Bol. Soc. Paran. Mat. (3s.) **v. 37** 2 (2019): 157–166.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.31658

On Derivations of Prime and Semi-prime Gamma Rings

L. Kamali Ardakani, B. Davvaz and Shuliang Huang

abstract: The concept of Γ-ring is a generalization of ring. Two important classes of Γ-rings are prime and semi-prime Γ-rings. In this paper, we consider the concept of derivations on prime and semi-prime Γ-rings and we study some of their properties.

Key Words: Γ-ring, derivation, bi-(σ, τ) derivation, trace.

Contents

1. Introduction

In [11], Nobusawa introduced Γ-rings as a generalization of ternary rings. Barnes [2] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [2], Luh [9] and Kyuno [8] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Since then, some papers have been published on the topic of Γ-rings [1,5,7,10,12]. In [6], Chakraborty and Pau defined an isomorphism, an anti-isomorphism and a Jordan isomorphism in a Γ-ring and developed some important results relating to these concepts, also see [13,14].

The Γ-rings are defined in [11] as follows. Let M and Γ be additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied: (i) $a\alpha b \in M$; (ii) $(a + b)\alpha c = a\alpha c + b\alpha c$; (iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$; (iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$; then M is called a Γ-*ring*, (in the sense of [2]). Every ring is a Γ-ring. A *right* (*left*, respectively) *ideal* of a Γ-ring M is an additive subgroup U of M such that $U\Gamma M \subseteq U$ ($M\Gamma U \subseteq U$, respectively). If U is both right and left ideal, then we say U is an *ideal* of M. If the following condition holds for a Γ-ring M, then M is called a *prime* Γ-ring: $a\Gamma M\Gamma b = 0 \Rightarrow a = 0$ or $b = 0$, where $a, b \in M$. Also, M is called a *semi prime* Γ-ring, if $a\Gamma M\Gamma a = 0$ implies that $a = 0$, where $a \in M$. We refer to [1,3,6,7,15,16] to see more results about prime and semiprime rings. Throughout this paper all Γ-rings will be associative. An additive mapping $D : M \longrightarrow M$ is called a *derivation* if $D(x\gamma y) = D(x)\gamma y + x\gamma D(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. A mapping $B(.,.) : M \times M \longrightarrow M$ is said to be *symmetric* if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in M$. A mapping $f : M \longrightarrow M$ defined by $f(x) = B(x, x)$, where $B(.,.) : M \times M \longrightarrow$

2010 *Mathematics Subject Classification:* 16Y99, 16N60.

Submitted April 15, 2016. Published March 29, 2017

M is a symmetric mapping, is called the *trace* of B. It is obvious that, in case $B(.,.) : M \times M \longrightarrow M$ is a symmetric mapping which is also bi-additive, then trace of B satisfies the relation $f(x+y) = f(x) + f(y) + 2B(x,y)$, for all $x, y \in M$. We shall use also the fact that trace of symmetric bi-additive mapping is an even function. Let σ, τ be endomorphisms of Γ-ring M. A symmetric bi-additive mapping $D(.,.) : M \times M \longrightarrow M$ is called a *symmetric bi-(σ,τ) derivation*, if $D(x\gamma y, z) = D(x,z)\gamma\sigma(y) + \tau(x)\gamma D(y,z)$ is fulfilled for all $x, y, z \in M$ and $\gamma \in \Gamma$. Obviously, in this case also the relation $D(x, y\gamma z) = D(x,y)\gamma\sigma(z) + \tau(y)\gamma D(x,z)$ holds for all $x, y, z \in M$ and $\gamma \in \Gamma$.

2. Main results

In the proofs of several theorems in [4], the authors used the following lemma:

(Lemma 1, [4]). Let $D : M \longrightarrow M$ be a derivation, where M is a prime Γ-ring. Let U be a nonzero right ideal of M. Suppose either

(i) $a\gamma D(x) = 0$, $x \in U$ and $\gamma \in \Gamma$,

(ii) $D(x)\gamma a = 0$, $x \in U$ and $\gamma \in \Gamma$,

holds. In both cases, we have $a = 0$ or $D = 0$.

Unfortunately, the proofs of some theorems are not correct. In order to correct them, we use the following lemmas instead of Lemma 1 in [4].

