Bol. Soc. Paran. Mat. (3s.) **v. 37** 2 (2019): 145–155.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.32321

The Cesàro Convergence of Triple Chi Sequence Spaces of Fuzzy Real Numbers Defined by a Sequence of Musielak-Orlicz Function

N. Subramanian

ABSTRACT: We have to find the necessary and sufficient Tauberian conditions of convergence follows form $[C, 1, 1, 1]-$ convergence of triple sequence spaces of χ^3 of fuzzy numbers.

Key Words: Analytic Sequence, Triple sequences, Musielak-Orlicz function, $p-$ metric space, Fuzzy number, Tauberian conditons, Cesàro convergence.

Contents

1. Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by
($sahiner$ et $al.$, 2007, 2008; Esi et $al.$, 2014, 2015; $Datta$ et $al.$, 2013;)
($Subramanian$ et $al.$, 2015; $Debnath$ et $al.$, 2015) and many others.
A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple chi sequence if

$$((m+n+k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

2010 *Mathematics Subject Classification:* 40A05; 40C05; 46A45; 03E72; 46B20.
Submitted June 16, 2016. Published March 18, 2017

The space of all triple chi sequences are usually denoted by χ^3.

A fuzzy number is a fuzzy set on the real axis, (i.e) a mapping $X : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to [0, 1]$ which satisfies the following four conditions.
(i) X is normal (i.e) there exists an $\bar{0} \in \mathbb{R}$ such that $X(\bar{0}) = 1$.
(ii) X is fuzzy convex, (i.e) $X[\lambda X + (1-\lambda)Y] \geq min\{X(x), X(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.
(iii) X is upper semi-continuous.
(iv) The set $[X] = \{X \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : X(x) > 0\}$, where

$$\{X \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : X(x) > 0\},$$

denotes the closure of the set $\{X \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : X(x) > 0\}$ in the usual topology of $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. The set of all fuzzy numbers on $\mathbb{R}$ is denoted by F and $\alpha-$ level sets $[X]_\alpha$ of $X \in F$ is defined by $[X]_\alpha = \left\{\frac{\{X\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:X(t)\geq\alpha\},(0<\alpha\leq 1)}{\{X\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:X(t)\geq\alpha\},(\alpha=0)}\right\}$.

Let X be a non-empty set, then a family of sets $I \subset 2^{X\times X\times X}$ (the class of all subsets of X) is called an ideal if and only if for each $A, B \in I$, we have $A \bigcup B \in I$ and for each $A \in I$ and each each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^{X\times X\times X}$ is a filter on X if and only if $\phi \notin F$, for each $A, B \in F$, we have $A \bigcap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^{X\times X\times X}$ is a non-trivial ideal if $F = F(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^{X\times X\times X}$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. Further details on ideals of $2^{X\times X\times X}$ can be found in Kostyrko. The notion was further investigated by Salat, et. al. and others. Throughout the ideals of $2^{N\times N\times N}$ and $2^{N\times N\times N}$ will be denoted by I and I_2 respectively.
A fuzzy real number X is a fuzzy set on R, a mapping $X : R\times R\times R \to L\times L\times L\,(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. The $\alpha-$ level set of a fuzzy real number $X, 0 < \alpha < 1$ denoted by $[X]^\alpha$ is defined as $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$. A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) \wedge X(v) = min(X(s), X(r), X(v))$, where $s < t < r < v$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. A fuzzy real X is said to be upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a+\epsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal convex fuzzy number is denoted by $L(R)$.
Throughout a fuzzy real valued triple sequence is denoted by (X_{mnk}) i.e a triple infinite array of fuzzy real number X_{mnk} for all $m, n, k \in \mathbb{N}$.
Every real number r can express as a fuzzy real number $\overline{r}$ as follows:

$$\overline{r} = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{otherwise} \end{cases}$$

