Bol. Soc. Paran. Mat. (3s.) v. 37 2 (2019): 167–177.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.33618

Some Results on Almost b-continuous Functions in a Bitopological Space

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ABSTRACT: The aim of this paper is to investigate some properties of almost b-continuous function in a bitopological space. Relationships with some other types of functions are investigated.

Key Words: Bitopological space, (i, j)-b-open set, (i, j)-δ-closed set, almost b-continuous function.

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1. Introduction

The notion of a bitopological space (X, τ_1, τ_2), where X is a non-empty set and τ_1, τ_2 are topologies on X, was introduced by Kelly [7]. In 1996, Andrijevic [2] introduced the concept of b-open set in a topological space. Later Al-Hawary and Al-Omari [1] defined the notion b-open set, b-continuity in a bitopological space and established several fundamental properties. Sengul [11] defined the notion of almost b-continuous function in a topological space and established relationships between several properties of this notion with other known results. In addition to this, Duszynski et al. [6] introduced the concept of almost b-continuous function in a bitopological space. The purpose of this paper is to study more on almost b-continuity of a bitopological space, in the light of Duszynski et al.[6].

Bitopological space and its properties have many useful applications in real world. In 2010, Salama [10] used lower and upper approximations of Pawlak's rough set by using a class of generalized closed set of bitopological space for data reduction of rheumatic fever data sets. Fuzzy topology integrated support vector machine (FTSVM)-classification method for remotely sensed images based on standard support vector machine (SVM) were introduced by using fuzzy topology by Zhang et al. [16]. One may refer to [10,14,16] for some recent applications of generalized forms of general topology or bitopology in fuzzy set theory, rough set

2010 *Mathematics Subject Classification:* 54E55.
Submitted September 27, 2016. Published March 30, 2017

theory etc.

2. Preliminaries

Throughout this paper, bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are represented by X and Y respectively; on which no separation axiom is assumed until it is stated clearly. (i, j) means the topologies τ_i and τ_j; where $i, j \in \{1, 2\}, i \neq j$. For $A \subseteq X$, i-$int(A)$ (respectively, i-$cl(A)$) denotes interior (resp. closure) of A with respect to the topology τ_i, where $i \in \{1, 2\}$.

Now, we list some definitions and results which will be used throughout this paper.

Definition 2.1. *Let (X, τ_1, τ_2) be a bitopological space, then a subset A of X is said to be*

(a) (i, j)-b-open ([1]) *if $A \subseteq i$-$int(j$-$cl(A)) \cup j$-$cl(i$-$int(A))$,*

(b) (i, j)-regular open ([3]) *if $A = i$-$int(j$-$cl(A))$,*

(c) (i, j)-regular closed ([4]) *if $A = i$-$cl(j$-$int(A))$.*

The complement of (i, j)-b-open set is said to be (i, j)-b-closed set.

Definition 2.2. ([1]) *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then,*

(a) (i, j)-b-closure of A; denoted by (i, j)-$bcl(A)$, is defined as the intersection of all (i, j)-b-closed sets containing A,

(b) (i, j)-b-interior of A; denoted by (i, j)-$bint(A)$, is defined as the union of all (i, j)-b-open sets contained in A.

Lemma 2.3. ([1]) *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then,*

(a) (i, j)-$bint(A)$ is (i, j)-b-open,

(b) (i, j)-$bcl(A)$ is (i, j)-b-closed,

(c) A is (i, j)-b-open if and only if $A = (i, j)$-$bint(A)$,

(d) A is (i, j)-b-closed if and only if $A = (i, j)$-$bcl(A)$.

Lemma 2.4. ([9]) *Let* (X, τ_1, τ_2) *be a bitopological space and* $A \subseteq X$. *Then,*

(a) $X \setminus (i,j)$-$bcl(A) = (i,j)$-$bint(X \setminus A)$,

(b) $X \setminus (i,j)$-$bint(A) = (i,j)$-$bcl(X \setminus A)$.