Lemma 2.1. *Let M be a prime Γ-ring, U be a nonzero ideal on M and σ, τ be automorphisms of M. Also, let $D : M \times M \longrightarrow M$ be a symmetric bi-(σ,τ)-derivation and d denotes the trace of D. Then, for all $x, y \in M$,*

(i) If $U\Gamma x = 0$ or $x\Gamma U = 0$, then $x = 0$;

(ii) If $x\Gamma U\Gamma y = 0$, then $x = 0$ or $y = 0$;

(iii) If $D(u,v) = 0$, for all $u, v \in U$, then $D = 0$;

(iv) If M is 2-torsion free and $d(U) = 0$, then $D = 0$.

Proof: (i) Suppose that $U\Gamma x = 0$, then $u\gamma M\Gamma x \subseteq U\Gamma x = \{0\}$, for all $u \in U$ and $\gamma \in \Gamma$. So $x = 0$, since M is prime and $U \neq 0$. In the case $x\Gamma U = 0$, the proof is similar.

(ii) Suppose that $x\Gamma U\Gamma y = 0$, then $x\Gamma U\Gamma M\Gamma y \subseteq x\Gamma U\Gamma y = \{0\}$. Hence $x\Gamma U = 0$ or $y = 0$, since M is prime. So by (i), $x = 0$ or $y = 0$.

(iii) Suppose that $D(u,v) = 0$, for all $u, v \in U$. Replace u by $r\gamma u$, where $r \in M$ and $\gamma \in \Gamma$, we get $0 = D(r\gamma u, v) = D(r,v)\gamma\sigma(u) + \tau(r)\gamma D(u,v) = D(r,v)\gamma\sigma(u)$. So, $D(r,v) = 0$, by (i). Replace v by $s\beta v$, where $s \in M$ and $\beta \in \Gamma$, we get $0 = D(r, s\beta v) = D(r,s)\beta\sigma(v) + \tau(s)\beta D(r,v) = D(r,s)\beta\sigma(v)$. So, $D(r,s) = 0$, by (i). This means that $D = 0$.

(iv) Suppose that $d(U) = 0$, then $2D(u,v) = d(u+v) - d(u) - d(v) = 0$, for all

$u, v \in U$. Thus, $D(u, v) = 0$, for all $u, v \in U$, since M is 2-torsion free. Now, (iii) completes the proof. □

Lemma 2.2. *Let M be a prime Γ-ring and $D : M \times M \longrightarrow M$ be a symmetric bi-(σ, τ) derivation, where σ, τ are automorphisms of M. Also, let U be a nonzero ideal of M and $a \in M$. Suppose either*

(i) $a\gamma D(u, v) = 0$, $u, v \in U$ and $\gamma \in \Gamma$,

(ii) $D(u, v)\gamma a = 0$, $u, v \in U$ and $\gamma \in \Gamma$,

holds. In both cases, we have $a = 0$ or $D = 0$.

Proof: (i) Writing $u\gamma r$ instead of u in $a\gamma D(u, v) = 0$, where $u, v \in U$ and $r \in M$. We get

$$\begin{aligned} 0 &= a\gamma D(u\gamma r, v) = a\gamma(D(u, v)\gamma\sigma(r) + \tau(u)\gamma D(r, v)) \\ &= a\gamma D(u, v)\gamma\sigma(r) + a\gamma\tau(u)\gamma D(r, v) = a\gamma\tau(u)\gamma D(r, v), \end{aligned}$$

for all $u, v \in U$, $r \in M$. So, $a\Gamma\tau(U)\Gamma D(r, v) = 0$. This implies that $a = 0$ or $D = 0$, by Lemma 2.1.

(ii) Writing $r\gamma u$ instead of u in $D(u, v)\gamma a = 0$, where $u, v \in U$ and $r \in M$. We get

$$\begin{aligned} 0 &= D(r\gamma u, v)\gamma a = (D(r, v)\gamma\sigma(u) + \tau(r)\gamma D(u, v))\gamma a \\ &= D(r, v)\gamma\sigma(u)\gamma a + \tau(r)\gamma D(u, v)\gamma a = D(r, v)\gamma\sigma(u)\gamma a, \end{aligned}$$

for all $u, v \in U$, $r \in M$. So, $D(r, v)\Gamma\sigma(U)\Gamma a = 0$. This implies that $a = 0$ or $D = 0$, by Lemma 2.1. □

Lemma 2.3. *Let M be a 2-torsion free prime Γ-ring and U be a nonzero ideal of M. Also, let $a, b \in M$ be fixed elements. If $a\gamma u\beta b + b\gamma u\beta a = 0$ is fulfilled for all $u \in U$ and $\alpha, \beta \in \Gamma$, then either $a = 0$ or $b = 0$.*

`(Theorem 1, [4])`. Let M be a prime Γ-ring of characteristic not two and U be nonzero ideal of M. Suppose there exist symmetric bi-(σ, τ) derivations $D_1(.,.) : M \times M \longrightarrow M$ and $D_2(.,.) : M \times M \longrightarrow M$ such that $D_1(d_2(u), u) = 0$ holds for all $u \in U$, where d_2 denotes the trace of D_2. In this case $D_1 = 0$ or $D_2 = 0$.

