almost periodic trajectories of the semiflow lie in the limit sets. This property of the limit sets reveals their aspect of stability, which is stated as follows.

Definition 2.1. *A nonempty subset $X \subset M$ is said to be* forward recursive *with respect to the nonempty subset $Y \subset M$ if for every $t > 0$, $\mu_t^+(Y) \cap X \neq \emptyset$; the set X is said to be* backward recursive *with respect to the set Y if for every $t > 0$, $\mu_t^-(Y) \cap X \neq \emptyset$. A point $x \in M$ is said to be* forward Poisson stable *if every neighborhood V of x is forward recursive with respect to the single set $\{x\}$, the point x is said to be* backward Poisson stable *if every neighborhood V of x is backward recursive with respect to the set $\{x\}$. The point x is* Poisson stable *if x is forward and backward Poisson stable.*

It is easily seen that the point $x \in M$ is forward Poisson stable if and only if $x \in \omega(x)$, which is equivalent to $\omega(x) = \mathrm{cls}\left(\mu_t^+(x)\right)$ for every $t > 0$. The singular points, the periodic points, and the almost periodic points are Poisson stable.

Let $\mathcal{U}, \mathcal{V}$ be two open coverings of M. We write $\mathcal{V} \leqslant \mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. This relation is a preorder on the set of all open coverings of M. We write $\mathcal{V} \leqslant \frac{1}{2}\mathcal{U}$ if for every $V, V' \in \mathcal{V}$, with $V \cap V' \neq \emptyset$, there is $U \in \mathcal{U}$ such that $V \cup V' \subset U$.

For a covering $\mathcal{U}$ of M and a subset $X \subset M$, the *star* of X with respect to $\mathcal{U}$ is the set

$$\mathrm{St}\left[X, \mathcal{U}\right] = \bigcup \left\{U \in \mathcal{U} : X \cap U \neq \emptyset\right\}.$$

Definition 2.2. *A family $\mathcal{O}$ of open coverings of M is said to be* admissible *if it satisfies the following properties:*

1. *For each $\mathcal{U} \in \mathcal{O}$, there is $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant \frac{1}{2}\mathcal{U}$;*
2. *Let $N \subset M$ be an open set and K a compact of M contained in N. Then there is an open covering $\mathcal{U} \in \mathcal{O}$ such that $\mathrm{St}\left[K, \mathcal{U}\right] \subset N$;*
3. *For any $\mathcal{U}, \mathcal{V} \in \mathcal{O}$, there is $\mathcal{W} \in \mathcal{O}$ such that $\mathcal{W} \leqslant \mathcal{U}$ and $\mathcal{W} \leqslant \mathcal{V}$.*

The space M is said to be admissible *if it admits an admissible family of open coverings.*

As mentioned in the introduction, the topological space M is admissible if and only if it is uniformizable. The stars $\mathrm{St}\left[x, \mathcal{U}\right]$, for $\mathcal{U}$ in the admissible family, form a neighborhood base at x in the topology of M.

From now on, we assume that M is an admissible space endowed with the admissible family $\mathcal{O}$ of open coverings of M.

Definition 2.3. *Let $x \in M$ and $t \in \mathbb{R}^+$. The* forward *and* backward t-prolongations *of x are respectively the sets*

$$D^+(x,t) = \bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu_t^+\left(\operatorname{St}[x,\mathcal{U}]\right)\right) \quad \text{and} \quad D^-(x,t) = \bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu_t^-\left(\operatorname{St}[x,\mathcal{U}]\right)\right).$$

We denote $D^+(x) = D^+(x,0)$ and $D^-(x) = D^-(x,0)$ for all $x \in M$. The forward and backward prolongations of a set $X \subset M$ are defined as $D^+(X,t) = \bigcup_{x\in X} D^+(x,t)$ and $D^-(X,t) = \bigcup_{x\in X} D^-(x,t)$, respectively. It is easily seen that $D^+(X,t)$ is forward invariant, because each $D^+(x,t)$ is forward invariant. Moreover, if X is compact, then $D^+(X,t)$ is closed ([3, Proposition 2.9]).

Definition 2.4. *The* forward prolongational limit set *of $x \in M$ is defined as*

$$J(x) = \bigcap_{t\in\mathbb{R}^+} D^+(x,t) = \bigcap_{t\in\mathbb{R}^+}\bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu_t^+\left(\operatorname{St}[x,\mathcal{U}]\right)\right)$$

and the backward prolongational limit set *of x as*

$$J^*(x) = \bigcap_{t\in\mathbb{R}^+} D^-(x,t) = \bigcap_{t\in\mathbb{R}^+}\bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu_t^-\left(\operatorname{St}[x,\mathcal{U}]\right)\right).$$

The forward and backward prolongational limit sets of a set $X \subset M$ are defined as $J(X) = \bigcup_{x\in X} J(x)$ and $J^*(X) = \bigcup_{x\in X} J^*(x)$, respectively. Since the admissible family $\mathcal{O}$ is directed by refinements we have

$$\begin{aligned} J(x) &= \left\{ \begin{array}{c} y \in M : \text{there are nets } t_\lambda \to +\infty \text{ and } x_\lambda \to x \\ \text{such that } \mu(t_\lambda, x_\lambda) \to y \end{array} \right\}, \\ J^*(x) &= \left\{ \begin{array}{c} y \in M : \text{there are nets } t_\lambda \to +\infty \text{ and } y_\lambda \to y \\ \text{such that } \mu(t_\lambda, y_\lambda) \to x \end{array} \right\}. \end{aligned}$$

In particular, $y \in J(x)$ if and only if $x \in J^*(y)$. This description shows that the concept of prolongational limit set does not depend on the admissible family of open coverings.

