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$H(i)$ Connected Ditopological Texture Space

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ABSTRACT: In this paper, we introduce the concept of $H(i)$ connected ditopological texture space. We develop some basic properties of bicontinuity and connectedness in term of ditopological texture space which will used in $H(i)$ connected ditopological texture space. We have established some correspondence related to known structure such as bitopological space, fuzzy lattice and topological space.

Key Words: Texture space;Fuzzy lattice; Bitopology; Ditopology; Direlation; Difunction; Connectedness; Component.

Contents

1. Introduction

Ditopological texture space may be regarded as a natural combination of texture space, topological space and bitopological space [14] but ditopology corresponds in a natural way to fuzzy topology. The texture is a generalisation of the fuzzy lattice. The notation of texture was introduced by Brown [7] in a point set setting for the study of a fuzzy set. It has been proved useful as a framework to discuss the complement-free mathematical concept. The motivation for the study of texture space is that they allow to represent a classical fuzzy set, L-fuzzy set [12], intuitionistic fuzzy set [1] and intuitionistic set, as a lattice of a crisp subset of some base set. Different fuzzy topological spaces have been studied by Tripathy and Debnath [19] , Tripathy and Ray [21,20] . A detailed analysis of the relation between texture space and the lattice of fuzzy sets of various kind is found in the works due to [1,4,5,6,3]. The concept of ditopological texture space is introduced by Brown [15]. This paper is totally devoted to the study on bicontinuity [5], connectedness [11] and their applications. In this paper we use the term $\omega - preserving$ [4] [lemma 3.4] point function and difunction [4][def 2.2]. Here we introduce the concept how to construct bicontinuous $\omega - preserving$ point function and explain

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different types of bicontinuity. We introduce the notion of order ditopology, cut point, $H(i)$- connected space, etc.

2. Preliminaries

Definition 2.1. *Let S be a non-empty set. Then $\Gamma \subseteq P(S)$ is called a texturing of S or S is said to be textured by Γ if*

1. $(\Gamma, \leq)$ is a complete lattice containing S, $\emptyset$ and $\{A_i \in \Gamma, i \in \Delta\}$, the meet $\bigwedge_{i\in\Delta} A_i$ and the join $\bigvee_{i\in\Delta} A_i$ in Γ are related to the intersection and union in $(P(S), \subseteq)$ by the equalities

[a] $\bigwedge_{i\in\Delta} A_i = \bigcap_{i\in\Delta} A_i$ where $\{A_i \in \Gamma | i \in \Delta$, the index set$\}$, while

[b] $\bigvee_{i\in\Delta} A_i = \bigcup_{i\in\Delta} A_i$ where $\{A_i \in \Gamma | i \in \Delta$, the finite index set $\}$.

2. Γ is completely distributive.

3. Γ separates the points of S. Given $s_1 \neq s_2$ in S we have $L \in \Gamma$ with $s_1 \in L, s_2 \notin L$ or $L \in \Gamma$ with $s_2 \in L, s_1 \notin L$.

If S is textured by Γ, then (S, Γ) is called a texture space or simply a texture. Hence a texture Γ on S is a set of ordinary crisp subset of S satisfying the above properties. We regard a texture as a framework.

A surjective mapping $\sigma : \Gamma \to \Gamma$ satisfying the condition $\sigma^2(A) = A$ for all $A \in \Gamma$ and for all $A, B \in \Gamma$, $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ is called a complementation on (S, Γ). A texture with a complementation is said to be complemented.

The sets $P_s = \bigwedge\{A | s \in A \in \Gamma\}$ and $Q_s = \bigvee\{A | s \notin A \in \Gamma\}$ are called p-sets and q-sets respectively.

For $A \in \Gamma$ the core $A^\flat$ of A is given by $A^\flat = \{s \in S | A \nsubseteq Q_s\}$. The set $A^\flat$ does not necessarily belong to Γ .

Example 2.2.

(a) If X is a set and $P(X)$ the power set of X, then $(X, P(X))$ is the discrete texture on X. For $x \in X,\ P_x = \{x\}$ and $Q_x = X - \{x\}$.

