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A New Approach to the Study of Fixed Point Theorems for Simulation Functions in G-Metric Spaces

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ABSTRACT: In this paper first of all, we introduce the mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$, called the simulation function and the notion of $\mathcal{Z}$-contraction with respect to ζ which generalize several known types of contractions. Secondly, we prove certain fixed point theorems using simulation functions in G-Metric spaces. An example is also given to support our results.

Key Words: Simulation function; Contraction mapping; $\mathcal{Z}$-contraction; Fixed point; G-Metric spaces.

Contents

1. Introduction

Let (X,d) be a metric space and $T : X \to X$ be a mapping, then T is called a contraction(Banach Contraction) on X if

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all $x,y \in X$.
Where λ is a real such that $\lambda \in [0,1)$. A point $x \in X$ is called a fixed point of T if $Tx = x$. The well-known Banach Contraction Principle[1] ensures the existence and uniqueness of a fixed point of a contraction on a complete metric space. After this principle, several authors generalized this principle by introducing the various contractions on metric spaces[2, 3-9]. In this work, we introduce a mapping namely simulation funtion and the notion of $\mathcal{Z}$-contraction. Among all the generalized metric spaces, the notion of G-Metric spaces was introduced by Mustafa and Sims in[10], where in the authors discuss the topological properties of this space and proved the analog of the Banach Contraction Principle in the context of G-Metric spaces.

Definition 1.1. A G-Metric space (X,G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Example 1.2. Let (X,d) be the usual metric space then the function $G : X \times X \times X \to [0,\infty)$ defined by $G(x,y,z) = max\{d(x,y), d(y,z), d(z,x)\}$ for all $x,y,z \in X$ is a G-Metric space.

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Definition 1.3. Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:
$(G1) G(x, y, z) = 0$ if $x = y = z$,
$(G2) 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
$(G3) G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
$(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),
$(G5) G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequlity).
Then, the function G is called a generalized metric, or, more specifically,a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.4. Let $(X, G), (X', G')$ be G-Metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Recently, Khojasteh et al. [11] introduced a new class of mappings called simulation functions. Later Argoubi et al. [12] slightly modified the definition of simulation functions in the definition of simulation functions by withdrawing a condition.

Let $\mathcal{Z}^*$ be the set of simulation functions in the sense of Argoubi et al.[12].

Definition 1.5. A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:
(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$
(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$lim_{n\to\infty}\{t_n\} = lim_{n\to\infty}\{s_n\} = l \in (0, \infty),$$

then

$$lim_{n\to\infty} sup \zeta(t_n, s_n) < 0.$$

2. Main Results

In this section, we define the simulation function, give some examples and prove a related fixed point result.

Definition 2.1. Let (X, G) be a G-Metric space, $f : X \to X$ a mapping and $\zeta \in \mathbb{Z}$. Then f is called a $\mathbb{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(G(fx, fy, fz), G(x, y, z)) \geq 0 \text{ for all } x, y, z \in X. \tag{2.1}$$

Lemma 2.2. Let (X, G) be a G-Metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathbb{Z}$. Then, f is asymptotically regular at every $x \in X$.

Proof: Let $x \in X$ be arbitrary. If for some $p \in \mathbb{N}$
we have $f^p x = f^{p+1} x$, that is $fy = y$, where $y = f^{p-1} x$, that is $fz = z$, where $z = f^{p-1} x$

then, $f^n y = f^{n-1} f y = f^{n-1} y = ... = fy = y$ for all $n \in \mathbb{N}$. Now for sufficient large $n \in \mathbb{N}$, we obtain

$$\begin{aligned} G(f^n x, f^{n+1} x, f^{n+1} x) &= G(f^{n-p+1} f^{p-1} x, f^{n-p+2} f^{p-1} x, f^{n-p+2} f^{p-1} x) \\ &= G(f^{n-p+1} y, f^{n-p+2} y, f^{n-p+2} y) \\ &= G(y, y, y) = 0 \end{aligned}$$

Therefore, $lim_{n\to\infty} G(f^n x, f^{n+1} x, f^{n+1} x) = 0$
Suppose, $f^n x \neq f^{n-1} x$ for all $n \in \mathbb{N}$, then it follows from (1) that

$$\begin{aligned} 0 &\leq \zeta(G(f^{n+1} x, f^n x, f^n x), G(f^n x, f^{n-1} x, f^{n-1} x)) \\ &= \zeta(G(f f^n x, f f^{n-1} x, f f^{n-1} x), G(f^n x, f^{n-1} x, f^{n-1} x)) \\ &\leq G(f^n x, f^{n-1} x, f^{n-1} x) - G(f^{n+1} x, f^n x, f^n x) \end{aligned}$$

The above inequality show that $\{G(f^n x, f^{n-1} x, f^{n-1} x)\}$ is a monotonically decreasing sequence of non-negative reals and so it must be convergent.