Note that the forward prolongational limit set $J(x)$ contains the limit set $\omega(x)$, while the backward prolongational limit set $J^*(x)$ contains the limit set $\omega^*(x)$. The following theorem is proved in [2, Theorem 4.3, Chapter II] for the case of dynamical systems on metric spaces. The proof for semiflows on admissible spaces follows analogously by using nets.

Theorem 2.5. *For any $x \in M$,*

1. $D^+(x) = \mu^+(x) \cup J(x) = \operatorname{cls}(\mu^+(x)) \cup J(x)$.
2. *If μ_t is an open map for every t, then $J^*(x)$ is backward invariant.*
3. *Both $J(x)$ and $J^*(x)$ are forward invariant.*

The following result is proved in [11, Proposition 2.2] and describes the limit sets and the prolongational limit sets under conjugation.

Proposition 2.6. *Let $\sigma : \mathbb{R}^+ \times N \to N$ be a semiflow on the admissible space N. Assume that $f : M \to N$ is a continuous semiconjugation between μ and σ, that is, $f(\mu(t,x)) = \sigma(t, f(x))$ for all $x \in M$ and $t > 0$. For $X \subset M$, the following inclusions hold:*

1. $f(\omega(X)) \subset \omega(f(X))$;
2. $f(D^+(X,t)) \subset D^+(f(X),t)$;
3. $f(J(X)) \subset J(f(X))$.

Another aspect of recursiveness is represented in terms of the nonwandering points. There are situations in which a point is Poisson unstable but lies in its prolongational limit set.

Definition 2.7. *A point $x \in M$ is* nonwandering *if every neighborhood U of x is forward recursive with respect to itself, that is, $\mu_t^+(U) \cap U \neq \emptyset$ for all $t > 0$.*

By [14, Theorem 4.1], the following statements are equivalent:

1. x is nonwandering.
2. $x \in J(x)$.
3. For every neighborhood U of x and $t > 0$, $\mu_t^-(U) \cap U \neq \emptyset$.
4. $x \in J^*(x)$.

The dispersiveness is characterized by the absence of recursiveness, as follows.

Definition 2.8. *Let $x \in M$.*

1. *The point x is called* forward Poisson unstable *if $x \notin \omega(x)$,* backward Poisson unstable *whenever $x \notin \omega^*(x)$, and* Poisson unstable *if it is both forward and backward Poisson unstable.*
2. *The point x is called* wandering *whenever $x \notin J(x)$.*

Definition 2.9. *The semiflow μ is said to be:*

1. Forward (backward) Poisson unstable *if each $x \in M$ is forward (backward) Poisson unstable, and* Poisson unstable *whenever it is both forward and backward Poisson unstable.*
2. completely unstable *if every $x \in M$ is wandering.*
3. dispersive *if for every pair of points $x, y \in M$ there are neighborhoods U_x of x and U_y of y such that U_x is not forward recursive with respect to U_y, that is, there is $t \geq 0$ such that $U_x \cap \mu_t^+(U_y) = \emptyset$.*

The following result shows how the prolongational limit sets play in dispersive semiflows.

Theorem 2.10. *The following statements are equivalent:*

1. *The semiflow μ is dispersive.*
2. *For each pair of points $x, y \in M$, $y \notin J(x)$.*
3. *For any point $x \in M$, $J(x) = \emptyset$.*
4. *$D^{+}(x) = \mu^{+}(x)$, for every $x \in M$, and there is no almost periodic point in M.*

Proof. See [14, Theorem 5.1]. □

We may also describe the dispersive semiflows by means of the Lagrange instability.

Definition 2.11. *Let $x \in M$. The motion μ_x is said to be **forward Lagrange stable** if $\mathrm{cls}(\mu^{+}(x))$ is a compact set; the motion μ_x is called **forward Lagrange unstable** if $\mathrm{cls}(\mu^{+}(x))$ is not compact. The semiflow μ is said to be **forward Lagrange unstable** if for each $x \in M$ the motion μ_x is forward Lagrange unstable.*

Theorem 2.12. *The semiflow μ is dispersive if and only if μ is forward Lagrange unstable and $D^{+}(x) = \mu^{+}(x)$, for every $x \in M$.*

Proof. Suppose that μ is forward Lagrange unstable and $D^{+}(x) = \mu^{+}(x)$, for every $x \in M$. If the semiflow was not dispersive, then a periodic trajectory $\mu_x(t)$ would exist. By taking $\tau > 0$ such that $\mu_x(\tau) = x$, it would follow that $\mathrm{cls}(\mu^{+}(x)) = \mu_x([0, \tau])$ is compact, and therefore μ_x would be forward Lagrange stable. As to the converse, suppose that μ is dispersive. Then $D^{+}(x) = \mu^{+}(x)$, for every $x \in M$, and there is no minimal set in M. As the forward orbits are forward invariant sets, it follows that there is no Lagrange stable motion. □

3. Semiflows on fiber bundles

This section contains the main results of the paper. We present a criteria for dispersive semiflow on fiber bundle.