(b) Setting $I = [0, 1],\ \Gamma = \{[0, r),\ [0, r] \mid r \in I\}$ gives the unit interval texture $\{I, \Gamma\}$. For $r \in I,\ P_r = [0, r]$ and $Q_r = [0, r)$.

(c) The texture $\{L, \Gamma\}$ is defined by $L = (0, 1],\ \Gamma = \{(0, r] \ | r \in I\}$. For $r \in L,\ P_r = (0, r] = Q_r$.

We procure the notation of relation, corelation and difunction. $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ will denote the p-set, q-set for the product $(S \times T, P(S) \otimes \Gamma_2)$ of the texture (S, Γ_1) and (T, Γ_2). $P_{(s,t)},\ Q_{(s,t)}$ will denote the p-set, q-set of $(S \times T, \Gamma_1 \otimes \Gamma_2)$. Note that $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = ((S \setminus s \times T) \cup (S \times Q_t))$.

Definition 2.3. *[4] Let (S, Γ_1) and (T, Γ_2) be textures. Then*

(1) $r \in P(S) \otimes \Gamma_2$ is called a relation from (S, Γ_1) to (T, Γ_2) if it satisfies

R1 $r \nsubseteq \overline{Q}_{(s,t)},\ P_{s'} \nsubseteq Q_s \Rightarrow r \nsubseteq \overline{Q}_{(s',t)}$.

R2 $r \nsubseteq \overline{Q}_{(s,t)} \Rightarrow$ there exists $s' \in S$ such that $P_s \nsubseteq Q_{s'}$ and $r \nsubseteq \overline{Q}_{(s',t)}$.

(2) $R \in P(S) \otimes \Gamma_2$ is called a corelation from (S, Γ_1) to (T, Γ_2) if it satisfies

CR1 $\overline{P}_{(s,t)} \nsubseteq R,\ P_s \nsubseteq Q_{s'} \Rightarrow \overline{P}_{(s',t)} \nsubseteq R$.

CR2 $\overline{P}_{(s,t)} \not\subseteq R \Rightarrow$ *there exists* $s' \in S$ *such that* $P_{s'} \not\subseteq Q_s$ *and* $\overline{P}_{(s',t)} \not\subseteq R$.

(3) A pair (r, R),*where* r *is a relation and* R *is a corelation from* (S, Γ_1) *to* (T, Γ_2) *is called a direlation from* (S, Γ_1) *to* (T, Γ_2).

Definition 2.4. *[4] Let* (S, Γ_1), (T, Γ_2) *be textures and* (r, R) *a direlation from* (S, Γ_1) *to* (T, Γ_2). *Then the direlation* $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ *from* (T, Γ_2) *to* (S, Γ_1) *defined by*

$r^{\leftarrow} = \bigcap\{\overline{Q}_{(t,s)} | r \not\subseteq \overline{Q}_{(s,t)}\}$, $\quad R^{\leftarrow} = \bigvee\{\overline{P}_{(t,s)} | \overline{P}_{(s,t)} \not\subseteq R\}$ *is called the inverse of* (r, R). *Similarly,* $r^{\leftarrow}$ *is called the inverse of* r *and* $R^{\leftarrow}$ *the inverse of* R. *It is easy to verify that* $R^{\leftarrow}$ *is a relation and* $r^{\leftarrow}$ *is a corelation from* (T, Γ_2) *to* (S, Γ_1).

Definition 2.5. *[4] Let* (f, F) *be a direlation from* (S, Γ_1) *to* (T, Γ_2). *Then* (f, F) *is called a difunction from* (S, Γ_1) *to* (T, Γ_2) *if it satisfies the following two condition:*

*DF*1. *For* $s,\ s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow$ *there exists* $t \in T$ *with* $f \not\subseteq \overline{Q}_{(s,t)}$ *and* $\overline{P}_{(s',t)} \not\subseteq F$.

*DF*2. *For* $t, t' \in T$ *and* $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ *and* $\overline{P}_{(s,t')} \not\subseteq F \Rightarrow P_{t'} \not\subseteq Q_t$.

Definition 2.6. *[8] Let* (S, Γ_1), (T, Γ_2) *be textures. Let* (f, F) *be a difunction from* (S, Γ_1) *to* (T, Γ_2) *and* $A \in \Gamma_1$.