Let $lim_{n\to\infty} G(f^n x, f^{n+1} x, f^{n+1} x) = r \geq 0$. If $r > 0$ then since f is $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore, we have

$$0 \leq lim_{n\to\infty} sup \zeta(G(f^{n+1} x, f^n x, f^n x), G(f^n x, f^{n-1} x, f^{n-1} x)) < 0.$$

This, contradiction shows that $r = 0$, that is, $lim_{n\to\infty} G(f^n x, f^{n+1} x, f^{n+1} x) = 0$. Thus, f is an asymptotically regular mapping at x.

Lemma 2.3. Let (X, G) be a G-Metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by f with initial value $x_0 \in X$ is a bounded sequence, where $x_n = f x_{n-1}$ for all $n \in \mathbb{N}$.

Proof: Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = f x_{n-1}$ for all $n \in \mathbb{N}$. On the contrary, assume that $\{x_n\}$ is not bounded. Without loss of generality we can assume that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Since $\{x_n\}$ is not bounded, there exists a subsequence $\{x_n\}$ such that $n_1 = 1$ and each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) > 1$$

and

$$G(x_m, x_{n(k)}, x_{n(k)}) \leq 1$$

for $n_k \leq m \leq n_{(k)+1} - 1$. Therefore, by the triangular inequality, we have

$$\begin{aligned} 1 <& G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\ \leq& G(x_{n(k)+1}, x_{n(k)+1} - 1, x_{n(k)+1} - 1) + G(x_{n(k)+1} - 1, x_{n(k)}, x_{n(k)}) \\ \leq& G(x_{n(k)+1}, x_{n(k)+1} - 1, x_{n(k)+1} - 1) + 1. \end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.2 we get

$$lim_{k\to\infty} G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) = 1$$

By (1), we get $G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \leq G(x_{n(k)+1} - 1, x_{n(k)-1}, x_{n(k)-1})$, therefore using the triangular inequality we obtain

$$\begin{aligned}1 < G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) &\leq G(x_{n(k)+1} - 1, x_{n(k)-1}, x_{n(k)-1}) \\ &\leq G(x_{n(k)+1} - 1, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\leq 1 + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1})\end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.2, we obtain

$$lim_{k\to\infty} G(x_{n(k)+1} - 1, x_{n(k)-1}, x_{n(k)-1}) = 1$$

Now, since f is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore, we have

$$\begin{aligned}0 &\leq lim_{k\to\infty} sup \zeta(G(fx_{n(k)+1} - 1, fx_{n(k)-1}, fx_{n(k)-1})) \\ &= lim_{k\to\infty} sup \zeta(G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}), G(x_{n(k)+1} - 1, x_{n(k)-1}, x_{n(k)-1})) < 0\end{aligned}$$

This contradiction proves result.

Theorem 2.4. Let (X, G) be a complete G-Metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then, f has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$ where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of f.

Proof: Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. We shall show that this sequence is a Cauchy sequence. For this, let

$$C_n = sup\{G(x_i, x_j, x_j) : i, j \geq n\}$$

Note that the sequence $\{x_n\}$ is a monotonically decreasing sequence of positive reals and by Lemma 2.3 the sequence $\{x_n\}$ is bounded, threrefore $C_n < \infty$ for all $n \in \mathbb{N}$. Thus, $\{C_n\}$ is monotonic bounded sequence, therefore convergent, that is, there exists $C \geq 0$ such that $lim_{n\to\infty} C_n = C$. We shall show that $C = 0$. If $C > 0$ then by the definition C_n, for every $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq C_k$$

Hence,

$$lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq C_k \tag{2.2}$$

Using (1) and the triangular inequality, we obtain

$$\begin{aligned}G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq & G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ \leq & G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1})\end{aligned}$$

Using Lemma 2.2, (2) and letting $k \to \infty$ in the above inequality we get

$$lim_{k\to\infty} G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) = C. \tag{2.3}$$

Since T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore using (1), (2), (3) and (ζ_2), we get

$$0 \leq lim_{k\to\infty} sup\zeta(G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}), G(x_{m(k)}, x_{n(k)}, x_{n(k)})) < 0$$

This contradiction proves that $C = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is a complete G-Metric space, there exists $u \in X$ such that $lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of f. Suppose $fu \neq u$ then $G(u, fu, fu) > 0$. Again, using (1), ζ_1, ζ_2, we have

$$\begin{aligned} 0 &\leq lim_{n\to\infty} sup\zeta(G(fx_n, fu, fu), G(x_n, u, u)) \\ &\leq lim_{n\to\infty} sup\zeta[G(x_n, u, u) - G(x_{n=1}, fu, fu)] \\ &= -G(u, fu, fu) \end{aligned}$$

This contradiction shows that $G(u, fu, fu) = 0$, that is, $fu = u$. Thus, u is a fixed point of f.

