Bol. Soc. Paran. Mat. (3s.) **v. 37** 2 (2019): 123–127.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i2.34308

On Nilpotency Of The Right Singular Ideal Of Semiring

Nabanita Goswami and Helen K. Saikia

abstract: We introduce the concept of nilpotency of the right singular ideal of a semiring. We discuss some properties of such nilpotency and singular ideals. We show that the right singular ideal of a semiring with a.c.c. for right annihilators, is nilpotent.

Key Words: Semiring, Singular ideal, Nilpotency.

Contents

1. Introduction

In 1934, H. S. Vandiver first introduced the notion of semiring [8]. Semiring has got many applications in diverse areas of mathematics and other branches of science and thus it has achieved an importance in recent development of theory. As semiring is still a necessary branch in the study of many different areas of information sciences and applied mathematics, it is always useful to generalize the results of ring theory to semiring.

The right singular ideal $Z(R)$ of a ring R is defined as $Z\,(R) = \{r \in R \mid rK = 0$ for some essential right ideal K of $R\}$. In ring and module theory, the concept of singularity plays an important role while studying various algebraic structures. It becomes easier to study the various properties of rings if the rings under consideration are either singular or non singular. Also, the notion of singularity is essential for the study of various properties of Goldie Rings. Likewise, in the study of structure theory of semirings, ideals play a central role. In [6] and [7], the concepts of essential ideal of semiring and essential subsemimodule of a semimodule are discussed. Dutta and Das [2] generalized the notion of singular ideal for a semiring using the concept of essentiality. In this paper, we extend some useful results related to nilpotency of the singular ideal. Motivation of this work is thus to extend the concept of Goldie Ring to a more general semiring case. In section 2 we give some basic definitions that are available in [3,4,5]. In the last section, we discuss some results of right singular ideal of a semiring.

2010 *Mathematics Subject Classification:* 16Y60, 12K10.
Submitted November 21, 2016. Published March 13, 2017

2. Preliminaries

In this section we give some basic definitions, notations and results which are needed in the subsequent section.

Definition 2.1. *A semiring is a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:*

(i) $(S, +)$ *is a commutative monoid with identity element* 0*;*

(ii) $(S, \cdot)$ *is a monoid with identity element* 1_S*;*

(iii) Multiplication distributes over addition from either side;

(iv) $0s = 0 = s0$ *for all* $s \in S$*.*

Definition 2.2. *A right ideal I of a semiring S is a nonempty subset of S satisfying the following conditions:*

(i) If $a, b \in I$ *then* $a + b \in I$*;*

(ii) If $a \in I$ *and* $s \in S$ *then* $as \in I$*;*

(iii) $I \neq S$*.*

A left ideal of S is defined in the analogous manner and an ideal of S is a subset which is both a left ideal and a right ideal of S. The ideals of a semiring are proper, namely S is not an ideal of itself.

Definition 2.3. *An ideal I of a semiring S is said to be an essential ideal of S if $I \cap K \neq 0$ for every nonzero ideal K of S. We shall denote an essential ideal I of a semiring S by $I \leqslant_e S$.*

Definition 2.4. *Let A be a nonempty subset of a semiring S. Right annihilator of A in S, denoted by $r(A)$, is defined as $r(A) = \{s \in S \mid as = 0 \text{ for all } a \in A\}$. For any element $a \in S$, right annihilator of a is defined as $r(a) = \{s \in S \mid as = 0\}$.*

Definition 2.5. *An element a of a semiring S is nilpotent if and only if there exists a positive integer n satisfying $a^n = 0$. The smallest such positive integer n is called the index of nilpotency of a.*

Definition 2.6. *An ideal I in a semiring S is said to be nilpotent if there exists a positive integer n (depending on I), such that $I^n = 0$.*

Definition 2.7. *An ideal I in a semiring S is said to be a nil ideal if each of its elements is nilpotent.*

Definition 2.8. *The ideal consisting of all nilpotent elements of a semiring S is called the nilradical of S. It is denoted by $Nil(S)$.*

Definition 2.9. *A nilpotent element x of a semiring S is said to be a central nilpotent element if $xy = yx \ \forall y \in S$.*