Let G be a topological group acting on the right on a topological space Q. We denote by $(q, g) \in Q \times G \to qg \in Q$ the right action of G on Q, and assume that the action is free. A *principal bundle* is a quadruplet (Q, π, B, G) where $\pi : Q \to B$ is an open and surjective map such that $\pi(qg) = \pi(q)$ for all $q \in Q$ and $g \in G$. The space B is the base space, the space Q is the total space, and G is the structure group. Assume that G acts on the left on a topological space F. Then G acts on the right on $Q \times F$ by $(q, v)g = \left(qg, g^{-1}v\right)$. The quotient space $E = Q \times_G F$ is called *bundle associated* to the principal bundle (Q, π, B, G), and is indicated by (E, π, B, F, Q). We denote an element of E as $[q, v]$. The projection $\pi_E : E \to B$

defined by $\pi([q,v]) = \pi(q)$. Each fiber $E_x = \pi_E^{-1}(x)$ of the associated bundle is homeomorphic with the topological space F, which is called the *typical fiber* of the associated bundle.

A principal bundle (Q, π, B, G) is *locally trivial*, that is, there is an open covering $\{U_i\}_{i\in I}$ of the base space B such that, for each $i \in I$, there is a homeomorphism $\psi_i : \pi^{-1}(U_i) \to U_i \times G$, $\psi_i = (\pi, \upsilon_i)$, where $\upsilon_i : \pi^{-1}(U_i) \to G$ is a continuous mapping satisfying $\upsilon_i(qg) = \upsilon_i(q)\,g$, for all $q \in \pi^{-1}(U_i)$ and $g \in G$. The family $\Psi = \{(U_i, \psi_i)\}_{i\in I}$ is called a *map* of M. For each $i \in I$, the application $\upsilon_i^E : \pi_E^{-1}(U_i) \to F$ given by $\upsilon_i^E([q,u]) = \upsilon_i(q)\,u$ is open, and $\psi_i^E : \pi_E^{-1}(U_i) \to U_i \times F$ given by $\psi_i^E = \left(\pi_E, \upsilon_i^E\right)$ is a homeomorphism. Thus, the associated bundle is locally trivial and the family $\Psi^E = \left\{\left(U_i, \psi_i^E\right)\right\}_{i\in I}$ is a map of E.

Let $\mu : \mathbb{R}^+ \times Q \to Q$ be a semiflow on the total space Q of the principal bundle (Q, π, B, G) which commutes with the right action of G, that is, $\mu_t(qg) = \mu_t(q)\,g$ for all $q \in Q$, $t \in \mathbb{R}^+$, and $g \in G$. Then μ is a right invariant semiflow and each map $\mu_t : Q \to Q$ is called an endomorphism of Q. The semiflow μ induces the semiflow $\mu_B : \mathbb{R}^+ \times B \to B$ on the base space defined by $\mu_B(t, \pi(q)) = \pi(\mu(t,q))$, and μ induces the semiflow $\mu_E : \mathbb{R}^+ \times E \to E$ on the associated bundle $E = Q \times_G F$ defined as $\mu_E(t, [q,v]) = [\mu(t,q), v]$. In particular, the projection π is a continuous conjugation between the semiflows μ_E and μ_B. We often indicate both the semiflows μ_B and μ_E by μ, if there is no confusion.

The semiflow μ induces a semigroup action on the fiber F of the associated bundle E. For $q \in Q$ we define the set

$$\mathbb{R}_q^+ = \{g \in G : \text{there is } t \in \mathbb{R}^+ \text{ such that } \mu(t,q) = qg\},$$

which is a subsemigroup of G. We often denote by g_t the element of $\mathbb{R}_q^+$ such that $\mu(t,q) = qg_t$. Then $g_0 = 1 \in \mathbb{R}_q^+$, where 1 is the identity of G. Moreover, $g_t g_s = g_{t+s}$ for all $g_t, g_s \in \mathbb{R}_q^+$.

There are situations where the semigroup $\mathbb{R}_q^+$ is an one-parameter subgroup of G.

Example 3.1. *Let G be a topological group and $H \subset G$ a closed subgroup of G. The quotient map $\pi : G \to G/H$ of G onto the homogeneous space G/H is a principal bundle with structural group H. Let $\gamma : \mathbb{R} \to H$ be a one-parameter subgroup of H, that is, a homomorphism of $\mathbb{R}$ into H. Define $\mu : \mathbb{R} \times G \to G$ by $\mu(t,g) = \gamma(t)\,g$. Then μ is a right invariant flow on the principal bundle $(G, \pi, G/H, H)$. If 1 is the identity in G, then $\mathbb{R}_1 = \{\gamma(t)\}_{t\in\mathbb{R}}$. For $h \in H$, we have $\mu(t,h) = \gamma(t)\,h = hh^{-1}\gamma(t)\,h$. Hence, $\mathbb{R}_h = \left\{h^{-1}\gamma(t)\,h\right\}_{t\in\mathbb{R}} = h^{-1}\mathbb{R}_1 h$, for every $h \in H$, that is, $\mathbb{R}_h$ is the conjugation of $\mathbb{R}_1$ by h. If H is a normal subgroup of G, then $\mathbb{R}_g = g^{-1}\mathbb{R}_1 g$, for every $g \in G$.*

Example 3.2. *Let $\pi : V \to B$ be an n-dimensional real vector bundle. For $x \in B$, a frame σ_x on x is an invertible linear map $\sigma_x : \mathbb{R}^n \to V_x$ where V_x is the fiber of V above x. The set of all frames is denoted by BV. The bundle of frames of V is the bundle $p : BV \to B$ defined as $p(\sigma_x) = x$. The structural group of BV is*