Lemma 2.5. ([1]) *Let* (X, τ_1, τ_2) *be a bitopological space and* $A \subseteq X$. *Then,* $x \in (i,j)$-$bcl(A)$ *if and only if for every* (i,j)-*b-open set* U *containing* x *such that* $U \cap A \neq \emptyset$.

Definition 2.6. *A function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is said to be*

(a) (i,j)-*b-continuous* ([1]) *if* $f^{-1}(A)$ *is* (i,j)-*b-open in* X; *for each* σ_i-*open set* A *of* Y,

(b) (i,j)-*weakly b-continuous* ([13]) *if for each* $x \in X$ *and each* σ_i-*open set* V *of* Y *containing* $f(x)$, *there exists an* (i,j)-*b-open set* U *containing* x *such that* $f(U) \subseteq j$-$cl(V)$.

Definition 2.7. ([8]) *Let* (X, τ_1, τ_2) *be a bitopological space. A point* $x \in X$ *is said to be an* (i,j)-δ-*cluster point of* A *if* $A \cap U \neq \emptyset$; *for every* (i,j)-*regular open set* U *containing* x. *The set of all* (i,j)-δ-*cluster points of* A *is called* (i,j)-δ-*closure of* A *and it is denoted by* (i,j)-$cl_\delta(A)$. *A subset* A *of* X *is said to be* (i,j)-δ-*closed if the set of all* (i,j)-δ-*cluster points of* A *is a subset of* A. *The complement of an* (i,j)-δ-*closed set is an* (i,j)-δ-*open. So, a subset of* X *is* (i,j)-δ-*open; if it is expressible as union of* (i,j)-*regular open sets.*

3. (i,j)-almost b-continuous functions

Definition 3.1. ([6]) *A function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is said to be* (i,j)-*almost b-continuous at a point* $x \in X$; *if for each* σ_i-*open set* V *of* Y *containing* $f(x)$, *there exists an* (i,j)-*b-open set* U *of* X *containing* x *such that* $f(U) \subseteq i$-$int(j$-$cl(V))$.

If f *is* (i,j)-*almost b-continuous at every point* x *of* X, *then it is called* (i,j)-*almost b-continuous.*

Theorem 3.2. *The following statements are equivalent for a function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$.

(a) f *is* (i,j)-*almost b-continuous,* *(b)* (i,j)*-bcl*$(f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(B))))) \subseteq f^{-1}(i\text{-}cl(B))$*; for every subset* B *of* Y*,*

(c) (i,j)*-bcl*$(f^{-1}(i\text{-}cl(j\text{-}int(G)))) \subseteq f^{-1}(G)$*; for every* (i,j)*-regular closed set* G *of* Y*,*

(d) (i,j)*-bcl*$(f^{-1}(i\text{-}cl(V))) \subseteq f^{-1}(i\text{-}cl(V))$*; for every* σ_j*-open set* V *of* Y*,*

(e) $f^{-1}(V) \subseteq (i,j)$*-bint*$(f^{-1}(i\text{-}int(j\text{-}cl(V))))$*; for every* σ_i*-open set* V *of* Y*.*

Proof. $(a) \Rightarrow$(b) Let $x \in X$ and B is any subset of Y. We assume that $x \in X \setminus f^{-1}(i\text{-}cl(B))$ and so, $f(x) \in Y \setminus i\text{-}cl(B)$. Then, there exists a σ_i-open set C of Y containing $f(x)$ such that $C \cap B = \emptyset$. Therefore, $C \cap i\text{-}cl(j\text{-}int(i\text{-}cl(B))) = \emptyset$. Hence, $i\text{-}int(j\text{-}cl(C)) \cap i\text{-}cl(j\text{-}int(i\text{-}cl(B))) = \emptyset$. By the given hypothesis, there exists an (i,j)-b-open set D such that $f(D) \subseteq i\text{-}int(j\text{-}cl(C))$. So, we have $D \cap f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(B)))) = \emptyset$. Therefore by Lemma 2.3, we have $x \in X \setminus (i,j)\text{-}bcl(f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(B)))))$. Hence, $(i,j)\text{-}bcl(f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(B))))) \subseteq f^{-1}(i\text{-}cl(B))$.