Now, we give a corrected version of the above theorem as follows:

Theorem 2.4. *Let M be a 2-torsion free prime Γ-ring, U be a nonzero ideal of M and σ, τ be automorphisms of M such that $\sigma(U) \subseteq U$ and $\sigma = \tau$. Also, let there exist symmetric bi-(σ, τ) derivations $D_1(.,.), D_2(.,.) : M \times M \longrightarrow M$ such that $D_1(d_2(u), \sigma(u)) = 0$ holds for all $u \in U$, where d_2 denotes the trace of D_2. In this case $D_1 = 0$ or $D_2 = 0$.*

Proof: By linearity of the relation

$$D_1(d_2(u), \sigma(u)) = 0, \tag{2.1}$$

one obtains

$$D_1(d_2(u) + d_2(v) + 2D_2(u, v), \sigma(u) + \sigma(v)) = 0, \text{for all } u, v \in U,$$

whence

$$D_1(d_2(u), \sigma(v)) + D_1(d_2(v), \sigma(u)) + 2D_1(D_2(u, v), \sigma(u)) + 2D_1(D_2(u, v), \sigma(v)) = 0$$

according to (2.1). Substituting in the equation above u by $-u$, we obtain by comparing this new equation with the equation above that

$$D_1(d_2(u), \sigma(v)) + 2D_1(D_2(u, v), \sigma(u)) = 0 \tag{2.2}$$

is fulfilled for all pairs $u, v \in U$. Let us replace in (2.2) v by $u\alpha v$, where $\alpha \in \Gamma$. Then,

$$\begin{aligned} 0 &= D_1(d_2(u), \sigma(u\alpha v)) + 2D_1(D_2(u, u\alpha v), \sigma(u)) \\ &= D_1(d_2(u), \sigma(u)\alpha\sigma(v)) + 2D_1(d_2(u)\alpha\sigma(v) + \sigma(u)\alpha D_2(u, v), \sigma(u)) \\ &= D_1(d_2(u), \sigma(u))\alpha\sigma^2(v) + \sigma^2(u)\alpha D_1(d_2(u), \sigma(v)) \\ &\quad +2D_1(d_2(u), \sigma(u))\alpha\sigma^2(v) + 2\sigma(d_2(u))\alpha D_1(\sigma(v), \sigma(u)) \\ &\quad +2d_1(\sigma(u))\alpha\sigma(D_2(u, v)) + 2\sigma^2(u)\alpha D_1(D_2(u, v), \sigma(u)) \\ &= 2\sigma(d_2(u))\alpha D_1(\sigma(v), \sigma(u)) + 2d_1(\sigma(u))\alpha\sigma(D_2(u, v)), \end{aligned}$$

where d_1 denotes the trace of D_1. In the above calculation, we used (2.1) and (2.2). Thus, we have

$$\sigma(d_2(u))\alpha D_1(\sigma(u), \sigma(v)) + d_1(\sigma(u))\alpha\sigma(D_2(u, v)) = 0, \tag{2.3}$$

for all $u, v \in U$ and $\alpha \in \Gamma$. Let us write in (2.3), $v\beta u$ instead of v, where $\beta \in \Gamma$. We have

$$\begin{aligned} 0 &= \sigma(d_2(u))\alpha D_1(\sigma(u), \sigma(v\beta u)) + d_1(\sigma(u))\alpha\sigma(D_2(u, v\beta u)) \\ &= \sigma(d_2(u))\alpha D_1(\sigma(u), \sigma(v)\beta\sigma(u)) + d_1(\sigma(u))\alpha\sigma(D_2(u, v)\beta\sigma(u) + \sigma(v)\beta d_2(u)) \\ &= \sigma(d_2(u))\alpha D_1(\sigma(u), \sigma(v))\beta\sigma^2(u) + \sigma(d_2(u))\alpha\sigma^2(v)\beta d_1(\sigma(u)) \\ &\quad +d_1(\sigma(u))\alpha\sigma(D_2(u, v))\beta\sigma^2(u) + d_1(\sigma(u))\alpha\sigma^2(v)\beta\sigma(d_2(u)) \\ &= \sigma(d_2(u))\alpha\sigma^2(v)\beta d_1(\sigma(u)) + d_1(\sigma(u))\alpha\sigma^2(v)\beta\sigma(d_2(u)). \end{aligned}$$

