Bol. Soc. Paran. Mat. (3s.) **v. 37** 2 (2019): 129–144.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.34310

Triple Almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ Lacunary Riesz $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ Sequence Spaces Defined by Orlicz Function

Nagarajan Subramanian and Ayhan Esi

ABSTRACT: In this paper we introduce a new concept for generalized almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ −Riesz spaces strong $P-$ convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of generalized almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ −Riesz space and also some inclusion theorems are discussed.

Key Words: Analytic sequence, Orlicz function, Double sequences, Riesz space, Riesz convergence, Pringsheim convergence.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in *Apostol [1]* and double sequence spaces is found in *Hardy [7]*, *Subramanian et al. [8]*, *Deepmala et al. [9,9]* and many others. Later on investigated by some initial work on triple sequence spaces is found in *sahiner et al. [11]*, *Esi et al. [2,3,4,5]*, *Savas et al. [6]* , *Subramanian et al. [12]*, *Prakash et al. [13,14]* and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one

2010 *Mathematics Subject Classification:* 40A05, 40C05, 40D05.
Submitted November 21, 2016. Published March 15, 2017

space is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m,n,k = 1,2,3,...).$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y) = \sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \quad (1.1)$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in Γ^3. Let $\phi = \{finitesequences\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all $m,n,k \in \mathbb{N}$,

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ \end{bmatrix}$$

with 1 in the $(m,n,k)^{th}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0,$$

as $m,n,k \to \infty$. The triple gai sequences will be denoted by χ^3.

2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$) (i.e) $((m+n+k)!\,|x_{mnk}|)^{1/m+n+k} \to 0$ as $m,n,k \to \infty$. We shall write more briefly as $P - convergent\ to\ 0$.

Definition 2.1. *A function $M : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it satisfies the following conditions*
(i) M is continuous, convex and non-decreasing;
(ii) $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark 2.2. *If the convexity of an Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.*

Definition 2.3. *Let $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ be sequences of positive numbers and*

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & ... & q_{1s} & 0... \\ q_{21} & q_{22} & ... & q_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & ... & q_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \\ & & & & \end{bmatrix}$$

where $q_{11} + q_{12} + \ldots + q_{rs} \neq 0$,

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & ... & \overline{q}_{1s} & 0... \\ \overline{q}_{21} & \overline{q}_{22} & ... & \overline{q}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & ... & \overline{q}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \\ & & & & \end{bmatrix}$$

where $\overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0$,

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & ... & \overline{\overline{q}}_{1s} & 0... \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & ... & \overline{\overline{q}}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & ... & \overline{\overline{q}}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \\ & & & & \end{bmatrix}$$

where $\overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0$. *Then the transformation is given by*

$$T_{rst} = \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)! \left|x_{mnk}\right|\right)^{1/m+n+k}$$

is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - \lim_{rst} T_{rst}(x) = 0$, $0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - \lim x = 0$.

Definition 2.4. *Let* $\lambda = (\lambda_{m_i}), \mu = (\mu_{n_\ell})$ *and* $\gamma = (\gamma_{k_j})$ *be three non-decreasing sequences of positive real numbers such that each tending to* ∞ *and* $\lambda_{m_i+1} \leq \lambda_{m_i} + 1, \lambda_1 = 1, \; \mu_{n_\ell+1} \leq \mu_{n_\ell} + 1, \mu_1 = 1 \; \gamma_{k_j+1} \leq \gamma_{k_j} + 1, \gamma_1 = 1$. *Let* $I_{m_i} = [m_i - \lambda_{m_i} + 1, m_i], I_{n_\ell} = [n_\ell - \mu_{n_\ell} + 1, n_\ell]$ *and* $I_{k_j} = [k_j - \gamma_{k_j} + 1, k_j]$. *For any set* $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, *the number*

$$\delta_{\lambda,\mu,\gamma}(K) = \lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \left| \left\{ (i,j) : i \in I_{m_i}, j \in I_{n_\ell}, k \in I_{k_j}, (i,\ell,j,) \in K \right\} \right|,$$

is called the $(\lambda,\mu,\gamma)-$ *density of the set* K *provided the limit exists.*

Definition 2.5. *A triple sequence* $x = (x_{mnk})$ *of numbers is said to be* $(\lambda,\mu,\gamma)-$ *statistical convergent to a number* ξ *provided that for each* $\epsilon > 0$,

$$\lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j} \left| \left\{ (i,\ell,j) \in I_{m_i n_\ell k_j} : q_m \overline{q}_n \overline{\overline{q}}_k \left| x_{mnk} - \xi \right| \geq \epsilon \right\} \right| = 0,$$