Let D be the set of all closed bounded intervals $X = \left[X^L, X^R\right]$. Then $X \leq Y$ if and only if $X^L \leq Y^L$ and $X^R \leq Y^R$.
Also $d(X, Y) = max\left(\left|X^L - Y^L\right|, \left|X^R - Y^R\right|\right)$. Then (D, d) is a complete metric space.
Let $\overline{d} : L(R)\times L(R)\times L(R) \to R\times R\times R$ be defined by

$$\overline{d}(X,Y) = sup_{0\leq\alpha\leq1} d\left([X]^{\alpha}, [Y]^{\alpha}\right) \text{ for } X, Y \in L(R).$$

Then $\overline{d}$ defined a metric on $L(R)$.

2. Definitions and Preliminaries

Definition 2.1. *An Orlicz function* $(Kamthan\, et\, al., 1981)$ *is a function* $M : [0,\infty) \to [0,\infty)$ *which is continuous, non-decreasing and convex with* $M(0) = 0$, $M(x) > 0$, *for* $x > 0$ *and* $M(x) \to \infty$ *as* $x \to \infty$. *If convexity of Orlicz function* M *is replaced by* $M(x+y) \leq M(x) + M(y)$, *then this function is called modulus function.*

$(Lindenstrauss\, et\, al., 1971)$ *used the idea of Orlicz function to construct Orlicz sequence space.*

A sequence $g = (g_{mn})$ *defined by*

$$g_{mn}(v) = sup\left\{|v|\, u - (f_{mnk})(u) : u \geq 0\right\}, m, n, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. *For a given Musielak-Orlicz function* f, $(Musielak, 1983)$ *the Musielak-Orlicz sequence space* t_f *is defined as follows*

$$t_f = \left\{x \in w^3 : I_f\left(|x_{mnk}|\right)^{1/m+n+k} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_f *is a convex modular defined by*

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(|x_{mnk}|\right)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f *equipped with the Luxemburg metric*

$$d(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Definition 2.2. *Let* X, Y *be a real vector space of dimension* w, *where* $n \leq m$. *A real valued function* $d_p(x_1, \ldots, x_n) = \|(d(x_1,0), \ldots, d(x_n,0))\|_p$ *on* X *satisfying the following four conditions:*

(i) $\|(d(x_1,0), \ldots, d_n(x_n,0))\|_p = 0$ *if and only if* $d(x_1,0), \ldots, d(x_n,0)$ *are linearly dependent,*

(ii) $\|(d(x_1,0), \ldots, d(x_n,0))\|_p$ *is invariant under permutation,*

(iii) $\|(\alpha d(x_1,0), \ldots, d(x_n,0))\|_p = |\alpha|\, \|(d(x_1,0), \ldots, d(x_n,0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p\left((x_1,y_1), (x_2,y_2) \cdots (x_n,y_n)\right) = \left(d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p}$ $for 1 \leq p < \infty$; *(or)*

(v) $d\left((x_1,y_1), (x_2,y_2), \cdots (x_n,y_n)\right) := \sup\left\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\right\}$,

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ *is called the* p *product metric of the Cartesian product of* n *metric spaces* $(subramanian\, et\, al., 2016)$.

Definition 2.3. *A triple sequence spaces of* $X = (X_{mnk})$ *of fuzzy numbers is a function* $X : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to F$. *The fuzzy numbers* X_{mnk} *denotes the value of the function at* $m, n, k \in \mathbb{N}$ *and is called the* $[m,n,k]^{th}$ *section of the triple sequence spaces. By* $w^3(F)$, *we denote the set of all triple sequence spaces of fuzzy real numers.*

Definition 2.4. *A triple sequence spaces* $(X_{mnk}) \subset w^3(F)$ *is called convergent with limit* $0 \in F$, *if and only if for every* $\epsilon > 0$ *there exists an* $m_0n_0k_0 = m_0n_0k_0(\epsilon) \in \mathbb{N}$ *such that* $D(X_{mnk}, \bar{0}) < \epsilon$ *for all* $m, n, k \geq m_0n_0k_0$.

