Bol. Soc. Paran. Mat. (3s.) **v. 37** 2 (2019): 59–65.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.31562

Solutions of Weakened Field Equations in Gödel Space-Time

Aditya Mani Mishra

ABSTRACT: We have solved Weakened field equations, collected work of Lovelock for cylindrically symmetric Gödel type spacetime. A comparative study of these solutions to solution of Einstein's field equation have shown. Conformality of Gödel spacetime has discussed with vanishing and non-vanishing scalar curvature of the spacetime.

Key Words: Weyl tensor, Conformal property, Bianchi identity, Alternative theory of gravitation.

Contents

1. Introduction

Many theories alternative to Einstein's theory have been formulated after Einstein and less or strong conditions have been appeared in literature. Lovelock has collected [1] these equations and called as Weakened Field Equations. These equations are weaker than the Einstein equations in the sense that spaces which are interpreted as the gravitational field in vacuum in the orthodox theory form only a sub-set of such spaces for these field equations. Einstein's equations for empty space form a particular subclass of these weakened field equations. The physical implications of these equations are not well-established. Kilmister and Thompson have shown that weakened field equations are too weak [2,3]. Lovelock has solved these equations and found a static solution which corresponds to the field of massless charged particle in rest although this solution has already been used by Mimura and Takeno (1962) in the theory of the atom based on wave geometry [4].

2010 *Mathematics Subject Classification:* 83D05, 83-02.
Submitted April 06, 2016. Published July 20, 2016

. Recently, Weakened Field Equations was studied for developing a generalised plane gravitational waves [5] and compared with its counterpart in General Relativity. It opens an opportunity to view gravitational waves in extended form. The solutions of Weakened Field Equations have been obtained for spherically symmetry [6], plane waves in plane symmetric spacetime [7,8] and Peres spacetime [9,10,11]. Cylindrical wave solutions of Weakened field equation(1.1) have been studied in [12]. A detailed discussion is given in [13,14].

. Vacuum field equations which have been suggested by Lovelock [1] as alternatives to the vacuum field equation of Einstein theory of general relativity They are

$$J_{ikl} \equiv \; R^{j}_{ikl;j} = 0 \tag{1.1}$$

$$\Xi^{rs} \equiv \; \sqrt{-g}[(g^{rs}g^{tu} - \frac{1}{2}g^{rt}g^{su} - \frac{1}{2}g^{ru}g^{st})R_{;ut} + R(R^{sr} - \frac{1}{4}g^{sr}R)] = 0 \tag{1.2}$$

$$H^{ij}_{k} \equiv \; R^{ij}_{;k} = 0 \tag{1.3}$$

$$\begin{aligned} E^{hk} \equiv & \sqrt{-g}[g^{hj}g^{ki}(2R_{jlim}R^{ml} + g^{ml}R_{ij;lm} - R_{;ij}) \\ & - \frac{1}{2}g^{hk}(R^{l}_{m}R^{m}_{l} - g^{lm}R_{;lm})] \\ & = 0 \end{aligned} \tag{1.4}$$

$$\begin{aligned} \Upsilon_{jk} \equiv & (-g)^{1/4}[g^{ih}R_{kj;ih} - g^{ih}R_{ij;kh} + \frac{1}{6}(R_{;kj} - g_{jk}g^{ih}R_{;ih}) \\ & - R^{ih}C_{jhik} + \frac{R}{6}g^{ij}C_{jhik}] \\ & = 0 \end{aligned} \tag{1.5}$$

here, g_{ij} is fundamental metric tensor, R_{ij} is Ricci curvature tensor which is obtained by contracting Riemannian tensor,R^{j}_{ikl}. C_{jhik} is Weyl conformal tensor.

. All of equation (1.1-1.5) are Weakened Field Equations and abbreviated to WFE. These equations are weaker than Einstein vacuum field equation in the sense that they each admit a solution for which $R_{ij} = 0$.

. Consequently, as equations (1.1-1.5) admit the Schwarzschild metric as a solution of spherically symmetric case, Gödel type spacetime [15,16] may also be a solution of these weakened field equations, which when taken together with geodesic hypothesis, give agreement with the usual experimental tests of general relativity.