Thus, we have

$$\sigma(d_2(u))\alpha\sigma^2(v)\beta d_1(\sigma(u)) + d_1(\sigma(u))\alpha\sigma^2(v)\beta\sigma(d_2(u)) = 0, \tag{2.4}$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$. So, by Lemma 2.3, we have

$$d_2(u) = 0 \text{ or } d_1(\sigma(u)) = 0, \text{ for all } u \in U, \tag{2.5}$$

since σ is an automorphism and $\sigma(U) \subseteq U$. We claim that $d_1(U) = 0$ and $d_2(U) = 0$. Suppose that $d_1(U) \neq 0$ or $d_2(U) \neq 0$. So, there exist elements $u_2, u_3 \in U$ such that $d_1(u_3) \neq 0$ and $d_2(u_2) \neq 0$. There exists element $u_1 \in U$ such that $\sigma(u_1) = u_3$, since σ is onto. Therefore, we have $d_1\sigma(u_1) \neq 0$ and $d_2(u_2) \neq 0$. So, $d_2(u_1) = 0$ and $d_1\sigma(u_2) = 0$, by (2.5). Let us replace u by u_2 in (2.3), we get $0 = \sigma d_2(u_2)\alpha D_1(\sigma(u_2), \sigma(v)) + d_1\sigma(u_2)\alpha\sigma D_2(u_2, v) = \sigma d_2(u_2)\alpha D_1(\sigma(u_2), \sigma(v))$, for all $v \in U$. Using Lemma 2.2, we obtain that $D_1(\sigma(u_2), \sigma(v)) = 0$, for all $v \in U$. In particular, we have $D_1(\sigma(u_1), \sigma(u_2)) = 0$. So, $d_1(\sigma(u_1 + u_2)) = d_1\sigma(u_1) + d_1\sigma(u_2) + 2D_1(\sigma(u_1), \sigma(u_2)) = d_1\sigma(u_1) \neq 0$. Now, replace u by u_1 in (2.3), we get $0 = \sigma d_2(u_1)\alpha D_1(\sigma(u_1), \sigma(v)) + d_1\sigma(u_1)\alpha\sigma D_2(u_1, v) = d_1\sigma(u_1)\alpha\sigma D_2(u_1, v)$, for all $v \in U$. Using Lemma 2.2, we obtain that $D_2(u_1, v) = 0$, for all $v \in U$. In particular, we have $D_2(u_1, u_2) = 0$. So, $d_2(u_1+u_2) = d_2(u_1)+d_2(u_2)+2D_2(u_1, u_2) = d_2(u_2) \neq 0$. If we set $v = u_1 + u_2$, then $d_2(v) \neq 0$ and $d_1\sigma(v) \neq 0$ and this is a contradiction with (2.5). So, $d_1(U) = 0$ or $d_2(U) = 0$. This implies that $D_1 = 0$ or $D_2 = 0$, by Lemma 2.1. $\square$

`(Theorem 2, [4])`. Let M be a 2-torsion free semi-prime Γ-ring and U be a nonzero ideal of M. Suppose there exists such a symmetric bi-(σ, τ) derivation $D(.,.) : M \times M \longrightarrow M$ that $D(d(u), u) = 0$ holds for all $u \in U$, where d denotes the trace of D. In this case we have $D = 0$.