Definition 2.10. *A semiring S is said to satisfy the ascending chain condition (a.c.c.) for left (right) ideals if for each sequence of left (right) ideals $I_1, I_2, I_3, \cdots$ of S with $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists a positive integer n (depending on the sequence) such that $I_n = I_{n+1} = I_{n+2} = \cdots$.*

Definition 2.11. *The intersection of all prime ideals of a semiring S is called the lower nil radical of the semiring and it is denoted by Nil_*S.*

Definition 2.12. *An ideal I of a semiring S is semiprime if and only if, for any ideal K of S, $K^2 \subseteq I$ only when $K \subseteq I$.*

Definition 2.13. *A semiring S is semiprime if 0 is a semiprime ideal.*

Lemma 2.14. *For any two non-empty subsets T_1, T_2 of a semiring S, $T_1 \subseteq T_2$ implies $r(T_1) \supseteq r(T_2)$.*

Lemma 2.15. *Let S be a semiring and Nil_*S be the lower nilradical of S, then $S/(Nil_*S)$ is semiprime.*

Lemma 2.16. *Let S be a semiring that satisfies the a.c.c. for right annihilators of elements. Then any nil one sided ideal A of S is contained in Nil_*S.*

Proof: We consider a nil right ideal A of S. If possible we suppose $A \nsubseteq Nil_*S$. We take $a \in A \setminus Nil_*S$ such that $r(a)$ is maximal. Now we claim that for any $x \in S$, $axa \in Nil_*S$. It can be assumed that $ax \neq 0$. As A is a nil right ideal, $ax \in A$ and thus there exists an integer $k > 0$ such that $(ax)^k = 0 \neq (ax)^{k-1}$. Now it is very easy to show that $x(ax)^{k-2} \in r(axa)$ but $x(ax)^{k-2} \notin r(a)$. Thus $r(a) \subsetneq r(axa)$. The maximality of $r(a)$ implies that $axa \in Nil_*S$. By lemma 2.15. we have $S/(Nil_*S)$ is semiprime. This implies that Nil_*S is a semiprime ideal and hence $a \in Nil_*S$, which is a contradiction. So we can conclude that A is contained in Nil_*S. Next, if A is a nil left ideal, then for any $b \in A$, bS is a nil right ideal. So, by the above $bS \subseteq Nil_*S$. Therefore we also have $A \subseteq Nil_*S$. □

3. Nilpotency of the Right Singular Ideal

We start this section with an important definition.

Definition 3.1. *Let S be a semiring. As defined in [2], the right singular ideal $Z(S)$ of S is given by $Z(S) = \{s \in S \mid sK = 0$ for some essential right ideal K of $S\}$. In other words, if $x \in S$ then $x \in Z(S)$ if and only if $r(x)$ is an essential right ideal of S.Also, S is called right non-singular if $Z(S) = 0$.*

Lemma 3.2. *If x is a central nilpotent element in a semiring S, then $x \in Z(S)$.*

Proof: In order to show $x \in Z(S)$ we need to show that $r(x) \leqslant_e S$. We consider any $y(\neq 0) \in S$. As x is a nilpotent element, there exists a smallest $n \geqslant 0$ such that $x^{n+1} = 0$, which implies that $x^{n+1}y = 0$. Then $x^n y \in r(x) \setminus 0$, as $x^n y \neq 0$. Since x

is a central nilpotent element, so $x^n y = yx^n$. Now, if we consider any nonzero left or right ideal I of S, then clearly $I \cap r(x) \neq 0$. Thus $r(x) \leqslant_e S$.

□

Theorem 3.3. *Let S be a semiring with the a.c.c. on right annihilators of elements, then $Z(S)$ is a nil ideal and $Z(S) \subseteq Nil_* S$.*

Proof: Let $x \in Z(S)$. To show that x is a nilpotent element. We know that $r(x) \subseteq r(x^2) \subseteq r(x^3) \subseteq \cdots$. So the assumption on S implies that $r(x^n) = r(x^{n+1}) = \cdots$ for some $n \geqslant 1 \cdots\cdots\cdots (i)$

We claim that $x^n = 0$. If $x^n \neq 0$, then $r(x^n) \cap x^n S$ would contain a nonzero element $x^n y (y \in S)$, with $x^n x^n y = 0$. This implies $y \in r(x^{2n})$

$\Longrightarrow y \in r(x^n)$ (Using (i))

$\Longrightarrow x^n y = 0$, a contradiction.