Definition 2.5. *A triple sequence spaces* $X = (X_{mnk})$ *of fuzzy real numbers is said to be Cauchy if for every* $\epsilon > 0$ *there exists a positive integer* $m_0n_0k_0$ *such that* $D(X_{mnk}, \bar{0}) < \epsilon$ *for all* $m, n, k \geq m_0n_0k_0$.

The Cesàro convergence of a triple sequence spaces of fuzzy numbers defined as follows:

Definition 2.6. *Let* $(X_{mnk})(m, n, k = 0, 1, 2, \cdots)$ *be a triple sequence spaces of fuzzy numbers. The arithmetic means* $\sigma_{rst}(X_{mnk})$ *is defined by*

$$\sigma_{rst} = \frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} X_{mnk}\,(r, s, t = 0, 1, 2, \cdots). \tag{2.1}$$

We say that the triple sequence spaces of (X_{mnk}) *is Cesàro convergent, which we will denote by* $((C, 1, 1, 1) - convergent)$, *to a fuzzy numer* $\bar{0}$ *if*

$$lim_{rst\to\infty}\sigma_{rst} = \bar{0} \tag{2.2}$$

Definition 2.7. *Let* A *be a particular limitation method. Any additional condition on a triple sequence spaces, which together with the* $A-$*limitability of that triple sequence spaces implies the convergence of that triple sequence spaces, is called a Tauberian condition for the limitation method. The theorem which establishes the validity of the condition is called a Tauberian theorem.*

In this paper, we introduce some Tauberian type of theorems for triple sequence spaces of fuzzy numbers and defined as following sets.
Let $f = (f_{mnk})$ *be a Musielak-Orlicz function, and*

$$\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$$

be a triple sequence spaces of fuzzy luxemburg p− metric spaces respectively. (i)

$$\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right] = \left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right],$$

where

$$\mu_{mnk}(X) = D\left(\left((m+n+k)!\,(\Delta^m X_{mnk})^{1/m+n+k}, \bar{0}\right)\right) \to \bar{0},$$

as $m, n, k \to \infty$ *and*

$$\widetilde{d}(x) = (d(x_1), d(x_2), \cdots, d(x_{n-1})).$$

3. Main Results

Theorem 3.1. *If* $\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is convergent then* $\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is* $[C,1,1,1]$ *convergent.*

Proof. Let $X = (X_{mnk}) \in \left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right]$. Then, there exists $\bar{0} \in F$ such that $\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] = 0$. Write the following inequality

$$D[\sigma_{rst}, \bar{0}] = D\left[\frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right]\right]$$
$$\leq \frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} D\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right].$$

Since $lim_{mnk\to\infty} D\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] = 0$,
$lim_{rst\to\infty}\frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} D\left[f_{rst}\left(\left\|\mu_{rst}(X), \widetilde{d}(x)\right\|_p\right)\right] = 0$. We obtain $lim_{rst\to\infty} D[\sigma_{rst}, \bar{0}] = 0$.
The fact that the converse does not hold follows from the following example:
Example: Consider the fuzzy triple sequence spaces $\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right]$ as follows:
$\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right] = (\mu_{000}(X), \mu_{000}(Y), \cdots)$

$$\mu_{000}(X(t)) = \begin{cases} 1-t, & if t \in [0,1], \\ 0, & otherwise, \end{cases}$$

and

$$\mu_{000}(Y(t)) = \begin{cases} 1+t, & if t \in [0,1], \\ 0, & otherwise. \end{cases}$$