Now, we give a corrected version of the above theorem as follows:

Theorem 2.5. *Let M be a 2-torsion free semi-prime Γ-ring, U be a nonzero ideal of M and σ, τ be automorphisms of M such that $\sigma(U) \subseteq U$ and $\sigma = \tau$. Also, let there exists symmetric bi-(σ, τ) derivation $D(.,.) : M \times M \longrightarrow M$ such that $d\sigma = \sigma d$ and $D(d(u), \sigma(u)) = 0$ holds for all $u \in U$, where d denotes the trace of D. In this case we have $D = 0$.*

Proof: In this case (2.4) of Theorem 2.4 reduces to

$$0 = \sigma(d(u))\alpha\sigma^2(v)\beta d(\sigma(u)) + d(\sigma(u))\alpha\sigma^2(v)\beta\sigma(d(u)) = 2\sigma(d(u))\alpha\sigma^2(v)\beta\sigma(d(u)),$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$. This implies that $d(U) = 0$, since M is 2-torsion free and semi-prime. Therefore $D = 0$, by Lemma 2.1. $\square$

`(Theorem 3, [4])`. Let M be prime Γ-ring of characteristic not two and three. Let $\tau(U) \subset U$, U be nonzero ideal of M and $\sigma\tau = \tau\sigma$ and $\sigma = \tau$. Let $D_1(.,.) : M \times M \longrightarrow M$ and $D_2(.,.) : M \times M \longrightarrow M$ be symmetric bi-(σ, τ) derivations. Suppose further that there exists a symmetric bi-additive mapping $B(.,.) : M \times M \longrightarrow M$ such that $d_1(d_2(u)) = f(u)$ holds for all $u \in U$, where d_1 and d_2 are the traces of D_1 and D_2 respectively, and f is the trace of B. Then either $D_1 = 0$ or $D_2 = 0$.

Now, we give a corrected version of the above theorem as follows:

Theorem 2.6. *Let M be a 6-torsion free prime Γ-ring, U be a nonzero ideal of M and σ, τ be automorphisms of M such that $\sigma(U) \subseteq U$ and $\sigma = \tau$. Also, let*

$D_1(.,.), D_2(.,.) : M\times M \longrightarrow M$ be symmetric bi-(σ,τ) derivations and there exists a symmetric bi-additive mapping $B(.,.) : M\times M \longrightarrow M$ such that $d_1(d_2(u)) = f(u)$ holds for all $u \in U$, where d_1 and d_2 are the traces of D_1 and D_2, respectively, and f is the trace of B. Then either $D_1 = 0$ or $D_2 = 0$.

Proof: The linearity of the relation

$$d_1(d_2(u)) = f(u), \text{ for all } u \in U, \tag{2.6}$$

gives us

$$d_1(d_2(u) + d_2(v) + 2D_2(u, v)) = f(u) + f(v) + 2B(u, v), \text{ for all } u, v \in U.$$

Then, we have

$$\begin{aligned}
&d_1(d_2(u) + d_2(v)) + 2d_1(D_2(u, v)) + 4D_1(d_2(u) + d_2(v), D_2(u, v))\\
&= d_1(d_2(u)) + d_1(d_2(v)) + 2D_1(d_2(u), d_2(v)) + 2d_1(D_2(u, v))\\
&\ \ +4D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v))\\
&= f(u) + f(v) + 2B(u, v).
\end{aligned}$$

Using (2.6), we arrive at

$$\begin{aligned}
&D_1(d_2(u), d_2(v)) + d_1(D_2(u, v)) + 2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v))\\
&= B(u, v).
\end{aligned}$$

Substituting in the equation above u by $-u$, we obtain by comparing this new equation with the equation above that

$$2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v)) = B(u, v), \tag{2.7}$$

holds for all $u, v \in U$. Let us replace in (2.7) u by $2u$. We have

$$8D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v)) = B(u, v), \tag{2.8}$$

By comparing (2.7) and (2.8), we obtain $6D_1(d_2(u), D_2(u, v)) = 0$, which leads to

$$D_1(d_2(u), D_2(u, v)) = 0, \tag{2.9}$$

since, we have assumed that M is 6-torsion free. From (2.9) it follows that both terms on the left side of the relation (2.7) are zero, which means that $B = 0$ on U. Hence (2.6) reduces to

$$d_1(d_2(u)) = 0, \text{ for all } u \in U. \tag{2.10}$$

Let in (2.9) v be $v\alpha u$, where $\alpha \in \Gamma$. We have

$$\begin{aligned}
0 \ &= D_1(d_2(u), D_2(u, v\alpha u)) = D_1(d_2(u), D_2(u, v)\alpha\sigma(u)) + D_1(d_2(u), \sigma(v)\alpha d_2(u))\\
&= D_1(d_2(u), D_2(u, v))\alpha\sigma^2(u) + \sigma(D_2(u, v))\alpha D_1(d_2(u), \sigma(u))\\
&\ \ +D_1(d_2(u), \sigma(v))\alpha\sigma(d_2(u)) + \sigma^2(v)\alpha d_1(d_2(u)).
\end{aligned}$$