$(b) \Rightarrow$(c) Let G be an (i,j)-regular closed set in Y. Therefore, $G = i\text{-}cl(j\text{-}int(G))$.

Now, $(i,j)\text{-}bcl(f^{-1}(i\text{-}cl(j\text{-}int(G)))) = (i,j)\text{-}bcl(f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(j\text{-}int(G))))))$ $\subseteq f^{-1}(i\text{-}cl(j\text{-}int(G))) = f^{-1}(G)$.

$(c) \Rightarrow$(d) Let V be a σ_j-open in Y. Therefore, $i\text{-}cl(V)$ is (i,j)-regular closed in Y. Hence by (c), $(i,j)\text{-}bcl(f^{-1}(i\text{-}cl(V))) \subseteq (i,j)\text{-}bcl(f^{-1}(i\text{-}cl(j\text{-}int(i\text{-}cl(V))) \subseteq f^{-1}(i\text{-}cl(V))$.

$(d) \Rightarrow$(e) Let V be a σ_i-open in Y and so, $Y \setminus j\text{-}cl(V)$ is σ_j-open in Y. Hence by (d), $(i,j)\text{-}bcl(f^{-1}(i\text{-}cl(Y \setminus j\text{-}cl(V)))) \subseteq f^{-1}(i\text{-}cl(Y \setminus j\text{-}cl(V)))$.

$\Rightarrow (i,j)\text{-}bcl(f^{-1}(Y \setminus i\text{-}int(j\text{-}cl(V)))) \subseteq f^{-1}(Y \setminus i\text{-}int(j\text{-}cl(V)))$

$\Rightarrow (i,j)\text{-}bcl(X \setminus f^{-1}(i\text{-}int(j\text{-}cl(V)))) \subseteq X \setminus f^{-1}(i\text{-}int(j\text{-}cl(V)))$

$\Rightarrow X \setminus (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(V)))) \subseteq X \setminus f^{-1}(i\text{-}int(j\text{-}cl(V))) \subseteq X \setminus f^{-1}(V)$
Hence, $f^{-1}(V) \subseteq (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(V))))$.

$(e) \Rightarrow$(a) Let $x \in X$ and V be a σ_i-open set in Y containing $f(x)$. Then, $x \in f^{-1}(V) \subseteq (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(V))))$. Putting $U = (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(V))))$ and by Lemma 2.1, we have U is (i,j)-b-open and $U \subseteq f^{-1}(i\text{-}int(j\text{-}cl(V)))$. So $f(U) \subseteq i\text{-}int(j\text{-}cl(V))$. Hence, f is (i,j)-almost b-continuous.
□

Theorem 3.3. *The following statements are equivalent for a function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$*.*

(a) f *is* (i,j)*-almost* b*-continuous,*

(b) $f((i,j)\text{-}bcl(A)) \subseteq (i,j)\text{-}cl_\delta(f(A))$*; for every subset* A *of* X*,*

(c) $(i,j)\text{-}bcl(f^{-1}(B)) \subseteq f^{-1}((i,j)\text{-}cl_\delta(B))$*; for every subset* B *of* Y*,*

(d) $f^{-1}(C)$ *is* (i,j)*-*b*-closed in* X*; for every* (i,j)*-*δ*-closed subset* C *of* Y*,*

(e) $f^{-1}(D)$ *is* (i,j)*-*b*-open in* X*; for every* (i,j)*-*δ*-open subset* D *of* Y*.*