$G = \mathrm{GL}(n, \mathbb{R})$ *that acts on the right on* BV *by* $\sigma g = \sigma \circ g$, $\sigma \in BV$, $g \in \mathrm{GL}(n, \mathbb{R})$. *The vector bundle* $\pi : V \to B$ *is recovered from* BV *as the associated bundle obtained by the standard linear action of* $\mathrm{GL}(n, \mathbb{R})$ *in* $\mathbb{R}^n$. *Let* $\mu : \mathbb{R}^+ \times V \to V$ *be a right invariant semiflow which is linear and invertible on fibers. Then* μ *lifts to a semiflow on* BV *by putting* $\mu_t(\sigma) = \mu_t \circ \sigma$. *It is easily seen that* $\mu_t(\sigma g) = \mu_t(\sigma) g$, *for all* $\sigma \in BV$ *and* $g \in \mathrm{GL}(n, \mathbb{R})$. *Suppose that the fiber* V_x *is invariant by* μ. *Then* μ_t *is an automorphism of* V_x *for all* $t \in \mathbb{R}^+$. *For a given frame* σ_x *on* x, *there is* $g_t \in \mathrm{GL}(n, \mathbb{R})$ *such that* $\mu_t \circ \sigma_x = \sigma_x \circ g_t$, *that is,* $\sigma_x^{-1} \circ \mu_t \circ \sigma_x = g_t$. *Hence, we have* $\mathbb{R}^+_{\sigma_x} = \left\{\sigma_x^{-1} \circ \mu_t \circ \sigma_x\right\}_{t \in \mathbb{R}^+}$, *that may be extended to a one-parameter subgroup* $\left\{\sigma_x^{-1} \circ \mu_t \circ \sigma_x\right\}_{t \in \mathbb{R}}$ *in* $\mathrm{GL}(n, \mathbb{R})$.

There are cases in which the semigroup $\mathbb{R}^+_q$ is the trivial subgroup $\{1\}$. If $q \in Q$ is a rest point, then $\mathbb{R}^+_q = \{1\}$ with $g_t = 1$ for all $t \in \mathbb{R}^+$; if q is not a rest point, but the trajectory through q is periodic, then $\mathbb{R}^+_q = \{1\}$ with $g_{n\tau} = 1$ for some $\tau > 0$ and every $n \in \mathbb{N}$. In general, if $\mathbb{R}^+_q \neq \{1\}$ then $\mu_B(t, \pi(q))$ is a periodic trajectory in the base space B.

For studying limit behavior for the action of $\mathbb{R}^+_q$ on F, we assume that there is $g_t \in \mathbb{R}^+_q$ with $t > 0$. Then $g_{nt} = g_t^n \in \mathbb{R}^+_q$ for every $n \in \mathbb{N}$, and $nt \to +\infty$. For a given set $K \subset F$, the ω-limit and the ω^*-limit set of K are defined as

$$\begin{aligned} \omega(K) &= \left\{ \begin{array}{c} v \in F : \text{there are nets } (t_\lambda) \text{ in } \mathbb{R}^+ \text{ and } (x_\lambda) \text{ in } K \\ \text{such that } t_\lambda \to +\infty \text{ and } g_{t_\lambda} x_{t_\lambda} \to v \end{array} \right\}, \\ \omega^*(K) &= \left\{ \begin{array}{c} v \in F : \text{there are nets } (t_\lambda) \text{ in } \mathbb{R}^+ \text{ and } (x_\lambda) \text{ in } K \\ \text{such that } t_\lambda \to +\infty \text{ and } g_{t_\lambda}^{-1} x_{t_\lambda} \to v \end{array} \right\}. \end{aligned}$$

For $v \in F$, the forward and the backward prolongational limit sets of v are defined by

$$\begin{aligned} J(v) &= \{u \in F : \text{there are nets } t_\lambda \to +\infty \text{ and } v_\lambda \to v \text{ such that } g_{t_\lambda} v_\lambda \to u\}, \\ J^*(v) &= \{u \in F : \text{there are nets } t_\lambda \to +\infty \text{ and } u_\lambda \to u \text{ such that } g_{t_\lambda} u_\lambda \to v\}. \end{aligned}$$

Since $\mathbb{R}^+_q$ is a subsemigroup of a group, we have

$$J^*(v) = \left\{u \in F : \text{there are nets } t_\lambda \to +\infty \text{ and } v_\lambda \to v \text{ such that } g_{t_\lambda}^{-1} v_\lambda \to u\right\}.$$

The following result is partially proved in the setting of semigroup actions on fiber bundles ([3, Theorem 4.1]). We complete the proof for semiflows.

Proposition 3.3. *Let* $\mu : \mathbb{R}^+ \times Q \to Q$ *be a right invariant semiflow on the total space* Q *of the principal bundle* (Q, π, B, G). *Then* $\omega(q) g = \omega(qg)$, $\omega^*(q) g = \omega^*(qg)$, $J(q) g = J(qg)$, *and* $J^*(q) g = J^*(qg)$, *for all* $q \in Q$ *and* $g \in G$.