Then the $\alpha-$ level set of the arithmetic means $\sigma_{rst}\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right]$ are

$$\left[\sigma_{2(rst)}\right]_\alpha = \left[\frac{-(rst)}{2(rst)+1}(1-\alpha), \frac{(rst)+1}{2(rst)+1}(1-\alpha)\right]$$

and

$$\left[\sigma_{2(rst)-1}\right]_\alpha = \left[-\frac{1}{2}(1-\alpha), \frac{1}{2}(1-\alpha)\right].$$

So, $\left[\sigma_{(rst)}\right]$ is convergent to $\mu_{000}(Z) = \frac{1}{2}[\mu_{000}(X) + \mu_{000}(Y)]$ but $\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right]$ is not convergent. □

Theorem 3.2. *If a triple sequence spaces* $\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is* $[C,1,1,1]-$ *convergent to a fuzzy number* $\bar{0}$*, then for each* $\lambda > 1$*,*

$$lim_{rst\to\infty}\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] = \bar{0} \quad (3.1)$$

and for each $0 < \lambda < 1$*,*

$$lim_{rst\to\infty}\frac{1}{(rst)-\lambda_{rst}}\sum_{m=\lambda_r+1}^{r}\sum_{n=\lambda_s+1}^{s}\sum_{k=\lambda_t+1}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] = \bar{0},$$

where λ_{rst} *we denote the integral part of the product* $\lambda(rst)$*, in symbol* $\lambda_{rst} := [\lambda\, rst]$.

Proof. Case $\lambda > 1$. If $\lambda > 1$ and (rst) is large in the sense that $\lambda_{rst} > rst$, then
$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right]\right] =$
$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X\ \sigma_{rst}), \widetilde{d}(x)\right\|_p\right)\right]\right] =$
$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X)\ \ \sigma_{rst}, \widetilde{d}(x)\right\|_p\right)\right]\right] +$
$D[\sigma_{rst}, \bar{0}]$ and so

$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X)\ \ \sigma_{rst}, \widetilde{d}(x)\right\|_p\right)\right]\right] =$
$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X)\ \ \sigma_{rst}, \widetilde{d}(x)\right\|_p\right)\right]\right] =$
$[D]$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X)\ \ \sigma_{rst}, \widetilde{d}(x)\right\|_p\right)\right],$
$\frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] =$
$[D]$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] +$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right],$
$\frac{1}{(rst)+1}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X)\ \ \sigma_{rst}, \widetilde{d}(x)\right\|_p\right)\right] +$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right] =$
$[D]$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right],$

$\frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} \frac{1}{(rst)+1} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] =$
$[D]$
$\frac{1}{\lambda_{rst}+1} \frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} \frac{1}{(rst)+1} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] =$
$\left[\frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} \right] D$
$\frac{1}{\lambda_{rst}+1} \sum_{m=0}^{\lambda_r} \sum_{n=0}^{\lambda_s} \sum_{k=0}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\frac{1}{(rst)+1} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] =$
$\frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} D \left[\sigma_{\lambda_{rst}}, \sigma_{rst} \right].$
Now, (3) follows from (2) and the fact that for large enough(rst),
$\frac{\lambda}{\lambda-1} = \frac{\lambda_{rst}}{\lambda(rst)-(rst)} < \frac{\lambda_{rst}+1}{\lambda_{rst}-(rst)} < \frac{\lambda(rst)+1}{\lambda(rst)-(rst)-1} \leq \frac{2\lambda}{\lambda-1}.$
In case $0 < \lambda < 1$ then the following inequality:
$D \left[\frac{1}{(rst)-\lambda_{rst}} \sum_{m=\lambda_r+1}^{r} \sum_{n=\lambda_s+1}^{s} \sum_{k=\lambda_t+1}^{t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] \right]$
$\leq \frac{\lambda_{rst}+1}{(rst)-\lambda_{rst}} D \left[\sigma_{\lambda_{rst}}, \sigma_{rst} \right] + D \left[\sigma_{rst}, \bar{0} \right].$
Suppose (rst) is large, in the sense that $\lambda_{rst} < rst$; then the inequality for large (rst), $\frac{\lambda_{rst}+1}{(rst)-\lambda_{rst}} \leq \frac{2\lambda}{\lambda-1}$. □