. Equation (1.1) was first suggested by Kilmister and Newman [2] and has been the subject of a detailed investigation by Thompson [3]. He has studied that equation (1.1) possesses certain unphysical metrics as solutions. Equation (1.2) was proposed by Eddington but has been studied by Buchdahl [17], Du Plessis [18] and Pechlaner and Sexl [19] etc. equation (1.2) can be derived by variational principle. Ξ^{rs} shows property $\Xi^{rs} = \Xi^{sr}$ and $\Xi^{rs}_{;r} = 0$.

. Rund [20] arrived a set of third order differential equations (1.3) using a completely different type of variational principle. He studied this equation in different aspects but equation (1.3) also works as weakened field equation. Equation (1.3) actually is a stronger form of equation (1.1). Takeno [4] has investigated it and admitted that the only spherically symmetric solution of equation (1.3) is also a solution of equation

$$R_{ij} = \lambda g_{ij} \tag{1.6}$$

here λ is a constant. Equation (1.4) appeared in Rund's work [21,22] with Du Plessis [18]. This equation also has been derived by variational principle but none of these authors have considered it as a weakened field equation of general relativity. The relevance of equation (1.4) is based on the fact that whether or not the solutions of equation (1.4) exist which are not solutions of equation $R_{ij} = 0$ or equation (1.6). Equation (1.4) also satisfies $E^{hk} = E^{kh}$ and $E^{hk;h} = 0$.

. Equation (1.5) was given by Pirani and this equation also has been used in Schouten [23] with a different thought. However, Υ_{jk} shows symmetry and conservation property as $\Upsilon_{jk} = \Upsilon_{kj}$ and $\Upsilon^{i}_{k;j} = 0$ respectively. In this article, we find the solutions of weakened field equations(1.1-1.5) in Gödel space-time defined by equation (2.1).

2. Solutions to Weakened Field Equations

Gödal type spacetime is represented by cylindrically symmetric metric

$$ds^2 = dt^2 – dr^2 - (B^2(r) - A^2(r))d\phi^2 - dz^2 + 2A(r)dtd\phi \tag{2.1}$$

where $A(r)$ and $B(r)$ are functions of r coordinates. Note that $g \equiv \ |g_{ij}| < 0$ always. Vacuum solution of Einstein's field equation for metric (2.1) are $A'(r) = 0$ and $B''(r) = 0$ while for non vanishing density of pressure free matter,

$$ds^2 = dt^2 - dr^2 + \frac{e^{2r}}{2}d\phi^2 - dz^2 + 2e^r dtd\phi \tag{2.2}$$

Gö del spacetime is a singularity free rotating solution of general relativity with non expanding universe. Existence of closed time-like curves is a very interesting property of this universe . The entire content of the G̈del universe (on cosmological scale) is in a state of uniform, rigid rotation. The light rays and free particles exhibit a kind of boomerang effect. It also allows the possibility of time-travel in a certain sense.

2.1. Solution to Weakened Field Equation (1.1)

Equation (1.1) is a three rank tensor obtained by contracting the Riemannian curvature tensor with covariant differentiation. We have the relations $R_{ijkl} = h_{ja}R^{a}_{ikl}$ and $R_{ij} = R^{l}_{ijl}$ Furthermore,R^{j}_{ikl} satisfies the Bianchi identity

$$R^{j}_{ikl;i} + R^{j}_{ikl;k} + R^{j}_{ikl;l} = 0 \tag{2.3}$$

from which it follows that

$$R^j_{ikl;j} = R_{im;l} - R_{il,m} \tag{2.4}$$

Using equation (2.4), we can find the solution of equation (1.1) in less effort. Solving equation (2.4), we get

$$\frac{A'}{B} = Constant = \mu \tag{2.5}$$

and

$$\frac{B''}{B} = Constant = \nu^2 \tag{2.6}$$

where the constants μ and ν are related to

$$\mu^2 - \nu^2 = 0 \tag{2.7}$$

We note that the solution of differential equation (2.5) and (2.6) in case of $\mu = \nu = 0$ is one which satisfies Einstein field equation. It reflects the weakness property of weakened field equation (1.1).

2.2. Solution to Weakened Field Equation (1.2)

In this sub-section, we solve weakened field equation (1.2). This field equation is identically satisfied by a zero curvature space-time. It leaves the case when curvature is non-zero. Since A and B defined in the metric (2.1) are functions of r only, the component that exists in $R_{;ut}$, is $R_{;rr}$ and for this value of u and t,

$$g^{rs}g^{tu} - \frac{1}{2}g^{rt}g^{su} - \frac{1}{2}g^{ru}g^{st} \tag{2.8}$$

vanishes. So, weakened field equation (1.2) reduces to Einstein field equation for the metric (2.1) and does not provide any extra solution.