Using (2.9) and (2.10), we arrive at

$$D_1(d_2(u),\sigma(v))\alpha\sigma(d_2(u))+\sigma(D_2(u,v))\alpha D_1(d_2(u),\sigma(u))=0,\ \text{for all } u,v\in U \quad (2.11)$$

Let us replace in (2.11), v by $u\beta v$, where $\beta\in\Gamma$. We have

$$\begin{aligned}0 &= D_1(d_2(u),\sigma(u)\beta\sigma(v))\alpha\sigma(d_2(u))+\sigma(D_2(u,u\beta v))\alpha D_1(d_2(u),\sigma(u))\\ &= D_1(d_2(u),\sigma(u))\beta\sigma^2(v)\alpha\sigma(d_2(u))+\sigma^2(u)\beta D_1(d_2(u),\sigma(v))\alpha\sigma(d_2(u))\\ &\quad+\sigma(d_2(u))\beta\sigma^2(v)\alpha D_1(d_2(u),\sigma(u))+\sigma^2(u)\beta\sigma(D_2(u,v))\alpha D_1(d_2(u),\sigma(u))\end{aligned}$$

Now, by (2.11) we arrive finally at

$$D_1(d_2(u),\sigma(u))\beta\sigma^2(v)\alpha\sigma(d_2(u))+\sigma(d_2(u))\beta\sigma^2(v)\alpha D_1(d_2(u),\sigma(u))=0, \quad (2.12)$$

for $u,v\in U$ and $\alpha,\beta\in\Gamma$. So, we have $d_2(u)=0$ or $D_1(d_2(u),\sigma(u))=0$, by Lemma 2.3. If $d_2(u)=0$, then $D_1(d_2(u),\sigma(u))=D_1(0,\sigma(u))=0$. So, in general we have $D_1(d_2(u),\sigma(u))=0$, for all $u\in U$. Therefore, $D_1=0$ or $D_2=0$, by Theorem 2.4. □

`(Theorem 4, [4])`. Let M be a semi-prime Γ-ring which is 2-torsion and 3-torsion free. Let $d(U)\subset U$, $\tau(U)\subset U$, $\sigma(U)\subset U$, U be nonzero ideal of M, $\sigma\tau=\tau\sigma$ and $\sigma=\tau$. Let $D(.,.):M\times M\longrightarrow M$ and $B(.,.):M\times M\longrightarrow M$ be a symmetric bi-(σ,τ) derivation and a symmetric bi-additive mapping, respectively. Suppose that $d(d(u))=f(u)$ holds for all $u\in U$, where d is the trace of D and f is the trace of B. In this case we have $D=0$.

Now, we give a corrected version of the above theorem as follows:

Theorem 2.7. *Let M be a 6-torsion free semi-prime Γ-ring, U be a nonzero ideal of M and σ, τ be automorphisms of M such that $\sigma(U)\subseteq U$ and $\sigma=\tau$. Also, let $D(.,.):M\times M\longrightarrow M$ be a symmetric bi-(σ,τ) derivation such that $\sigma D(.,.)=D(\sigma(.),\sigma(.))$ and $B(.,.):M\times M\longrightarrow M$ be a symmetric bi-additive mapping. Suppose that $d(d(u))=f(u)$ holds for all $u\in U$, where d and f are the trace of D and B, respectively. In this case, we have $D=0$.*

Proof: Obviously, we can use the beginning of the proof of Theorem 2.6. In this case, relations (2.9) and (2.10) can be written in the form

$$D(d(u),D(u,v))=0,\ \text{for all } u,v\in U \quad (2.13)$$

and

$$d(d(u))=0,\ \text{for all } u\in U. \quad (2.14)$$

Let us write in (2.13), $v\alpha w$ instead of v, where $w\in M$ and $\alpha\in\Gamma$. We have

$$\begin{aligned}0 &= D(d(u),D(u,v\alpha w))=D(d(u),D(u,v)\alpha\sigma(w))+D(d(u),\sigma(v)\alpha D(u,w))\\ &= D(d(u),D(u,v))\alpha\sigma^2(w)+\sigma(D(u,v))\alpha D(d(u),\sigma(w))\\ &\quad+D(d(u),\sigma(v))\alpha\sigma(D(u,w))+\sigma^2(v)\alpha D(d(u),D(u,w)).\end{aligned}$$