Proof. If $p \in \omega(q)$ then there is $t_\lambda \to +\infty$ such that $\mu(t_\lambda, q) \to p$. For $g \in G$, it follows that $\mu(t_\lambda, qg) \to pg$, and therefore $pg \in \omega(qg)$. Hence, $\omega(q) g \subset \omega(qg)$. On the other hand, $\omega(qg) = \omega(qg) g^{-1} g \subset \omega(q) g$, and therefore $\omega(q) g = \omega(qg)$. If $p \in \omega^*(q)$ and $g \in G$, take an open neighborhood U of pg. Then Ug^{-1} is an open neighborhood of p. For $t > 0$, we have $Ug^{-1} \cap \mu_t^- (q) \neq \emptyset$, which implies that $U \cap \mu_t^- (qg) \neq \emptyset$. Hence, $pg \in \text{cls}\left(\mu_t^- (qg)\right)$. It follows that $pg \in \omega^* (qg)$, and therefore $\omega^* (q) g \subset \omega^* (qg)$. The inclusion $\omega^* (qg) \subset \omega^* (q) g$ is easy of checking. Now, take $p \in J (q)$ and $g \in G$. There are nets $t_\lambda \to +\infty$ and $q_\lambda \to q$ such that $\mu (t_\lambda, q_\lambda) \to p$. Hence, $q_\lambda g \to qg$ and $\mu (t_\lambda, q_\lambda g) \to pg$, and therefore $pg \in J (qg)$. Thus, $J (q) g = J (qg)$. Finally, if $p \in J^* (q)$, then $q \in J (p)$. Hence, $qg \in J (pg)$, and therefore $pg \in J^* (qg)$. It follows that $J^* (q) g = J^* (qg)$. □

In other words, the collection of the limit sets and the prolongational limit sets is invariant by G. This property implies the following results.

Corollary 3.4. *Let $\mu : \mathbb{R}^+ \times Q \to Q$ be a right invariant semiflow on the total space Q of the principal bundle (Q, π, B, G). The following statements hold:*

1. *If $q \in Q$ is forward (backward) Poisson stable, then all points in the fiber qG are forward (backward) Poisson stable;*
2. *If $q \in Q$ is nonwandering, then all points in the fiber qG are nonwandering.*

Corollary 3.5. *Assume that G acts transitively on the right on Q (Q is a homogeneous space of G) and let $\mu : \mathbb{R}^+ \times Q \to Q$ be a right invariant semiflow.*

1. *If there exists a forward (backward) Poisson stable point in Q, then all the points in Q are forward (backward) Poisson stable;*
2. *If there exists a nonwandering point in Q, then all the points in Q are nonwandering.*
3. *If Q is compact, then every point in Q is Poisson stable.*
4. *If there exists a forward (backward) Poisson unstable point in Q, then the semiflow is forward (backward) Poisson unstable*
5. *If there exists a wandering point in Q, then the semiflow is completely unstable.*
6. *If $J (q) = \emptyset$ for some $q \in Q$, then the semiflow is dispersive.*

Example 3.6. *Consider the differential equation $g' = Ag$ on $G = \mathrm{GL}(2, \mathbb{R})$ with A skew-symmetric. The flow associated to this equation is given by $\mu (t, g) = \exp (tA) g$, where $\exp (tA) \in \mathrm{SO}(2, \mathbb{R})$. Hence, the trajectories of this flow are periodic, and therefore every point in G is Poisson stable.*

Example 3.7. *Consider the differential equation $g' = Ag$ on $G = \mathrm{GL}(n, \mathbb{R})$ where A is a diagonalizable matrix with non-negative eigenvalues. The flow associated to this equation is given by $\mu (t, g) = \exp (tA) g$, where $\exp (tA) \to 0$ as $t \to -\infty$. Let 1 be the identity matrix. If $x \in J^* (1)$, then there are sequences $t_n \to -\infty$ and $x_n \to 1$ such that $\exp (t_n A) x_n \to x$. As $\exp (t_n A) \to 0$, it follows that $x = 0$. Hence, $J^* (1) = \emptyset$, and therefore $J^* (g) = \emptyset$ for all $g \in G$. Thus, the flow is dispersive.*

The following result is proved in [11, Proposition 3.2].

Theorem 3.8. *Suppose that* (E, π, B, F, Q) *is an associated bundle. Fix* $q \in Q$ *and consider the action of* $\mathbb{R}_q^+$ *on the fiber* F. *For* $[q, v] \in E$, *the following statements hold:*

1. $[q, \omega(v)] \subset \omega([q, v])$;
2. $[q, J(v)] \subset J([q, v])$;
3. $[q, J^*(v)] \subset J^*([q, v])$.

We have the following consequence from Theorem 3.8.

Corollary 3.9. *Suppose that* (E, π, B, F, Q) *is an associated bundle. Fix* $q \in Q$ *and consider the action of* $\mathbb{R}_q^+$ *on the fiber* F. *The following statements hold:*

1. *If* $v \in F$ *is forward Poisson stable, then* $[q, v] \in E$ *is forward Poisson stable;*
2. *If* $v \in F$ *is nonwandering, then* $[q, v] \in E$ *is nonwandering.*
3. *If* $[q, v] \in E$ *is forward Poisson unstable, then* v *is forward Poisson unstable.*
4. *If* $[q, v] \in E$ *is wandering, then* v *is wandering.*
5. *If the semiflow on* E *is completely unstable, then the semigroup action on* F *is completely unstable.*
6. *If the semiflow on* E *is dispersive, then the semigroup action on* F *is dispersive.*
7. *If the fiber* F *is compact, then there is a forward Poisson stable point in* E. *In particular, the semiflow is not dispersive.*

Proof. Items $(1) - (6)$ are immediate consequences from Theorem 3.8. For item (7), there is an $\mathbb{R}_q^+$-minimal set $M \subset F$, since F is compact. Hence, every point in M is forward Poisson stable. Thus, every point in $[q, M] \subset E$ is forward Poisson stable. □

Because of item 7 of Corollary 3.9, for a dispersive semiflow on fiber bundle with compact typical fiber, $g_t \in \mathbb{R}_q^+$ if and only if $t = 0$.