Proof. (a) $\Rightarrow$(b) Let A be a subset of X containing x and V be a σ_i-open set of Y containing $f(x)$. Since, f is (i,j)-almost b-continuous, there exists an (i,j)-b-open set U containing x such that, $f(U) \subseteq i\text{-}int(j\text{-}cl(V))$. Let $x \in (i,j)\text{-}bcl(A)$, then by Lemma 2.3, we have $U \cap A \neq \emptyset$; hence $\emptyset \neq f(U) \cap f(A) \subseteq i\text{-}int(j\text{-}cl(V)) \cap f(A)$. Since, V is σ_i-open in Y hence, $V \subseteq i\text{-}int(j\text{-}cl(V))$ and $i\text{-}int(j\text{-}cl(V))$ is (i,j)-regular open in Y. Hence, $f(x) \in (i,j)\text{-}cl_\delta f(A)$. Consequently, $(i,j)\text{-}bcl(A) \subseteq f^{-1}((i,j)\text{-}cl_\delta(f(A)))$. It implies that $f((i,j)\text{-}bcl(A)) \subseteq (i,j)\text{-}cl_\delta(f(A))$.

(b) $\Rightarrow$ (c) Suppose, B is any subset of Y. Then by (b), $f((i,j)\text{-}bcl(f^{-1}(B))) \subseteq (i,j)\text{-}cl_\delta(f(f^{-1}(B))) \subseteq (i,j)\text{-}cl_\delta(B)$. It implies $(i,j)\text{-}bcl(f^{-1}(B)) \subseteq f^{-1}((i,j)\text{-}cl_\delta(B))$.

(c) $\Rightarrow$ (d) Let C be an (i,j)-δ-closed subset of Y. Then by (c), $(i,j)\text{-}bcl(f^{-1}(C)) \subseteq f^{-1}(C)$ and so, $f^{-1}(C)$ is (i,j)-b-closed in X.

(d) $\Rightarrow$ (e) Let D be an (i,j)-δ-open subset of Y. Then, $Y \setminus D$ is (i,j)-δ-closed in Y. By (d), $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$ is (i,j)-b-closed in X. Hence, $f^{-1}(D)$ is (i,j)-b-open in X.

(e) $\Rightarrow$ (a) Let A be a σ_i-open subset of Y containing $f(x)$. Then, $i\text{-}int(j\text{-}cl(A))$ is (i,j)-regular open in Y containing $f(x)$. Since, $i\text{-}int(j\text{-}cl(A))$ is (i,j)-δ-open in Y, thus by (e), $f^{-1}(i\text{-}int(j\text{-}cl(A)))$ is (i,j)-b-open in X. Now, $A \subseteq i\text{-}int(j\text{-}cl(A))$. This implies that, $f^{-1}(A) \subseteq f^{-1}(i\text{-}int(j\text{-}cl(A)) = (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(A))))$. Hence, by theorem 3.1, f is (i,j)-almost b-continuous.
□

Definition 3.4. ([15]) *A function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is said to have* (i,j)*-*b *interiority condition, if* $(i,j)\text{-}bint(f^{-1}(j\text{-}cl(V))) \subseteq f^{-1}(V)$*; for every* σ_i*-open subset* V *of* Y*.*

Theorem 3.5. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a function. If* f *is* (i,j)*-almost* b*-continuous which satisfies* (i,j)*-*b *interiority condition, then* f *is* (i,j)*-*b*-*

continuous.

Proof. Let U be a σ_i-open subset of Y. By hypothesis, f is (i,j)-almost b-continuous. Therefore by theorem 3.1, we have $f^{-1}(U) \subseteq (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(U)))) \subseteq (i,j)\text{-}bint(f^{-1}(j\text{-}cl(U)))$. Again by the (i,j)-b interiority condition of f, we get $(i,j)\text{-}bint(f^{-1}(j\text{-}cl(U))) \subseteq f^{-1}(U)$. Thus, $f^{-1}(U) = (i,j)\text{-}bint(f^{-1}(j\text{-}cl(U)))$. So, by Lemma 2.1 $f^{-1}(U)$ is (i,j)-b-open. Hence, f is (i,j)-b-continuous.
□

Definition 3.6. ([7]) *A bitopological space (X,τ_1,τ_2) is said to be pairwise Hausdorff or pairwise T_2, if for each pair of distinct points x and y of X, there exist a τ_i-open set U containing x and a τ_j-open set V containing y such that $U \cap V = \emptyset$.*