Theorem 3.3. *If a triple sequence spaces* $\left[\chi_f^{3(F)}, \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \right]$ *is* $[C,1,1,1]-$ *convergent to a fuzzy number* $\bar{0}$*, then*

$$\lim_{rst \to \infty} \left[\chi_f^{3(F)}, \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \right] = \bar{0}$$

if and only if one of the following two conditions are satisfied

$$\lim_{rst \to \infty} D \left[\frac{1}{\lambda_{rst} - (rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] \right] = 0,$$

is extended real number (or)

$$\lim_{rst \to \infty} D \left[\frac{1}{(rst) - \lambda_{rst}} \sum_{m=\lambda_r+1}^{r} \sum_{n=\lambda_s+1}^{s} \sum_{k=\lambda_t+1}^{t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] \right] = 0,$$

is extended real number.

Proof. (Necessity) The necessity of condition (3) follows from theorem (4.1).
(Sufficiency): Suppose that condition (3) holds. Then, for any given $\epsilon > 0$, there exists $\lambda > 0$ such that,
$lim_{rst \to \infty}$

$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]\right]<\epsilon.$

On the other hand, since $D\left[\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]\right]=$
$[D]\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]+$
$\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]$
$\leq\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]+$
$D\left[\frac{1}{\lambda_{rst}-(rst)}\sum_{m=r+1}^{\lambda_r}\sum_{n=s+1}^{\lambda_s}\sum_{k=t+1}^{\lambda_t}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]\right].$
We conclude that $\lim_{rst\to\infty}D\left[\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]\leq\epsilon\right]$ is an extended real number, since ϵ is arbitrary. □

Remark 3.4. *The triple sequence spaces convergent of fuzzy numbers is slowly oscillating, which follows from the Cauchy criterion. On the other hand, the sequence*

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(Y\right),\widetilde{d}(x)\right\|_p\right)\right],$$

where
$\mu_{mnk}\left(Y\left(t\right)\right)=\left\{\begin{array}{cc}1-\left(\left(mnk\right)+1\right)t, & \text{if }\left(0\leq t\leq\frac{1}{(mnk)+1}\right);\\ \bar{0}, & \text{otherwise}\end{array}\right\}$
is not convergent, but μ is slowly oscillating since for all

$$\left(r_0s_0t_0\right)\leq\left(rst\right)\leq\left(mnk\right)\leq\lambda\left(rst\right),$$

with $1<\lambda\leq1+\epsilon$,
$D\left[\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]\right]=$
$D\left[\sum_{u=0}^{m}\sum_{v=0}^{n}\sum_{w=0}^{k}\left[f_{mnk}\left(\left\|\mu_{mnk}\left(Y\right),\widetilde{d}(x)\right\|_p\right)\right]\right]\leq$
$\sum_{u=r+1}^{m}\sum_{v=s+1}^{n}\sum_{w=t+1}^{k}D\left[\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]\right]=$
$\sum_{u=r+1}^{m}\sum_{v=s+1}^{n}\sum_{w=t+1}^{k}\frac{1}{(uvw)+1}\leq\left(\frac{(mnk)}{(rst)}-1\right)\leq(\lambda-1)\leq\epsilon.$

Proposition 3.5. *A triple sequence spaces (X_{mnk}) of fuzzy numbers be slowly oscillating. Then*
$lim_{rst\to\infty}\sigma_{rst}=\bar{0}\Longrightarrow lim_{rst\to\infty}[\left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right),\widetilde{d}(x)\right\|_p\right)\right]=\bar{0}.$

Proof. If the triple sequence spaces (X_{mnk}) is slowly oscillating, then the following from the inequality