Hence by (2.13), we have

$$\sigma(D(u,v))\alpha D(d(u),\sigma(w))+D(d(u),\sigma(v))\alpha\sigma(D(u,w))=0.$$

In particular, for $w=\sigma^{-1}d(u)$ we obtain

$$\sigma(D(u,v))\alpha D(d(u),\sigma(\sigma^{-1}d(u)))+D(d(u),\sigma(v))\alpha\sigma(D(u,\sigma^{-1}d(u)))=0.$$

So,

$$D(d(u),\sigma(v))\alpha\sigma(D(u,\sigma^{-1}d(u)))=0,\ \text{for all } u,v\in U. \tag{2.15}$$

according to (2.14). Replace in (2.15), v by $u\beta v$, where $\beta\in\Gamma$. We have

$$\begin{aligned}
0\ &= D(d(u),\sigma(u\beta v))\alpha\sigma(D(u,\sigma^{-1}d(u)))\\
&= D(d(u),\sigma(u)\beta\sigma(v))\alpha\sigma(D(u,\sigma^{-1}d(u)))\\
&= D(d(u),\sigma(u))\beta\sigma^2(v)\alpha\sigma(D(u,\sigma^{-1}d(u)))\\
&\quad +\sigma^2(u)\beta D(d(u),\sigma(v))\alpha\sigma(D(u,\sigma^{-1}d(u)))\\
&= D(d(u),\sigma(u))\beta\sigma^2(v)\alpha\sigma(D(u,\sigma^{-1}d(u)))\\
&= D(d(u),\sigma(u))\beta\sigma^2(v)\alpha D(\sigma(u),d(u))\\
&= D(d(u),\sigma(u))\beta\sigma^2(v)\alpha D(d(u),\sigma(u)),
\end{aligned}$$

according to (2.15) and finally

$$D(d(u),\sigma(u))=0,\ \text{for all } u\in U,$$

since we have assumed that M is semi-prime Γ-ring. Now, Theorem 2.5 completes the proof. □

Remark 2.8. *In Example 1 in [4], d is not a symmetric bi-(σ,τ) derivation.*

Let σ, τ be endomorphisms of Γ-ring M. A symmetric bi-additive mapping $D(.,.):M\times M\longrightarrow M$ is called a *symmetric bi-(σ,τ) derivation*, if

$$D(x\gamma y,z)=D(x,z)\gamma\sigma(y)+\tau(x)\gamma D(y,z)$$

is fulfilled for all $x,y,z\in M$ and $\gamma\in\Gamma$.

Example 2.1. *Let $(R,+,\circ)$ be a commutative ring and $\Gamma=\{\circ\}$. Then $M=(R,+,\Gamma)$ is Γ-ring. Also, let the functions $\sigma,\tau:M\longrightarrow M$ and $D(.,.):M\times M\longrightarrow M$ be defined as $\sigma(x)=x$, $\tau(x)=0$ and $D(x,y)=x\circ y$, for all $x,y\in M$. Then, σ,τ are endomorphisms and D is symmetric bi-(σ,τ) derivation.*

Example 2.2. *Let $(R,+,\circ)$ be a ring of characteristic 2 and $\Gamma=\{\circ\}$. Consider Γ-ring $M=(R,+,\Gamma)$. Define the functions $\sigma,\tau:M\longrightarrow M$ and $D(.,.):M\times M\longrightarrow M$ as $\sigma(x)=\tau(x)=x$ and $D(x,y)=x\circ y+y\circ x$, for all $x,y\in M$. Then, σ,τ are endomorphisms and D is symmetric bi-(σ,τ) derivation.*

Example 2.3. *Let R be a commutative and unitary ring. Put*

$$M = \left\{ \begin{pmatrix} x & x \end{pmatrix} \mid x \in R \right\} \subseteq M_{1\times 2}(R) \text{ and } \Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

Then, M is Γ-ring [5]. Define the functions $\sigma, \tau : M \longrightarrow M$ and $D(.,.) : M \times M \longrightarrow M$ as $\sigma(X) = X$, $\tau(X) = \overline{0}$ and $D(X, Y) = X \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y$, for all $X, Y \in M$. It is clear that if $X = \begin{pmatrix} x & x \end{pmatrix}$ and $Y = \begin{pmatrix} y & y \end{pmatrix}$, then

$$D(X, Y) = \begin{pmatrix} xy & xy \end{pmatrix}.$$

It is easily to check that σ, τ are endomorphisms and D is symmetric bi-(σ, τ) derivation.