Definition 3.7. ([15]) *A bitopological space (X,τ_1,τ_2) is said to be pairwise b-T_2, if for each pair of distinct points x and y of X, there exist a (i,j)-b-open set U containing x and a (j,i)-b-open set V containing y such that $U \cap V = \emptyset$.*

Theorem 3.8. *Let $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a function and Y is pairwise T_2. If for any two distinct points x and y of X, following conditions are hold*
(a) $f(x) \neq f(y)$,
(b) f is (i,j)-weakly b-continuous at x,
(c) f is (j,i)-almost b-continuous at y,
then, X is a pairwise b-T_2 space.

Proof. Let $x, y \in X$ such that $x \neq y$. Suppose, Y is pairwise T_2. Therefore, there exist a σ_i-open set U and a σ_j-open set V such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \emptyset$. Since $U \cap V = \emptyset$, so we have $j\text{-}cl(U) \cap (j\text{-}int(i\text{-}cl(V))) = \emptyset$. Again since f is (i,j)-weakly b-continuous at x and (j,i)-almost b-continuous at y, therefore there exists an (i,j)-b- open set F in X such that $x \in F$, $f(F) \subseteq j\text{-}cl(U)$ and there exists a (j,i)-b-open set G in X such that $y \in G$, $f(G) \subseteq j\text{-}int(i\text{-}cl(V))$. Thus, $F \cap G = \emptyset$. Hence, X is a pairwise b-T_2 space.
□

Definition 3.9. ([4]) *A bitopological space (X,τ_1,τ_2) is said to be pairwise Urysohn, if for each pair of distinct points x and y of X, there exist a τ_i-open set U containing x and a τ_j-open set V containing y such that $j\text{-}cl(U) \cap i\text{-}cl(V) = \emptyset$.*

Theorem 3.10. *Let $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a function and Y is a pairwise Urysohn space. If f is pairwise almost b-continuous, then X is pairwise b-T_2 space.*

Proof. Let $x, y \in X$ such that $x \neq y$. Therefore, $f(x) \neq f(y)$. Since, Y is pairwise Urysohn, therefore there exist a σ_i-open set U containing $f(x)$ and a σ_j-open set V containing $f(y)$ such that $j\text{-}cl(U) \cap i\text{-}cl(V) = \emptyset$. It implies $i\text{-}int(j\text{-}cl(U)) \cap j\text{-}int(i\text{-}cl(V)) = \emptyset$. Hence, $f^{-1}(i\text{-}int(j\text{-}cl(U))) \cap f^{-1}(j\text{-}int(i\text{-}cl(V))) = \emptyset$ and so, $(i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(U)))) \cap (j,i)\text{-}bint(f^{-1}(j\text{-}int(i\text{-}cl(V)))) = \emptyset$. Since f is pairwise almost b-continuous, therefore by theorem 3.1, we have $x \in f^{-1}(U) \subseteq (i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(U))))$ and $y \in f^{-1}(V) \subseteq (j,i)\text{-}bint(f^{-1}(j\text{-}int(i\text{-}cl(V))))$. Hence, X is a pairwise b-T_2 space.
□

Theorem 3.11. *Let $f : (X_1, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-weakly b-continuous, $g : (X_2, \psi_1, \psi_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-almost b-continuous and Y is pairwise Hausdorff, then the set $\{(x,y) \in X_1 \times X_2 : f(x) = g(y)\}$ is an (i,j)-b-closed in $X_1 \times X_2$.*