Proposition 3.10. *Suppose that* (E, π, B, F, Q) *is an associated bundle. The following statements hold:*

1. *If* $[q, v] \in E$ *is forward Poisson stable, then* $\pi(q) \in B$ *is forward Poisson stable;*
2. *If* $[q, v] \in E$ *is nonwandering, then* $\pi(q) \in B$ *is nonwandering.*

Proof. Since π is a continuous conjugation between the semiflows μ_E and μ_B, the result follows by Proposition 2.6. □

The following results are consequences from Theorem 3.8 and Proposition 3.10.

Corollary 3.11. *Suppose that (E, π, B, F, Q) is an associated bundle.*

1. *If the semiflow on the base space B is forward Poisson unstable, then both the semiflow on E and the semigroup action on F are forward Poisson unstable;*
2. *If the semiflow on the base space B is completely unstable, then both the semiflow on E and the semigroup action on F are completely unstable;*
3. *If the semiflow on the base space B is dispersive, then both the semiflow on E and the semigroup action on F are dispersive.*

Corollary 3.12. *Let $\mu : \mathbb{R}^+ \times V \to V$ be a right invariant semiflow on the n-dimensional vector bundle $\pi : V \to B$ which is linear and invertible on fibers. Assume that the fibers are μ-invariant. Then every point in the section $\{[\sigma, 0]\}_{\sigma \in BV}$ is forward Poisson stable.*

Proof. Consider the notation as in Example 3.2. For each $\sigma \in BV$, we have the one-parameter subgroup $\mathbb{R}^+_\sigma = \left\{\sigma^{-1} \circ \mu_t \circ \sigma\right\}_{t\in\mathbb{R}}$ of $\mathrm{GL}\,(n, \mathbb{R})$. Hence, $\omega\,(0) = \{0\}$. By Theorem 3.8, it follows that $[\sigma, 0] \in \omega\,([\sigma, 0])$. □

Example 3.13. *Consider the trivial bundle $\pi : B \times \mathrm{GL}\,(n, \mathbb{R}) \to B$ and define the flow $\mu : \mathbb{R} \times B \times \mathrm{GL}\,(n, \mathbb{R}) \to B \times \mathrm{GL}\,(n, \mathbb{R})$ by $\mu\,(t, (x, g)) = (x, e^t g)$. This flow is dispersive. For indeed, suppose that $(y, h) \in J^*\,(x, g)$. Then, there are nets $t_\lambda \to -\infty$ and $(x_\lambda, g_\lambda) \to (x, g)$ such that $(x_\lambda, e^{t_\lambda} g_\lambda) \to (y, h)$. Then $y = x$ and $h = 0$. But $(x, 0) \notin B \times \mathrm{GL}\,(n, \mathbb{R})$. Hence, $J^*\,(x, g) = \emptyset$, for all $(x, g) \in B \times \mathrm{GL}\,(n, \mathbb{R})$. It follows that $J\,(x, g) = \emptyset$ for all $(x, g) \in B \times \mathrm{GL}\,(n, \mathbb{R})$, that is, the flow on $B \times \mathrm{GL}\,(n, \mathbb{R})$ is dispersive. However, since the flow induced on the base space B is trivial, then it is Poisson stable on B. Thus, the converse to Proposition 3.10 is not true.*

Example 3.14. *Let M be an n-dimensional manifold and TM the tangent bundle. Then the projection $\pi : TM \to M$ is an n-dimensional vector bundle. Denote by BM the bundle of frames of TM. Let*

$\mu : \mathbb{R} \times BM \to BM$ be the right invariant flow given by $\mu(t, \sigma) = e^t \sigma$. This flow induces the flow $\mu_t\,([\sigma, v]) = [e^t\sigma, v] = [\sigma, e^t v]$ on $TM = BM \times_{\mathrm{Gl}(n,\mathbb{R})} \mathbb{R}^n$, which is linear on fibers. As in Example 3.13, this flow is dispersive on BM. Nevertheless, the flow is not dispersive on the tangent bundle. In fact, for $\sigma \in BV$, $\mathbb{R}_\sigma$ is the one-parameter subgroup $\{e^t 1\}_{t\in\mathbb{R}}$. It is easily seen that $\omega^*\,(0) = \omega\,(0) = \{0\}$. Hence, 0 is Poisson stable. Take $u, v \in \mathbb{R}^n$ such that $u \in J^*\,(v)$. Then there are nets $t_\lambda \to -\infty$ and $v_\lambda \to v$ such that $e^{t_\lambda} v_\lambda \to u$. As $e^{t_\lambda} v_\lambda \to 0$, we have $u = 0$. Hence, $J^*\,(v) = \{0\}$ for every $v \in \mathbb{R}^n$. But $e^{t_\lambda} v \to 0$ as $t_\lambda \to -\infty$. Thus, $\omega^*\,(v) = J^*\,(v) = \{0\}$ for every $v \in \mathbb{R}^n$. Now, suppose that $J\,(v) \neq \emptyset$

and take $u \in J(v)$*. Then* $v \in J^{*}(u) = \{0\}$*. Thus,* $J(v) \neq \emptyset$ *if and only if* $v = 0$*. Moreover,* $J(0) = \mathbb{R}^{n}$*. For indeed, if* $u \in \mathbb{R}^{n}$*, we have* $e^{n}e^{-n}u = u \to u$*, and* $e^{-n}u \to 0$ *as* $n \to +\infty$*. By Theorem 3.8, we have* $[\sigma, 0] \in \omega([\sigma, 0])$ *and* $[\sigma, \mathbb{R}^{n}] \subset J([\sigma, 0])$*. Thus, the points in the section* $\{[\sigma, 0]\}_{\sigma \in BM}$ *are Poisson stable. Moreover,* $J(\{[\sigma, 0]\}) = TM$*.*