(i.e) the set

$$K(\epsilon) = \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j} \left| \left\{ (i,\ell,j) \in I_{m_i n_\ell k_j} : q_m \overline{q}_n \overline{\overline{q}}_k \left| x_{mnk} - \xi \right| \geq \epsilon \right\} \right|$$

has $(\lambda,\mu,\gamma)-$ *density zero. In this case the number* ξ *is called the* $(\lambda,\mu,\gamma)-$ *statistical limit of the sequence* $x = (x_{mnk})$ *and we write* $St_{(\lambda,\mu,\gamma)} \lim_{m,n,k\to\infty} = \xi$.

Definition 2.6. *The triple sequence* $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ *is called triple lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty,$$

$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty$$

and

$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, *and* $\theta_{i,\ell,j}$ *is determine by*

$$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\},$$

$$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ *be a triple lacunary sequence and* $q_m \overline{q}_n \overline{\overline{q}}_k$ *be sequences of positive real numbers such that*

$$Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$$

and

$$H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}.$$

Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. *If the Riesz transformation of triple sequences is RH-regular, and* $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ *as* $i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$ *as* $\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$ *as* $j \to \infty$, *then* $\theta^{'}_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ *is a triple lacunary sequence. If the assumptions* $Q_r \to \infty$ *as* $r \to \infty$, $\overline{Q}_s \to \infty$ *as* $s \to \infty$ *and* $\overline{\overline{Q}}_t \to \infty$ *as* $t \to \infty$ *may be not enough to obtain the conditions* $H_i \to \infty$ *as* $i \to \infty, \overline{H}_\ell \to \infty$ *as* $\ell \to \infty$ *and* $\overline{\overline{H}}_j \to \infty$ *as* $j \to \infty$ *respectively. For any lacunary sequences* $(m_i), (n_\ell)$ *and* (k_j) *are integers. Throughout the paper, we assume that*

$$\begin{aligned} Q_r &= q_{11} + q_{12} + \ldots + q_{rs} \to \infty \, (r \to \infty), \\ \overline{Q}_s &= \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty \, (s \to \infty), \\ \overline{\overline{Q}}_t &= \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \to \infty \, (t \to \infty), \end{aligned}$$

such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ *as* $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ *as* $\ell \to \infty$ *and* $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ *as* $j \to \infty$. *Let* $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j$,

$$I^{'}_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\},$$

$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ *and* $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. *and* $V_{i\ell j} = V_i\overline{V}_\ell\overline{\overline{V}}_j$.

If we take $q_m = 1, \overline{q}_n = 1$ *and* $\overline{\overline{q}}_k = 1$ *for all* m, n *and* k *then* $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ *and* $I^{'}_{i\ell j}$ *reduce to* $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ *and* $I_{i\ell j}$.

Let f *be an Orlicz function and* $p = (p_{mnk})$ *be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \left\{P - \lim_{i,\ell,j\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{p_{mnk}}\right]\right\} = 0,$$

uniformly in i, ℓ *and* j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \left\{x = (x_{mnk}) : P - \sup_{i,\ell,j} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}}\right] < \infty\right\},$$

uniformly in i, ℓ *and* j.

Let f *be an Orlicz function,* $p = p_{mnk}$ *be any factorable double sequence of strictly positive real numbers and* $q_m, \overline{q}_n$ *and* $\overline{\overline{q}}_k$ *be sequences of positive numbers and* $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ *and* $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$. *If we choose* $q_m = 1, \overline{q}_n = 1$ *and* $\overline{\overline{q}}_k = 1$ *for all* m, n *and* k, *then we obtain the following sequence spaces.*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \left\{P - \lim_{i,\ell,j \to \infty} m \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left((m+n+k)! \left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0\right\},$$

uniformly in i, ℓ *and* j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \left\{P - \sup_{i,\ell,j} \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left((m+n+k)! \left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] < \infty\right\},$$

uniformly in i, ℓ *and* j.