Proof. Let $G = \{(x,y) \in X_1 \times X_2 : f(x) = g(y)\}$ and $(x,y) \in (X_1 \times X_2) \setminus G$. Thus, we get $f(x) \neq f(y)$. Since Y is pairwise Hausdorff, therefore there exist a σ_i-open set U_1 and a σ_j-open set U_2 of Y such that $f(x) \in U_1$, $g(y) \in U_2$ and $U_1 \cap U_2 = \emptyset$. Since, U_1 and U_2 are disjoint, hence $j\text{-}cl(U_1) \cap (i\text{-}int(j\text{-}cl(U_2)) = \emptyset$. Since, f is (i,j)-weakly b-continuous; there exists an (i,j)-b-open set V_1 containing x such that $f(V_1) \subseteq j\text{-}cl(U_1)$. Again g is (i,j)-almost b-continuous, thus there exists an (i,j)-b-open set V_2 containing y such that $g(V_2) \subseteq i\text{-}int(j\text{-}cl(U_2))$. Thus, we obtain $(x,y) \in V_1 \times V_2 \subseteq (X_1 \times X_2) \setminus G$ and $V_1 \times V_2$ is (i,j)-b-open in $X_1 \times X_2$. It implies G is an (i,j)-b-closed in $X_1 \times X_2$.
□

Definition 3.12. ([13]) *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-almost regular, if for every $x \in X$ and for every τ_i-open set V of X, there exists a τ_i-open set U containing x such that $x \in U \subseteq j\text{-}cl(U) \subseteq i\text{-}int(j\text{-}cl(V))$.*

Lemma 3.13. ([6]) *For a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent.*

(a) *f is (i,j)-almost b-continuous,*

(b) *$f^{-1}(i\text{-}int(j\text{-}cl(V)))$ is (i,j)-b-open set in X; for each σ_i-open set V in Y,*

(c) *$f^{-1}(i\text{-}cl(j\text{-}int(F)))$ is (i,j)-b-closed set in X; for each σ_i-closed set F in Y,*

(d) *$f^{-1}(F)$ is (i,j)-b-closed set in X; for each (i,j)-regular closed set F of Y,*

(e) *$f^{-1}(V)$ is (i,j)-b-open set in X; for each (i,j)-regular open set V of Y.*

Theorem 3.14. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function and Y is (i,j)-almost regular. Then, f is (i,j)-almost b-continuous if and only if f is (i,j)-weakly b-continuous.*

Proof. Necessity : It is obvious that (i,j)-almost b-continuity implies (i,j)-weakly b-continuity.

Sufficiency : Assume that f is (i,j)-weakly b-continuous. Let U be an (i,j)-regular open set in Y such that, $x \in f^{-1}(U)$. This implies $f(x) \in U$. Since Y is (i,j)-almost regular, therefore there exists a (i,j)-regular open set V in Y such that $f(x) \in V \subseteq j\text{-}cl(V) \subset U$. Since f is (i,j)-weakly b-continuous, therefore there exists an (i,j)-b-open set W in X containing x such that $f(W) \subseteq j\text{-}cl(V) \subseteq U$. Thus, we get $W \subseteq f^{-1}(U)$. Thus, $x \in W = (i,j)\text{-}bint(W) \subseteq (i,j)\text{-}bint(f^{-1}(U))$. Hence, $f^{-1}(U) \subseteq (i,j)\text{-}bint(f^{-1}(U))$. Consequently, $f^{-1}(U) = (i,j)\text{-}bint(f^{-1}(U))$ and so, $f^{-1}(U)$ is (i,j)-b-open. By Lemma 3.1, f is (i,j)-almost b-continuous.
□

Definition 3.15. ([12]) *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-semi regular, if for every $x \in X$ and for every τ_i-open set V of X, there exists a τ_i-open set U containing x such that $x \in U \subseteq i\text{-}int(j\text{-}cl(U)) \subseteq V$.*

Theorem 3.16. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function and Y is (i,j)-semi regular. If f is (i,j)-almost b-continuous, then f is (i,j)-b-continuous.*