2.3. Solution to Weakened Field Equation (1.3)

Field equation (1.3) has tensorial rank three that provides solutions:

$$\frac{A'}{B} = Constant = \mu \tag{2.9}$$

and

$$\frac{B''}{B} = Constant = \nu^2 \tag{2.10}$$

where the constants μ and ν are related to

$$\mu^2 - 2\nu^2 = 0 \tag{2.11}$$

It again lead to solution of Einstein field equation for $\mu = \nu = 0$. Weakened field equation (1.3) can be taken as analogue to the Maxwell's field equations $F^{ij}_{;j} = 0$ and, therefore, resembles as electromagnetic field tensor $F_{ij} = -F_{ji}$ when charged current vector J_i is zero.

2.4. Solution to Weakened Field Equation (1.4)

After some simplification in equation (1.4), we get

$$[g^{hj}g^{ki}(2R_{jlim}R^{ml}+g^{ml}R_{ij;lm})-\frac{1}{2}g^{hk}R^{l}_{m}R^{m}_{l}]+(\frac{1}{2}g^{hk}g^{ij}-g^{hj}g^{ki})R_{;ij}=0 \quad (2.12)$$

Since R is scalar curvature depends only upon the distance from source, on contraction of square bracket quantity in equation (2.12), we get

$$\frac{1}{4B^4}[3A'^4+A'^2(B'^2-9BB'')+B^2A'A'''-B4B'^2B''+B(A''^2-10B''^2-4B'B''')$$
$$+2\mathrm{B}^2B^{(iv)}]=0 \quad (2.13)$$

Here superscripts $', '', '''$ and iv denote first, second, third and fourth order derivatives of the variables with respect to r coordinates respectively.

. Thus, Gödel space-time (2.1) will be a solution of weakened field equation (1.4) under the condition (2.4). In particular for zero curvature,

$$\frac{\sqrt{3}}{2}A'^2=BA''-A'B' \quad (2.14)$$

2.5. Solution to Weakened Field Equation (1.5)

We multiply equation (1.5) to g^{jk} and summing over j and k;

$$R^{jk}R_{jk}+\frac{R''}{2}+\frac{R}{6}g^{jk}g^{il}C_{jlik}=0 \quad (2.15)$$

Substituting the values, equation (2.15) becomes

$$\frac{1}{24B^6}[(1+A^2+21B^2)A'^4-2ABA'^3B'-2BA'^2B(24B'^2-AA'')+3(1+A^2)B''$$
$$+27\,\mathrm{B}^2B''+4B^2A'(18BB'A''+2AB'B''-3B^2A''')+8B^2B(6B'^2-AA'')B''$$
$$+(1+\mathrm{A}^2)B''^2-3B^2(A''^2-2B''^2+2B'B''')+3B^3B^{iv}]=0 \quad (2.16)$$

This is a highly non-linear equation containing fourth order derivatives. Gödel metric will be a solution of weakened field equation (1.5) under the condition given by equation (2.5).

For zero space curvature, equation (2.5) reduces to equation (2.14). It is worthless to mention that conformally flat space-time with zero scalar curvature trivially satisfies weakened field equation (1.5) by virtue of Weyl tensor properties.

3. Conformal Property of Gödel Type spacetime

The Weyl tensor is defined as

$$C_{jlik} = R_{jlik} - \frac{1}{2}(g_{lk}R_{ji} - g_{jk}R_{li} + g_{ij}R_{lk} - g_{li}R_{jk}) + \frac{R}{6}(g_{ij}g_{lk} - g_{li}g_{jk}) \quad (3.1)$$

For zero space curvature

$$R_{jlik} = \frac{1}{2}(g_{lk}R_{ji} - R_{li}g_{jk} + g_{ij}R_{lk} - g_{li}R_{jk}) \quad (3.2)$$

which after contraction;

$$R^{j}_{iml;j} = \frac{1}{2}(R_{im;l} - R_{il;m}) \quad (3.3)$$

This equation reduces to

$$R^{j}_{iml;j} = 0 \quad (3.4)$$

or

$$R_{im;l} = R_{il;m} \quad (3.5)$$

using Bianchi identity. Note that equation (3.4) and equation (3.5) are weakened field equation (1.1). Gödel space-time define by metric (2.1) will be conformally flat if Weyl tensor is zero.