Example 2.4. *Let R be a commutative and unitary ring. Put $M = M_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$. Then, M is Γ-ring [5]. Define the functions $\sigma, \tau : M \longrightarrow M$ and $D(.,.) : M \times M \longrightarrow M$ as $\sigma(X) = \overline{0}$, $\tau(X) = \tau\left(\begin{pmatrix} x_1 & x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 & 0 \end{pmatrix}$ and $D(X, Y) = X \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y + Y \begin{pmatrix} 1 \\ 0 \end{pmatrix} X$, for all $X, Y \in M$. It is clear that if $X = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}$, then*

$$D(X, Y) = \begin{pmatrix} x_1 y_1 + y_1 x_1 & x_1 y_2 + y_1 x_2 \end{pmatrix}.$$

It is easily to check that σ, τ are endomorphisms and D is symmetric bi-(σ, τ) derivation.

Example 2.5. *The Abelian group $\mathbb{Z}$ with usual multiplication of numbers is a commutative $\mathbb{Z}$-ring. Define the functions $\sigma, \tau : \mathbb{Z} \longrightarrow \mathbb{Z}$ and $D(.,.) : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ as $\sigma(x) = x$, $\tau(x) = 0$ and $D(x, y) = xy$, for all $x, y \in \mathbb{Z}$. Then, σ, τ are endomorphisms and D is symmetric bi-(σ, τ) derivation.*

References

1. Ashraf, M.J., Malik, R., *Some differential identities in prime Γ-rings*, Bol. Soc. Parana. Mat. (3), 32, 193-205, (2014).
2. Barnes, W.E., *On the Γ-rings of Nobusawa*, Pacific J. Math. 18, 411-422, (1966).
3. Bell, H.E., Martindale, W.S. *Centralizing mappings of semiprime rings*, Canad. Math. Bull. 30, 92-101, (1987).
4. Ceran, S., Asci, M., *Symmetric bi-(σ, τ) derivations of prime and semi prime Gamma rings*, Bull. Korean Math. Soc. 43, 9-16, (2006).
5. Ceven, Y., Ozturk, M.A., *On Jordan generalized derivations in Gamma rings*, Hacettepe J. Math. Stati. 33, 11-14, (2004).
6. Chakraborty, S., Paul, A.C., *On Jordan isomorphisms of 2-torsion free prime gamma rings*, Novi Sad J. Math. 40, 1-5, (2010).
7. Halder, A.K., Paul, A.C., *Commutativity of two torsion free σ-prime gamma rings with nonzero derivations*, J. Phys. Sci. 15, 27-32, (2011).

8. Kyuno, S., *On prime Γ-rings*, Pacific J. Math.75, 185-190, (1978).

9. Luh, J., *On the theory of simple Γ-rings*, Michigan Math. J. 16, 65-75, (1969).

10. Majeed, A.H., Majeed, A.A., *Lie and Jordan structure in prime Γ-rings with Γ-centralizing derivations*, Iraq Journal of Science, 54, 182-190, (2013).

11. Nobusawa, N., *On a generalization of the ring theory*, Osaka J. Math. 1, 81-89, (1964).

12. Öztürk, M.A., Yazarli, H., *Some results on symmetric bi-(σ,τ)-derivations in near-rings*, Miskolc Math. Notes 11, 169-173, (2010).

13. Özden, D., Öztürk, M.A., Jun, Y.B., *Permuting tri-derivations in prime and semi-prime gamma rings*, Kyungpook Math. J. 46, 153-167, (2006).

14. Paul, A.C., Uddin, S., *On Artinian gamma rings*, Aligarh Bull. Math. 28, 15-19, (2009).

15. Rehman, N.U., Al-Omary, R.M., *On commutativity of 2-torsion free $*$-prime rings with generalized derivations*, Mathematica 53(76), 171-180, (2011).

16. Vukman, J., *Symmetric bi-derivations on prime and semi-prime rings*, Aequationes Math. 38, 245-254, (1989).

L. Kamali Ardakani,
Department of Engineering Sciences,
Ardakan University,
Ardakan, Iran.
E-mail address: l.kamali@ardakan.ac.ir

and

B. Davvaz,
Department of Mathematics,
Yazd University,
Yazd, Iran.
E-mail address: davvaz@yazd.ac.ir

and

Shuliang Huang,
Department of Mathematics,
Chuzhou University,
Chuzhou Anhui, China.
E-mail address: shulianghuang@sina.com