3. Main Results

Theorem 3.1. *If* f *be any Orlicz function and a bounded factorable positive triple number sequence* p_{mnk} *then* $\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$ *is linear space.*

Proof: The proof is easy. Therefore omit the proof. □

Theorem 3.2. *For any Orlicz function f, we have*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subset \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof: Let $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right]$ so that for each i, ℓ and j

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \left\{ P - \lim_{i,\ell,j\to\infty} \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}}\sum_{\ell\in I_{i\ell j}}\sum_{j\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0 \right\},$$

uniformly in i, ℓ and j. Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following,

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}}\left(h_{i\ell j}\epsilon\right) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}-0|>\delta} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]$$

$$\frac{1}{h_{i\ell j}}\left(h_{i\ell j}\epsilon\right) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} K\delta^{-1} f(2)\, h_{i\ell j}\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Hence i, ℓ and j goes to infinity, we are granted $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$. □

Theorem 3.3. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence and $q_i, \overline{q}_\ell\overline{\overline{q}}_j$ with $\liminf_i V_i > 1,\ \liminf_\ell \overline{V_\ell} > 1$ and $\liminf_j V_j > 1$ then for any Orlicz function f,*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, f, q, p\right] \subseteq \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof: Suppose $\liminf_i V_i > 1,\ \liminf_\ell \overline{V_\ell} > 1$ and $\liminf_j \overline{\overline{V}}_j > 1$ then there exists $\delta > 0$ such that $V_i > 1 + \delta,\ \overline{V_\ell} > 1 + \delta$ and $\overline{\overline{V}}_j > 1 + \delta$. This implies $\frac{H_i}{Q_{m_i}} \ge \frac{\delta}{1+\delta},\ \frac{\overline{H_\ell}}{\overline{Q}_{n_\ell}} \ge \frac{\delta}{1+\delta}$ and $\frac{\overline{\overline{H}}_j}{\overline{\overline{Q}}_{k_j}} \ge \frac{\delta}{1+\delta}$ Then for $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, f, q, p\right]$, we can write for each i, ℓ and j.

$$A_{i,\ell,j} = \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$$= \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} -$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{i-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} -$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} -$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{H_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$

$$= \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{Q}_{k_j}}{H h_{i\ell j}}$$

$$\left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right) -$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$

$$\left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{kj-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{k_{j-1}}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{Q}_{n_{\ell-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{n_{\ell-1}}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right) -$$

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{\overline{Q}}_{m_{k-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{m_{k-1}}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right).$$

Since $x \in \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, f, q, p \right]$, the last three terms tend to zero uniformly in

m, n, k in the sense, thus, for each i, ℓ and j

$$A_{i,\ell,j} = \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}{H_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)-$$
$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$
$$+O(1).$$

Since $\frac{1}{\lambda_i\mu_\ell\gamma_j}H_{i\ell j} = \frac{1}{\lambda_i\mu_\ell\gamma_j}Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j} - \frac{1}{\lambda_i\mu_\ell\gamma_j}Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}$ we are granted for each i, ℓ and j the following

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}{H_{i\ell j}} \le \frac{1+\delta}{\delta} \text{ and } \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \le \frac{1}{\delta}.$$

The terms

$$\left(\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

and

$$\left(\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} q_m\overline{q}_n\overline{\overline{q}}_k\right.$$
$$\left.\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

are both gai sequences for all r, s and u. Thus $A_{i\ell j}$ is a gai sequence for each i, ℓ and j. Hence $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right]$. □

Theorem 3.4. *Let* $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ *be a triple lacunary sequence and* $q_m\overline{q}_n\overline{\overline{q}}_k$ *with* $\limsup_i V_i < \infty$, $\limsup_\ell \overline{V}_\ell < \infty$ *and* $\limsup_j \overline{\overline{V}}_j < \infty$ *then for any Orlicz function* f,

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subseteq \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right].$$

Proof: Since $\limsup_i V_i < \infty$, $\limsup_\ell \overline{V}_\ell < \infty$ and $\limsup_j \overline{\overline{V}}_j < \infty$ there exists $H > 0$ such that $V_i < H$, $\overline{V_\ell} < H$ and $\overline{\overline{V}}_j < H$ for all i, ℓ and j. Let $x \in \left[\chi^3_{R_{\lambda_i \mu_\ell \gamma_j}}, \theta_{i\ell j}, q, f, p\right]$ and $\epsilon > 0$. Then there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$, $b \geq \ell_0$ and $c \geq j_0$ and for all i, ℓ and j.

$$A^{'}_{abc} = \frac{1}{\lambda_i \mu_\ell \gamma_j H_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} q_m \overline{q}_n \overline{\overline{q}}_k \cdot$$
$$\cdot \left[f\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \to 0 \text{ as } m, n, k \to \infty.$$