Proof. Let U be a σ_i-open set of Y containing $f(x)$. Therefore, $x \in f^{-1}(U)$. Since, Y is (i,j)-semi regular thus there exists a σ_i-open set V such that $f(x) \in V \subset i\text{-}int(j\text{-}cl(V)) \subseteq U$. Since f is (i,j)-almost b-continuous, hence there exists an (i,j)-b-open set W in X containing x such that $f(W) \subseteq i\text{-}int(j\text{-}cl(V)) \subset U$. So, $x \in W = (i,j)\text{-}bint(W) \subseteq (i,j)\text{-}bint(f^{-1}(U))$ and hence $f^{-1}(U) \subseteq (i,j)\text{-}bint(f^{-1}(U))$. Hence, $f^{-1}(U) = (i,j)\text{-}bint(f^{-1}(U))$. Now by Lemma 2.1, $f^{-1}(U)$ is an (i,j)-b-open in X. Thus, f is a (i,j)-b-continuous function.
□

Definition 3.17. *A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-almost b -open if $f(U) \subseteq i\text{-}int(j\text{-}cl(f(U)))$; for every (i,j)-b-open set U of X.*

Theorem 3.18. *If a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-almost b-open and (i,j)-weakly b-continuous, then f is (i,j)-almost b-continuous.*

Proof. Let V be a σ_i-open set of Y containing $f(x)$. Since f is (i,j)-weakly b-continuous, thus there exists an (i,j)-b-open set U in X containing x such

that $f(U) \subseteq j\text{-}cl(V)$. Also f is (i,j)-almost b-open, therefore $f(U) \subseteq i\text{-}int(j\text{-}cl(f(U))) \subseteq i\text{-}int(j\text{-}cl(V))$. Hence, f is (i,j)-almost b-continuous.
□

Lemma 3.19. ([6]) *For a function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent.*

(a) f is (i,j)-almost b-continuous,

(b) for each $x \in X$ and each (i,j)-regular open set V of Y containing $f(x)$, there exists an (i,j)-b-open U in X containing x such that $f(U) \subseteq V$,

(c) for each $x \in X$ and each (i,j)-δ-open set V of Y containing $f(x)$, there exists an (i,j)-b-open U in X containing x such that $f(U) \subseteq V$.

Theorem 3.20. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function and $g : X \longrightarrow X \times Y$ be the function defined by $g(x) = (x, f(x))$, for every $x \in X$, then g is (i,j)-almost b-continuous if and only if f is (i,j)-almost b-continuous.*

Proof. Let $x \in X$ and V be an (i,j)-regular open set of Y such that $f(x) \in V$. Then, $g(x) = (x, f(x)) \in X \times V$ and $X \times V$ is (i,j)-regular open in $X \times Y$. Since g is (i,j)-almost b-continuous, thus there exists an (i,j)-b-open set U containing x such that $g(U) \subseteq X \times Y$. Thus, we have $f(U) \subseteq V$. Hence by Lemma 3.2, we have f is (i,j)-almost b-continuous function.

Conversely, let $x \in X$ and W be an (i,j)-regular open set of $X \times Y$ such that $g(x) = (x, f(x)) \in X \times Y$. Then, there exists an (i,j)-regular open set V in Y such that $U \times V \subseteq W$. Since f is (i,j)-almost b-continuous, hence there exists an (i,j)-b-open set A containing x such that $f(A) \subseteq V$. Let $B = U \cap A$, then B is an (i,j)-b-open set containing x and so; $g(B) \subseteq U \times V \subseteq W$. Hence, g is (i,j)-almost b-continuous.
□

Theorem 3.21. *If $g : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-almost b-continuous and A is (i,j)-δ-closed set in $X \times Y$, then $P_X(A \cap G(g))$ is (i,j)-b-closed in X; where P_X denotes the projection of $X \times Y$ onto X and $G(g)$ denotes the graph of g.*