This shows that $Z(S)$ is a nil ideal.

Using lemma 2.16. we get $Z(S) \subseteq Nil_* S$.

□

The following theorem is a generalization of a result of Mewborn and Winton [1].

Theorem 3.4. *Let S be a semiring with the a.c.c. for right annihilators, then the right singular ideal of S is nilpotent.*

Proof: Let $J = Z(S)$, where $Z(S)$ is the right singular ideal of S. We have $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ and so $r(J) \subseteq r(J^2) \subseteq r(J^3) \subseteq \cdots$. Therefore by the given hypothesis there is a positive integer n such that $r(J^n) = r(J^{n+1}) \cdots\cdots\cdots (ii)$.

We suppose that $J^{n+1} \neq 0$; this will give us a contradiction.

Since $J^{n+1} \neq 0$, there is an element a of J such that $J^n a \neq 0$.Let $a \in J$ be such an element with $r(a)$ as large as possible. Let $b \in J$ then by the definition of J, $r(b)$ is an essential right ideal of S. This implies $r(b) \cap aS \neq 0$ as aS is a non-zero right ideal of S. So we have $s \in S$ such that $as \neq 0$ and $as \in r(b)$.

Since J is an ideal of S, $ba \in J$. Also we have $r(a) \subseteq r(ba)$.

But by the above $as \neq 0$ and $bas = 0$. Therefore $r(a) \subset r(ba)$.

It follows from the choice of a that $J^n ba = 0$.

But b is an arbitrary element of J. Hence $J^{n+1} a = 0$, which implies $a \in r(J^{n+1})$.

Using (ii), we have $a \in r(J^n)$ so that $J^n a = 0$.

But this contradicts our supposition and hence $J^{n+1} = 0$.

So the right singular ideal of S is nilpotent.

□

We now provide some consequences of Theorem 3.3 and Theorem 3.4.

Corollary 3.5. *Let S be a semiprime semiring that satisfies a.c.c. on right annihilators of elements. Then S is right nonsingular.*

Proof: We know that if S is semiprime, $Nil_{*}S = 0$. So the result follows from theorem 3.3. □

The following corollary can be obtained from theorem 3.4 using lemma 3.2.

Corollary 3.6. *Let S be a commutative semiring that satisfies a.c.c. on right annihilators of elements. Then $Z(S) = Nil(S)$.*

Acknowledgments

The first author would like to acknowledge Department of Science and Technology, Government of India for financial support. The authors would like to thank the reviewers for their valuable suggestions.

References

1. Chatters, A. W., Hajarnavis, C. R. *Rings with Chain Conditions*, Boston: Pitman Advanced Publishing Program, 1980.
2. Dutta, T. K., Das, M. L., *On strongly prime semiring*, Bull. Malays. Sci. Soc., 30(2), 135-141, (2007).
3. Glazek, K., *A guide to the Literature on Semirings and Their Applications in Mathematics and Information Sciences*, Dordrecht: Kluwer Academic Publishers, 2002.
4. Golan, J. S., *Semirings and Their Applications*, Netherlands: Springer, 1999.
5. Hebisch, U., Weinert, H. J., *Semirings: Algebraic Theory and Applications in Computer Sciences*, World Scientific Publishing Co. Pte. Ltd, 1998.
6. Pawar, K. F., Deore, R. P., *On essential ideal and radical class*, Int. J. of Pure and Appl. Math. Sci. 5(1), 1-5, (2012).
7. Pawar, K., *A note on essential subsemimodules*, New Trends in Mathematical Sciences. 1, 18-21, (2013).
8. Vandiver, H. S., *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Am. Math. Soc.,40, 914-920, (1934).

Nabanita Goswami
Department of Mathematics,
Gauhati University,
India.
E-mail address: nabanita01goswami@gmail.com

and

Helen K. Saikia,
Department of Mathematics,
Gauhati University, India.
E-mail address: hsaikia@yahoo.com