Let $G^{'} = \max \left\{ A^{'}_{a,b,c} : 1 \leq a \leq i_0,\ \ 1 \leq b \leq \ell_0 \ \textit{and} \ 1 \leq c \leq j_0 \right\}$ and p, r and t be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < r \leq n_\ell$ and $k_{j-1} < t \leq k_j$. Thus we obtain the following:

$$\frac{1}{\lambda_i \mu_\ell \gamma_j Q_p \overline{Q}_r \overline{\overline{Q}}_t} \sum_{m=1}^{p} \sum_{n=1}^{r} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$

$$\leq \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left[\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$

$$\leq \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$
$$\left(\sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$= \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{Q}_{k_{j-1}}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} H_{a,b,c} A^{'}_{a,b,c} +$$
$$\frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} H_{a,b,c} A^{'}_{a,b,c}$$

$$\leq \frac{G^{'} Q_{m_{i_0}} \overline{Q}_{n_{i_0}} \overline{\overline{Q}}_{k_{i_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} H_{a,b,c} A^{'}_{a,b,c}$$

$$\leq \frac{G^{'} Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j m_{i-1} n_{\ell-1} k_{j-1}} + \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{j_{j-1}}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} H_{a,b,c} A^{'}_{a,b,c}$$

$$G^{'} Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}$$

$$\leq \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}+$$

$$\left(sup_{a\geq i_0\bigcup b\geq \ell_0\bigcup c\geq j_0}A^{'}_{a,b,c}\right)\frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}\sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} H_{a,b,c}$$

$$\leq \frac{G^{'}Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}+\frac{\epsilon}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}\sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} H_{a,b,c}$$

$$= \frac{G^{'}Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}+V_i\overline{V}_\ell\overline{\overline{V}}_j\epsilon$$

$$\leq \frac{G^{'}Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}+\epsilon H^3.$$

Since $Q_{m_{i-1}}\ \overline{Q}_{n_{\ell-1}}\ \overline{\overline{Q}}_{k_{j-1}}\to\infty$ as $i,\ell,j\to\infty$ approaches infinity, it follows that

$$\frac{1}{\lambda_i\mu_\ell\gamma_j Q_p\overline{Q}_r\overline{\overline{Q}}_t}\sum_{m=1}^{p}\sum_{n=1}^{q}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}=0,$$

uniformly in i,ℓ and j. Hence $x\in\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}},q,f,p\right]$. □

Corollary 3.5. *Let $\theta_{i,\ell,j}=\{m_i,n_\ell,k_j\}$ be a triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ be sequences of positive numbers. If $1<\lim_{i\ell j}V_{i\ell j}\leq\lim_{i\ell j}\sup_{i\ell j}<\infty$, then for any Orlicz function f,*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}},\theta_{i\ell j},q,f,p\right]=\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}},q,f,p\right].$$

Definition 3.6. *Let $\theta_{i,\ell,j}=\{m_i,n_\ell,k_j\}$ be a triple lacunary sequence. The triple number sequence x is said to be $S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}},\theta_{i\ell j}\right]}-P$ convergent to 0 provided that for every $\epsilon>0$,*

$$P-\lim_{i\ell J}\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}sup_{i\ell j}\left|\left\{(m,n,k)\in I^{'}_{i\ell j}:\right.\right.$$
$$\left.\left.q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((m+n+k)!\left|x_{mnk}\right|\right)^{1/m+n+k},\bar{0}\right]\right\}\geq\epsilon\right|=0.$$

In this case we write $S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}},\theta_{i\ell j}\right]}-P-\lim x=0$.

Theorem 3.7. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence. If $I'_{i,\ell,j} \subseteq I_{i,\ell,j}$, then the inclusion*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right] \subset S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$$

is strict and

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right] - P - \lim x = S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - \lim x = 0.$$

Proof: Let

$$K_{Q_{i\ell j}}(\epsilon) = \left|\left\{(m,n,k) \in I'_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq \epsilon\right| \tag{3.1}$$

Suppose that $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]$. Then for each i, ℓ and j

$$P - \lim_{i\ell j} \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] = 0.$$

Since

$$\frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]$$
$$\geq \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]$$
$$= \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}}$$

for all i, ℓ and j, we get $P - \lim_{i,\ell,j} \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} = 0$ for each i, ℓ and j. This implies that $x \in S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$.