$[D] \frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] =$
$[D] \frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] \leq$
$\frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} [D] \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{rst} \left(\left\| \mu_{rst}(X), \widetilde{d}(x) \right\|_p \right) \right] \leq$
$[D] \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{rst} \left(\left\| \mu_{rst}(X), \widetilde{d}(x) \right\|_p \right) \right]$. Hence the equation (3) holds. □

Proposition 3.6. *Let* $\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]$ *be a triple sequence spaces of fuzzy numbers. Then*
$[D] \left[f_{rst} \left(\left\| \mu_{rst}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{r-1s-1t-1} \left(\left\| \mu_{r-1s-1t-1}(X), \widetilde{d}(x) \right\|_p \right) \right] = O\left(\frac{1}{rst}\right)$ *implies that the triple sequence spaces* (X_{mnk}) *is slowly oscillating.*

Proof. If the triple sequence spaces (X_{mnk}) is slowly oscillating, then the following from the inequality
$[D] \frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] =$
$[D] \frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right] \leq$
$\frac{1}{\lambda_{rst}-(rst)} \sum_{m=r+1}^{\lambda_r} \sum_{n=s+1}^{\lambda_s} \sum_{k=t+1}^{\lambda_t} [D] \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{rst} \left(\left\| \mu_{rst}(X), \widetilde{d}(x) \right\|_p \right) \right] \leq$
$[D] \left[f_{mnk} \left(\left\| \mu_{mnk}(X), \widetilde{d}(x) \right\|_p \right) \right],$
$\left[f_{rst} \left(\left\| \mu_{rst}(X), \widetilde{d}(x) \right\|_p \right) \right]$. Hence the equation (3) holds. □

Proposition 3.7. *Let* $\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *be a triple sequence spaces of fuzzy numbers. Then*
$[D]\left[f_{rst}\left(\|\mu_{rst}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$,
$\left[f_{r-1s-1t-1}\left(\|\mu_{r-1s-1t-1}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = O\left(\frac{1}{rst}\right)$ *implies that the triple sequence spaces* (X_{mnk}) *is slowly oscillating.*

Proof. Let $[D]\left[f_{rst}\left(\left\|\mu_{rst}(X), \widetilde{d}(x)\right\|_p\right)\right]$,
$\left[f_{r-1s-1t-1}\left(\left\|\mu_{r-1s-1t-1}(X), \widetilde{d}(x)\right\|_p\right)\right] = O\left(\frac{1}{rst}\right)$.
Then, there exists $B > 0$ such that
$[D]\left[f_{rst}\left(\left\|\mu_{rst}(X), \widetilde{d}(x)\right\|_p\right)\right]$,
$\left[f_{r-1s-1t-1}\left(\left\|\mu_{r-1s-1t-1}(X), \widetilde{d}(x)\right\|_p\right)\right] \leq \frac{B}{rst}$ for $r, s, t \in \mathbb{N}$.
So, for all $1 < (r_0 s_0 t_0) \leq (rst) < (mnk) \leq \lambda_{rst}$, we obtain
$[D]\left[f_{rst}\left(\left\|\mu_{rst}(X), \widetilde{d}(x)\right\|_p\right)\right]$,
$\left[f_{r-1s-1t-1}\left(\left\|\mu_{r-1s-1t-1}(X), \widetilde{d}(x)\right\|_p\right)\right] \leq$
$\sum_{u=r+1}^{m}\sum_{v=s+1}^{n}\sum_{w=t+1}^{k}[D]\left[f_{uvw}\left(\left\|\mu_{uvw}(X), \widetilde{d}(x)\right\|_p\right)\right]$,
$\left[f_{u-1v-1w-1}\left(\left\|\mu_{u-1v-1w-1}(X), \widetilde{d}(x)\right\|_p\right)\right] \leq$
$\sum_{u=r+1}^{m}\sum_{v=s+1}^{n}\sum_{w=t+1}^{k}\left(\frac{B}{uvw}\right) \leq B\left(\frac{(mnk)-(rst)}{rst}\right) = B\left(\frac{mnk}{rst} - 1\right) < B(\lambda - 1)$.
Hence, for each $\epsilon > 0$ and $1 \leq \lambda \leq 1 + \frac{\epsilon}{B}$ we get for all $(r_0 s_0 t_0) \leq (rst) < (mnk) \leq \lambda_{rst}$. We have
$[D]\left[f_{mnk}\left(\left\|\mu_{mnk}(X), \widetilde{d}(x)\right\|_p\right)\right]$,
$\left[f_{rst}\left(\left\|\mu_{rst}(X), \widetilde{d}(x)\right\|_p\right)\right] \leq \epsilon$. Hence $\left[\chi_f^{3(F)}, \left\|\widetilde{d}(x)\right\|_p\right]$ is slowly oscillating. □