We now discuss the situation in what a dispersive semiflow on the associated bundle yields a dispersive semiflow on the base space of a fiber bundle. Assume that G acts on the left on a compact metric space (F, d) and let $\pi_{E} : E \to B$ be the bundle associated to π, where $E = Q \times_{G} F$. Consider a trivializing covering $\{U_i\}_{i \in I}$ of Q and take the map $\Psi^{E} = \left\{\left(U_i, \psi_i^{E}\right)\right\}_{i \in I}$ of E. The total space E is locally compact, since it is locally trivial and F is compact.

Let $\mathcal{O}$ be the family of all open coverings of B. Since B is paracompact, $\mathcal{O}$ is admissible. Given $\varepsilon > 0$ and $\mathcal{U} \in \mathcal{O}$, a Ψ*-adapted covering* of E is defined as

$$\mathcal{U}_{\varepsilon} = \left\{\left(\psi_i^{E}\right)^{-1}\left(\left(U \cap U_i\right) \times B_{\varepsilon}(u)\right) : U \in \mathcal{U}, i \in I, u \in F\right\}.$$

We denote $\mathcal{O}_{\Psi}(E)$ the family of all Ψ-adapted coverings. The family $\mathcal{O}_{\Psi}(E)$ is admissible (see [9, Section 3.2]).

Lemma 3.15. *For any* $q \in Q$ *and* $\mathcal{U}_{\varepsilon} \in \mathcal{O}_{\Psi}(E)$*, one has the inclusions*

$$\pi_{E}^{-1}\left(\mathrm{St}\left[\pi(q), \mathcal{V}\right]\right) \subset \mathrm{St}\left[E_{\pi(q)}, \mathcal{U}_{\varepsilon}\right] \subset \pi_{E}^{-1}\left(\mathrm{St}\left[\pi(q), \mathcal{U}\right]\right)$$

where $\mathcal{V}$ *refines both* $\mathcal{U}$ *and* $\{U_i\}_{i \in I}$*.*

Proof. If $[p, u] \in \pi_{E}^{-1}(\mathrm{St}[\pi(q), \mathcal{V}])$, then $\pi(p) \in \mathrm{St}[\pi(q), \mathcal{V}]$. Hence, there are $V \in \mathcal{V}$ such that $\pi(p), \pi(q) \in V$. As $\mathcal{V}$ refines both $\mathcal{U}$ and $\{U_i\}_{i \in I}$, there are $U \in \mathcal{U}$ and $i \in I$ such that $V \subset U \cap U_i$. Thus, $\pi(p), \pi(q) \in U \cap U_i$. Take $w = v_i(p)u$ and $v = v_i(q)^{-1}w$. We have $\psi_i^{E}([p, u]) = (\pi(p), w)$ and $\psi_i^{E}([q, v]) = (\pi(q), w)$. Hence, $[p, u], [q, v] \in \left(\psi_i^{E}\right)^{-1}((U \cap U_i) \times B_{\varepsilon}(w))$, and therefore $[p, u] \in \mathrm{St}[[q, v], \mathcal{U}_{\varepsilon}]$. It follows that $\pi_{E}^{-1}(\mathrm{St}[\pi(q), \mathcal{V}]) \subset \mathrm{St}\left[E_{\pi(q)}, \mathcal{U}_{\varepsilon}\right]$. Now, let $[p, u] \in \mathrm{St}\left[E_{\pi(q)}, \mathcal{U}_{\varepsilon}\right]$. Then, there are $U \in \mathcal{U}$, $i \in I$, $w \in F$, and $[q, v] \in E_{\pi(q)}$ such that $[p, u], [q, v] \in \left(\psi_i^{E}\right)^{-1}((U \cap U_i) \times B_{\varepsilon}(w))$. It follows that $\pi_{E}([p, u]), \pi_{E}([q, v]) \in U \cap U_i$, that is, $\pi_{E}([p, u]) \in \mathrm{St}[\pi(q), \mathcal{U}]$. Therefore, $\mathrm{St}\left[E_{\pi(q)}, \mathcal{U}_{\varepsilon}\right] \subset \pi_{E}^{-1}(\mathrm{St}[\pi(q), \mathcal{U}])$. □

Proposition 3.16. *Let the assumptions be as in the last paragraph above. For each* $q \in Q$*,*

$$\pi_{E}^{-1}\left(D^{+}(\pi(q))\right) = \bigcap_{\mathcal{U}_{\varepsilon} \in \mathcal{O}_{\Psi}(E)} \mathrm{cls}\left(\mu^{+}\left(\mathrm{St}\left[E_{\pi(q)}, \mathcal{U}_{\varepsilon}\right]\right)\right).$$

Proof. It is easily seen that

$$\begin{aligned} \pi_E^{-1}\left(D^{+}\left(\pi\left(q\right)\right)\right) &= \pi_E^{-1}\left(\bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)\right) \\ &= \bigcap_{\mathcal{U}\in\mathcal{O}} \pi_E^{-1}\left(\operatorname{cls}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)\right). \end{aligned}$$