To show that this inclusion is strict, let $x = (x_{mnk})$ be defined as

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \ldots \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 0 & \ldots \\ 1 & 2 & 3 & \ldots \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 0 & \ldots \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 2 & 3 & \ldots \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 0 & \ldots \\ \cdot & & & & & \\ \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \ldots \; 0 & 0 & \ldots \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{bmatrix};$$

and $q_m = 1; \overline{q}_n = 1; \overline{\overline{q}}_k = 1$ for all m, n and k. Clearly, x is unbounded sequence. For $\epsilon > 0$ and for all i, ℓ and j we have

$$\left|\left\{(m,n,k) \in I'_{i\ell j} : q_m\overline{q}_n\overline{\overline{q}}_k \left[((m+n+k)!\,|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}, \bar{0}\right]\right\} \geq \epsilon\right|$$

$$= P - \lim_{i\ell j}\left(\frac{\lambda_i\mu_\ell\gamma_j(m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k}$$

$$= 0.$$

Therefore $x \in S_{\left[\chi^3_{R_{\lambda_{m_i}n_\ell k_j}}, \theta_{i\ell j}\right]}$ with the $P - \lim = 0$. Also note that

$$P - \lim_{i\ell j}\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[((m+n+k)!\,|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}, \bar{0}\right]$$

$$= P - \frac{1}{2}\left(\lim_{i\ell j}\left(\frac{\lambda_i\mu_\ell\gamma_j(m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k} + 1\right)$$

$$= \frac{1}{2}.$$

Hence $x \notin \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]$. □

Theorem 3.8. *Let $I'_{i\ell j} \subseteq I_{i\ell j}$. If the following conditions hold, then*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu \subset S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$$

and

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu - P - \lim x = S_{\left[\chi^3_{R_{\lambda_i \mu_\ell \gamma_j}}, \theta_{i\ell j}\right]} - P - \lim x = 0.$$

(1) $0 < \mu < 1$ *and* $0 \leq \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right] < 1.$

(2) $1 < \mu < \infty$ *and* $1 \leq \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right] < \infty.$

Proof: Let $x = (x_{mnk})$ be strongly $\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{n_j}}}, \theta_{i\ell j}, q\right]_\mu$ − almost $P-$ convergent to the limit 0. Since

$$q_m \overline{q}_n \overline{\overline{q}}_k \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right]^\mu \geq q_m \overline{q}_n \overline{\overline{q}}_k \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right]$$

for (1) and (2), for all i, ℓ and j, we have

$$\frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right]^\mu$$
$$\geq \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right]$$
$$\geq \frac{\epsilon \left| K_{Q_{i\ell j}}(\epsilon) \right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}}$$

where $K_{Q_{i\ell j}}(\epsilon)$ is as in (3.1). Taking limit $i, \ell, j \to \infty$ in both sides of the above inequality, we conclude that $S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - \lim x = 0.$ □

Definition 3.9. *A triple sequence* $x = (x_{mnk})$ *is said to be Riesz lacunary of* χ *almost* $P-$ *convergent 0 if* $P - \lim_{i,\ell,j} w^{i\ell j}_{mnk}(x) = 0$, *uniformly in* i, ℓ *and* j, *where*

$$w^{i\ell j}_{mnk}(x) = w^{i\ell j}_{mnk}$$
$$= \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right|)^{1/m+n+k}, \bar{0} \right].$$

Definition 3.10. *A triple sequence* (x_{mnk}) *is said to be Riesz lacunary* χ *almost statistically summable to 0 if for every* $\epsilon > 0$ *the set*

$$K_\epsilon = \left\{ (i, \ell, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \left| w^{i\ell j}_{mnk}, \bar{0} \right| \geq \epsilon \right\}$$

has triple natural density zero, (i.e) $\delta_3(K_\epsilon) = 0$. *In this we write*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]_{st_2} - P - \lim x = 0.$$

That is, for every $\epsilon > 0$,

$$P - \lim_{rst} \frac{1}{rst}\left|\left\{i \le r, \ell \le s, j \le t : \left|w^{i\ell j}_{mnk}, \bar{0}\right| \ge \epsilon\right\}\right| = 0, \text{ uniformly in } i, \ell \text{ and } j.$$

Theorem 3.11. *Let* $I'_{i\ell j} \subseteq I_{i\ell j}$ *and* $q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((m+n+k)!|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}, \bar{0}\right] \le M$ *for all* $m, n, k \in \mathbb{N}$ *and for each* i, ℓ *and* j. *Let* $x = (x_{mnk})$ *be*

$$S_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - \lim x = 0.$$

Let

$$K_{Q_{i\ell j}}(\epsilon) = \left|\left\{(m,n,k) \in I'_{i\ell j} : q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}, \bar{0}\right]\right\} \ge \epsilon\right|.$$