Corollary 3.8. *A triple sequence spaces* (X_{mnk}) *of fuzzy numbers which is* $[C,1,1,1]-$ *convergent a fuzzy number* $\bar{0}$*. Then*
$[D]\left[f_{rst}\left(\|\mu_{rst}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$,
$\left[f_{r-1s-1t-1}\left(\|\mu_{r-1s-1t-1}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = O\left(\frac{1}{rst}\right) \Longrightarrow$
$\left[\chi_f^{3(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is convergent to a fuzzy number of* $\bar{0}$.

4. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

5. Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The authors are thankful to the editor(s) and reviewers of Boletim Sociedade Paranaense de Matemática.

References

1. Datta, A. J., Esi, A. and Tripathy, B.C., *Statistically convergent triple sequence spaces defined by Orlicz function , Journal of Mathematical Analysis, 4(2)*, 16-22, (2013).
2. Debnath, S., Sarma, B. and Das, B.C., *Some generalized triple sequence spaces of real numbers,* Journal of nonlinear analysis and optimization, Vol. 6, No. 1, 71-79, (2015).
3. Esi, A., *On some triple almost lacunary sequence spaces defined by Orlicz functions*, Research and Reviews:Discrete Mathematical Structures,1(2), 16-25, (2014).
4. Esi, A. and Necdet Catalbas, M., *Almost convergence of triple sequences*, Global Journal of Mathematical Analysis, 2(1), 6-10, (2014).
5. Esi, A. and Savas, E., *On lacunary statistically convergent triple sequences in probabilistic normed space*, Applied Mathematics and Information Science, 9 (5) , 2529-2534, (2015).
6. Kamthan, P.K. and Gupta, M., *Sequence spaces and series, Lecture notes, Pure and Applied Mathematics*, 65 Marcel Dekker, In c. New York,(1981).
7. Lindenstrauss, J. and Tzafriri, L., *On Orlicz sequence spaces*, Israel Journal of Mathematics,10, 379-390, (1971).
8. Musielak, J., *Orlicz Spaces,Lectures Notes in Math.,1034*, Springer-Verlag,(1983).
9. Sahiner, A., Gurdal, M. and Duden, F.K., *Triple sequences and their statistical convergence*, ,Selcuk Journal of Applied Mathematics, 8 No. (2), 49-55, (2007).
10. Sahiner, A., Tripathy, B.C., *Some I related properties of triple sequences* ,Selcuk Journal of Applied Mathematics, 9 No. (2), 9-18, (2008).
11. Subramanian, N. and Esi, A., *The generalized tripled difference of* χ^3 *sequence spaces*,Global Journal of Mathematical Analysis, 3 (2), 54-60, (2015).
12. Subramanian, N. and Murugesan, C., *The entire sequence over Musielak* $p-$ *metric space*, Journal of the Egyptian Mathematical Society, Vol. 24 (2), 233-238, (2016).

N. Subramanian
Department of Mathematics,
SASTRA University,Thanjavur-613 401.
India.
E-mail address: nsmaths@yahoo.com