Proof. Let A be an (i,j)-δ-closed set in $X \times Y$. Consider $x \in (i,j)\text{-}bcl(P_X(A \cap G(g)))$. Again, let U be a τ_i-open set of X containing x and V be a σ_i-open set of Y containing $g(x)$. Since g is (i,j)-almost b-continuous, therefore by theorem 3.1, $x \in g^{-1}(V) \subseteq (i,j)\text{-}bint(g^{-1}(i\text{-}int(j\text{-}cl(V))))$ and $U \cap (i,j)\text{-}bint(g^{-1}(i\text{-}int(j\text{-}cl(V))))$ is (i,j)-b-open in X containing x. Since $x \in (i,j)\text{-}bcl(P_X(A \cap G(g)))$,

therefore $[U\cap(i,j)\text{-}bint(g^{-1}(i\text{-}int(j\text{-}cl(V))))]\cap P_X(A\cap G(g))$ contains one point say y of X, which implies $(y,g(y))\in A$ and $g(y)\in i\text{-}int(j\text{-}cl(V))$. Then, $\emptyset\neq(U\times(i\text{-}int(j\text{-}cl(V))))\cap A\subseteq i\text{-}int(j\text{-}cl(U\times V))\cap A$ and hence, $(x,g(x))\in(i,j)\text{-}cl_\delta(A)$. Since A is (i,j)-δ-closed, $(x,g(x))\in A\cap G(g)$ and $x\in P_X(A\cap G(g))$. Therefore, $(i,j)\text{-}bcl(P_X(A\cap G(g)))\subseteq P_X(A\cap G(g))$. Hence, $P_X(A\cap G(g))$ is (i,j)-b-closed in X.
□

Definition 3.22. ([3]) *Let (X,τ_1,τ_2) be a bitopological space and $A\subseteq X$, then A is said to be (i,j)-quasi H-closed relative to X; if for each cover $\{B_\alpha:\alpha\in\Delta\}$ of A by τ_i-open subsets of X, there exists a finite subset Δ_0 of Δ such that $A\subseteq\bigcup\{j\text{-}cl(B_\alpha):\alpha\in\Delta_0\}$, where Δ is an index set.*

Definition 3.23. *Let (X,τ_1,τ_2) be a bitopological space and $A\subseteq X$, then A is said to be (i,j)-b-compact relative to X, if every cover of A by (i,j)-b-open sets of X has a finite subcover.*

Theorem 3.24. *If a function $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is (i,j)-almost b-continuous and A is (i,j)-b-compact relative to X, then $f(A)$ is (i,j)-quasi H-closed relative to Y.*

Proof. Let A be (i,j)-b-compact relative to X and $\{B_\alpha:\alpha\in\Delta\}$ be any cover of $f(A)$ by σ_i-open subsets of Y. Therefore, $f(A)\subseteq\bigcup\{B_\alpha:\alpha\in\Delta\}$ and so; $A\subseteq\bigcup\{f^{-1}(B_\alpha:\alpha\in\Delta\}$. Since, f is (i,j)-almost b-continuous, therefore by theorem 3.1, we have $f^{-1}(B_\alpha)\subseteq(i,j)\text{-}bint(f^{-1}(i\text{-}int(j\text{-}cl(B_\alpha))))\subseteq(i,j)\text{-}bint(f^{-1}(j\text{-}cl(B_\alpha)))$. Then, $A\subseteq\bigcup\{(i,j)\text{-}bint(f^{-1}(j\text{-}cl(B_\alpha))):\alpha\in\Delta\}$. Since A is (i,j)-b-compact relative to X and $(i,j)\text{-}bint(f^{-1}(j\text{-}cl(B_\alpha)))$ is (i,j)-b-open for each $\alpha\in\Delta$, therefore there exists a finite subset Δ_0 of Δ such that $A\subseteq\bigcup\{(i,j)\text{-}bint(f^{-1}(j\text{-}cl(B_\alpha))):\alpha\in\Delta_0\}$. This implies $f(A)\subseteq\bigcup\{f((i,j)\text{-}bint(f^{-1}(j\text{-}cl(B_\alpha)))):\alpha\in\Delta_0\}\subseteq\bigcup\{f(f^{-1}(j\text{-}cl(B_\alpha))):\alpha\in\Delta_0\}\subseteq\bigcup\{j\text{-}cl(B_\alpha):\alpha\in\Delta_0\}$. Hence, $f(A)$ is (i,j)-quasi H-closed relative to Y.

□

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