Then

$$\left|w^{i\ell j}_{mnk}, \bar{0}\right|$$

$$= \left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}, \bar{0}\right]\right|$$

$$\le \left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m\in I'_{i\ell j}} \sum_{n\in I'_{i\ell j}} \sum_{k\in I'_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}, \bar{0}\right]\right|$$

$$\le \frac{M\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} + \epsilon$$

for each i, ℓ *and* j, *which implies that* $P - \lim_{i,\ell,j} w^{i\ell j}_{mnk}(x) = 0$, *uniformly* i, ℓ *and* j. *Hence,* $St_2 - P - \lim_{i\ell j} w^{i\ell j}_{mnk} = 0$ *uniformly in* i, ℓ, j. *Hence* $\left[\chi^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}\right]_{st_2} - P - \lim x = 0$. *To see that the converse is not true, consider the triple lacunary sequence* $\theta_{i\ell j}\left\{\left(2^{i-1}3^{\ell-1}4^{j-1}\right)\right\}$, $q_m = 1, \overline{q}_n = 1, \overline{\overline{q}}_k = 1$ *for all* m, n *and* k, *and the triple sequence* $x = (x_{mnk})$ *defined by* $x_{mnk} = \frac{(-1)^{m+n+k}}{(m+n+k)!}$ *for all* m, n *and* k.

Competing Interests: The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References

1. T. Apostol, *Mathematical Analysis*. Addison-Wesley, London, 1978.

2. A. Esi, *On some triple almost lacunary sequence spaces defined by Orlicz functions*. Research and Reviews: Discrete Mathematical Structures 1(2), 16-25, (2014).

3. A. Esi and M. Necdet Catalbas, *Almost convergence of triple sequences*. Global Journal of Mathematical Analysis 2(1), 6-10, (2014).

4. A. Esi and E. Savas, *On lacunary statistically convergent triple sequences in probabilistic normed space*. Appl. Math. and Inf. Sci. 9(5), 2529-2534, (2015).

5. A. Esi, *Statistical convergence of triple sequences in topological groups*. Annals of the University of Craiova, Mathematics and Computer Science Series 40(1), 29-33, (2013).

6. E. Savas and A. Esi, *Statistical convergence of triple sequences on probabilistic normed space*. Annals of the University of Craiova, Mathematics and Computer Science Series 39(2), 226-236, (2012).

7. G. H. Hardy, *On the convergence of certain multiple series*. Proc. Camb. Phil. Soc. 19, 86-95, (1917).

8. N. Subramanian, C. Priya and N. Saivaraju, *The* $\int \chi^{2I}$ *of real numbers over Musielak metric space*. Southeast Asian Bulletin of Mathematics 39(1), 133-148, (2015).

9. Deepmala, N. Subramanian and V. N. Mishra, *Double almost* $(\lambda_m \mu_n)$ *in* χ^2- *Riesz space*. Southeast Asian Bulletin of Mathematics 35, 1-11, (2016).

10. Deepmala, L. N. Mishra and N. Subramanian, *Characterization of some Lacunary* $\chi^2_{A_{uv}}-$ *convergence of order* α *with* $p-$ *metric defined by mn sequence of moduli Musielak*. Appl. Math. Inf. Sci. Lett. 4(3), 119-126, (2016).

11. A. Sahiner, M. Gurdal and F. K. Duden, *Triple sequences and their statistical convergence*. Selcuk J. Appl. Math. 8(2), 49-55, (2007).

12. N. Subramanian and A. Esi, *Some new semi-normed triple sequence spaces defined by a sequence of moduli*. Journal of Analysis & Number Theory 3(2), 79-88, (2015).

13. T. V. G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, *The Random of Lacunary statistical on* Γ^3 *over metric spaces defined by Musielak Orlicz functions*. Modern Applied Science 10(1), 171-183, (2016).

14. T. V. G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, *Lacunary Triple sequence* Γ^3 *of Fibonacci numbers over probabilistic* $p-$ *metric spaces*. International Organization of Scientific Research 12(1), 10-16 (2016).

15. H. Nakano, *Concave modulars*. Journal of the Mathematical Society of Japan 5, 29-49, (1953).

Nagarajan Subramanian
Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India.
E-mail address: nsmaths@yahoo.com

and

Ayhan Esi
Department of Mathematics,
Adıyaman University,
02040, Adiyaman, Turkey.
E-mail address: aesi23@hotmail.com