Example 2.5. Let $X = [0, 1]$ and $G : X \times X \to \mathbb{R}$ be defined by $G(x, y, z) = max\{|x - y|, |y - z|, |z - x|\}$. Then, (X, G) is a complete G-Metric space. Define a mapping $f : X \to X$ as $fx = \frac{x}{x+1}$ for all $x \in X$. f is a continuous function but it is not a Banach contraction. But it is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, where

$$\zeta(t, s) = \frac{s}{s + 1} - t \text{ for all } t, s \in [0, \infty).$$

Indeed, if $x, y \in X$, then by a simple calculation it can be shown that

$$\zeta(G(fx, fy, fy), G(x, y, y)) \geq 0.$$

Clearly, 0 is the fixed point of f.

Corollary 2.6. Let (X, G) be a complete G-Metric space and $f : X \to X$ be a mapping satisfying the following condition: $G(fx, fy, fy) \leq \lambda G(x, y, y)$ for all $x, y, y \in X$, where $\lambda \in [0, 1]$. Then, f has a unique fixed point in X.

Proof: Define $\zeta_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_B(t, s, s) = \lambda s - t$ for all $s, t \in [0, \infty)$. Note that, the mapping f is a $\mathcal{Z}$-contraction with respect to $\zeta_B \in \mathcal{Z}$. Therefore, the result follows by taking $\zeta = \zeta_B$ in Theorem 2.4.

Corollary 2.7. Let (X, G) be a complete G-Metric space and $f : X \to X$ be a mapping satisfying the following condition: $G(fx, fy, fy) \leq G(x, y, y) - \varphi(G(x, y, y))$ for all $x, y, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous function and $\varphi^{-1}(0) = \{0\}$. Then, f has a unique fixed point in X.

Proof: Define $\zeta_R : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_R(t, s, s) = s - \varphi(s) - t$ for all $s, t \in [0, \infty)$. Note that, the mapping f is a $\mathcal{Z}$-contraction with respect to $\zeta_R \in \mathcal{Z}$. Therefore, the result follows by taking $\zeta = \zeta_R$ in Theorem 2.4.

Corollary 2.8. Let Let (X, G) be a complete G-Metric space and $f : X \to X$ be a mapping satisfying the following condition: $G(fx, fy, fy) \leq \varphi(G(x, y, y))\times$

$\times G(x,y,y)$ for all $x,y,y \in X$, where $\varphi : [0,+\infty) \to [0,1)$ be a mapping such that $limsup_{t \to r^+} \varphi(t) < 1$, for all $r > 0$. Then, f has a unique fixed point.

Proof: Define $\zeta_R : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_R(t,s,s) = s\varphi(s) - t$ for all $s,t \in [0,\infty)$. Note that, the mapping f is a $\mathcal{Z}$-contraction with respect to $\zeta_R \in \mathcal{Z}$. Therefore, the result follows by taking $\zeta = \zeta_R$ in Theorem 2.4.

Corollary 2.9. Let Let (X,G) be a complete G-Metric space and $f : X \to X$ be a mapping satisfying the following condition: $G(fx,fy,fy) \leq \eta(G(x,y,y))$ for all $x,y,y \in X$, where $\eta : [0,+\infty) \to [0,+\infty)$ be an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$. Then, f has a unique fixed point.

Proof: Define $\zeta_B W : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_B W(t,s,s) = s\eta(s) - t$ for all $s,t \in [0,\infty)$. Note that, the mapping f is a $\mathcal{Z}$-contraction with respect to $\zeta_B W \in \mathcal{Z}$. Therefore, the result follows by taking $\zeta = \zeta_B W$ in Theorem 2.4.

Corollary 2.10. Let Let (X,G) be a complete G-Metric space and $f : X \to X$ be a mapping satisfying the following condition: $\int \binom{G(fx,fy,fy)}{0} \phi(t)dt \leq G(x,y,y)$ for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int \binom{t}{0} \phi(t)dt$ exists and $\int \binom{c}{0} \phi(t)dt > \epsilon$, for each $\epsilon > 0$. Then, f has a unique fixed point.

Proof: Define $\zeta_K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_K(t,s,s) = s - \int \binom{t}{0} \phi(u)du$ for all $s,t \in [0,\infty)$. Then, $\zeta_K \in \mathcal{Z}$. Therefore, the result follows by taking $\zeta = \zeta_K$ in Theorem 2.4.

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