Since π_E is an open map, we have

$$\pi_E^{-1}\left(D^{+}\left(\pi\left(q\right)\right)\right) = \bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\pi_E^{-1}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)\right).$$

Now, let $[p,u] \in \pi_E^{-1}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)$. Then, $\pi\left(p\right) \in \mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)$, that is, there are $t \in \mathbb{R}^{+}$ and $\pi\left(r\right) \in \operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]$ such that $\pi\left(p\right) = \pi\left(\mu_t\left(r\right)\right)$. Hence, there is $g \in G$ such that $p = \mu_t\left(rg\right)$. It follows that $[p,u] = \mu_t\left([rg,u]\right)$ with $\pi_E\left([rg,u]\right) = \pi\left(r\right) \in \operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]$. Hence, $[p,u] \in \mu_t\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)$, and therefore $\pi_E^{-1}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right) \subset \mu^{+}\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)$. On the other hand, as π_E is a conjugation, we have $\pi_E\left(\mu^{+}\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)\right) = \mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)$. Hence, $\mu^{+}\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right) \subset \pi_E^{-1}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)$, and therefore

$$\pi_E^{-1}\left(\mu^{+}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right) = \mu^{+}\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right).$$

We now have

$$\pi_E^{-1}\left(D^{+}\left(\pi\left(q\right)\right)\right) = \bigcap_{\mathcal{U}\in\mathcal{O}} \operatorname{cls}\left(\mu^{+}\left(\pi_E^{-1}\left(\operatorname{St}\left[\pi\left(q\right),\mathcal{U}\right]\right)\right)\right),$$

and the proof follows by Lemma 3.15. □

This result implies that the prolongations in the base space relate to the prolongations in the total space, as follows.

Theorem 3.17. *For each $x \in B$,*

$$\pi_E^{-1}\left(D^{+}\left(x\right)\right) = D^{+}\left(E_x\right).$$

Proof. Let $x = \pi\left(q\right)$. Since the fiber $E_{\pi(q)}$ is compact, the prolongation $D^{+}\left(E_{\pi(q)}\right)$ is closed. By Proposition 2.6, we have $\pi_E\left(D^{+}\left(E_{\pi(q)}\right)\right) \subset D^{+}\left(\pi\left(q\right)\right)$. Hence, $D^{+}\left(E_{\pi(q)}\right) \subset \pi_E^{-1}\left(D^{+}\left(\pi\left(q\right)\right)\right)$. On the other hand, take $[p,u] \in \pi_E^{-1}\left(D^{+}\left(\pi\left(q\right)\right)\right)$ and neighborhood N of $[p,u]$. Since E is locally compact, there is a neighborhood V of $[p,u]$ such that $\operatorname{cls}\left(V\right)$ is compact and $\operatorname{cls}\left(V\right) \subset N$. For $\mathcal{U}_\varepsilon \in \mathcal{O}_\Psi\left(E\right)$, there are $t_{\mathcal{U}_\varepsilon} \in \mathbb{R}^{+}$ and $[q_{\mathcal{U}_\varepsilon}, v_{\mathcal{U}_\varepsilon}] \in \operatorname{St}\left[E_{\pi(q)}, \mathcal{U}_\varepsilon\right]$ such that $\mu\left(t_{\mathcal{U}_\varepsilon}, [q_{\mathcal{U}_\varepsilon}, v_{\mathcal{U}_\varepsilon}]\right) \in V \cap \mu^{+}\left(\operatorname{St}\left[E_{\pi(q)}, \mathcal{U}_\varepsilon\right]\right)$, because of Proposition 3.16. Since $\operatorname{cls}\left(V\right)$ and the fiber $E_{\pi(q)}$ are compact sets, we can assume that the net $\left([q_{\mathcal{U}_\varepsilon}, v_{\mathcal{U}_\varepsilon}]\right)$ converges to some $[q,v] \in E_{\pi(q)}$ and $\left(\mu\left(t_{\mathcal{U}_\varepsilon}, [q_{\mathcal{U}_\varepsilon}, v_{\mathcal{U}_\varepsilon}]\right)\right)$ converges to some $x \in \operatorname{cls}\left(V\right)$. Hence, $x \in D^{+}\left([q,v]\right) \cap N$, and therefore $[p,u] \in \operatorname{cls}\left(D^{+}\left(E_{\pi(q)}\right)\right) = D^{+}\left(E_{\pi(q)}\right)$. □

Finally, we present the necessary and sufficient conditions to the dispersiveness on the base space of the fiber bundle.

Theorem 3.18. *The semiflow on the base space B is dispersive if and only if there is no almost periodic point and the semiflow on the associated bundle E is dispersive.*

Proof. If the semiflow on B is dispersive, then there is no almost periodic point, because the limit sets are empty. Furthermore, Corollary 3.11 assures that the semiflow on E is dispersive. As to the converse, suppose that there is no almost periodic point and the semiflow on E is dispersive. By Theorem 2.5, $D^{+}([q,u]) = \mu^{+}([q,u])$, for every $[q,u] \in E$. Hence, $D^{+}(E_x) = \mu^{+}(E_x)$, for every $x \in B$. By Theorem 3.17, it follows that $D^{+}(x) = \mu^{+}(x)$ for every $x \in B$. Since there is no almost periodic point, Theorem 2.10 says that the semiflow in B is dispersive. □

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