4. Conclusion

We introduced Weakened Field Equations which are a set of five different field equations in vacuum with respective properties and found solutions for Gödel type spacetime. Solution-space of these field equations vary for different Weakened Field Equations but each one contains (or equals to) solutions to Einstein's field equation and hence, agree with classical result. In the last section, the conformal property of spacetime has discussed and it is shown that for zero curvature, Gödel spacetime is conformally flat.

References

1. Lovelock D.: Weakened field equations in general relativity admitting an "unphysical" metric, Comm. Math. Phys., 5, 205-214 (1967).

2. Kilmister C. L., and Newman D.J.: The use of algebraic structure in Physics, Proc. Cambridge Phil. Soc. 57, 851-864(1961).

3. Thompson A. H.: The investigation of a set of weakened field equations for general relativity, Ph. D. thesis, U. S. A. F.(1963).

4. Takeno H.: The mathematical theory of Plane gravitational waves in general relativity, Sci. Res. Ins. Theor., Phys., Hiroshima uni., 1 (1961).

5. Katore S. D., Rane R. S.:Generalized plane gravitational waves of weakened field equations in general relativity, Journal of Mathematical Physics, 50(5), 053504-053504-9 (2009).

6. Lovelock D.: A spherically symmetric solution of Maxwell-Einstein equations, Comm. Math. Phys, 5, 257-261 (1967).

7. Bhoyar S. R., Deshmukh A. G.: Plane Wave Solutions of Weakened Field Equations in a Plane Symmetric Space-time – I, The African Review of Physics, 7 (0022), 219-224 (2012).

8. Bhoyar Sanjay R., Deshmukh A. G.: Plane Wave Solutions of Weakened Field Equations in a Plane Symmetric Space-time-II, , Prespacetime Journal, 2 (13), 2060 -2070 (2011).

9. Pandey S. N., Lal K. B.: On plane wave-like solutions of weakened field equations in Peres space-time, Tensor N. S., 29 (1975).

10. Pandey S. N.: On weakened field equations in general relativity, Indian J. Pure. Appl. Math. (1975).

11. Pandey S. N.: Plane wave solutions of weakened field equations in general relativity, Tensor N. S., 29 (1975).

12. Lal K. B., and Singh T.: On cylindrical wave solutions of Kilmister and Newman's weakened field equations in general relativity, Tensor N. S., 27, 1973 (287-290).

13. Michel Janssen, Jürgen Renn, John Norton, Tilmann Sauer, John Stachel: The Genesis of General Relativity: Sources and Interpretations, Springer Publications (2007).

14. Hesselbach B.: Uder die Erhaltungssatze der Konforman Geometrie, Math. Nachr., 3, 107-126 (1949).

15. Godel K.: A Remark about the relationship between relativity and idealistic philosophy, p 557-462 (1949).

16. Godel K.: An example of a new type of cosmological solutions of Einstein field equations of gravitation, Rev Mod Phys, 21, 447-450 (1949).

17. Buchdahl H. A.: On gravitational field equations arising from the square of Gaussian curvature, Nuovo Cimento 23, 141-157 (1952).

18. Plessis J. C. Du: n-variational properties of variational principles in general relativity, Ph. D. thesis, University of South Africa (1965).

19. Pechlaner E. and Sexl R.: On quadratic Lagrangians in general relativity, Comm. Math. Phys., 2, 165-175 (1966).

20. Rund H.: Invariant theory of variational problems for geometric objects, Tensor N. S., 19(1967).

21. Rund H.: Variational problems in which the unknown functions are tensor components, University of South Africa (1964).

22. Rund H.: Variational problems involving combined tensor fields, Math. Sem. Uni. Hamburg, 29, 243-262 (1966).

23. Schouten J. A.: Ricci Calculus, 2nd Edition, Berlin-Gottingen-Heidelberg, Springer (1954).

Acknowledgments

The author thanks to Prof. S. N. Pandey for his valuable suggestion and intuitive support for this manuscript.

Dr. A. M. Mishra,
Department of HEAS,
Rajasthan Technical University Kota,
India.
E-mail address: `adm.nita@gmail.com`