The image $f^{\rightarrow}A$ *and coimage* $F^{\rightarrow}A$ *are defined by,*

$f^{\rightarrow}A = \bigcap\{Q_t |$*for all* $s, f \not\subseteq \overline{Q}_{(s,t)} \Rightarrow A \subseteq Q_s\}$.

$F^{\rightarrow}A = \bigvee\{P_t |$*for all* $s, \overline{P}_{(s,t)} \not\subseteq F \Rightarrow P_s \subseteq A\}$.

Definition 2.7. *[8] Let* (S, Γ_1), (T, Γ_2) *be textures. Let* (f, F) *be a difunction from* (S, Γ_1) *to* (T, Γ_2) *and* $B \in \Gamma_2$. *The inverse image and inverse coimage are defined by,*

$f^{\leftarrow}B = \bigvee\{P_s|$ *for all* $t,\ f \not\subseteq \overline{Q}_{(s,t)} \Rightarrow P_t \subseteq B\} \in \Gamma_1$.

$F^{\leftarrow}B = \bigcap\{Q_s|$ *for all* $t,\ \overline{P}_{(s,t)} \not\subseteq F \Rightarrow B \subseteq Q_t\} \in \Gamma_1$.

Definition 2.8. *[7,15]* (Γ, τ, k) *is called a ditopological texture space on* S *if*

(1) $\tau \subseteq \Gamma$ *satisfies*

(a) S , $\emptyset \in \tau$.

(b) $G_1, G_2 \in \tau \Rightarrow G_1 \bigcap G_2 \in \tau$.

(c)$G_\alpha \in \tau, \alpha \in \Delta \Rightarrow \bigvee_{\alpha \in \Delta} G_\alpha \in \tau$.

and

(2) $k \subseteq \Gamma$ *satisfies*

(a) $S, \emptyset \in k$

(b) $F_1, F_2 \in k \Rightarrow F_1 \bigcup F_2 \in k$

(c) $F_\alpha \in k, \alpha \in \Delta \Rightarrow \bigwedge_{\alpha \in \Delta} G_\alpha \in k$.

The elements of τ *are called open set and element of* k *are called closed set. In the case of complemented texture space,* τ *and* k *are connected by the relation* $k = \{K(G)|G \in \tau\}$, *where* $K(G)$ *denote the complementation of* G. (Γ, τ, k, K) *is called the complemented ditopological texture space on a non empty set* S.

Definition 2.9. *[8] Let the difunction* (f, F) *from* $(S, \Gamma_1, \tau_1, k_1)$ *to* $(S, \Gamma_2, \tau_2, k_2)$ *of ditopological texture spaces. Then*

(1) (f, F) *is open (co-open) if* $G \in \tau_1 \Rightarrow f^{\rightarrow}G \in \tau_2 (F^{\rightarrow}G \in \tau_2)$.

(2) (f, F) *is closed (co-closed) if* $K \in k_1 \Rightarrow f^{\rightarrow}K \in k_2 (F^{\rightarrow}K \in k_2)$.

Definition 2.10. *[11] Let* $Z \subseteq S$. *Define*

$int(Z) = \bigvee\{G | G \in \tau \text{ and } G \subset Z\}$ *(interior)*

$ext(Z) = \bigvee\{G | G \in \tau \text{ and } G \cap Z = \emptyset\}$ *(exterior)*

$[Z] = \bigcap\{F | Z \subseteq F \ , F \in k\}$ *(closure)*

Definition 2.11. *[6] Let* (S, Γ, τ, k) *be a ditopological texture space.*

(1) If $s \in S^{\flat}$, *a neighbourhood of* s *is a set* $N \in \Gamma$ *for which there exists* $G \in \tau$ *satisfying* $P_s \subseteq G \subseteq N \nsubseteq Q_s$.

(2) If $s \in S$, *a coneighbourhood of* s *is a set* $M \in \Gamma$ *for which there exists* $K \in k$ *satisfying* $P_s \nsubseteq M \subseteq K \subseteq Q_s$.

Definition 2.12. *[5] Let* $(S_1, \Gamma_1, \tau_1, k_1)$ *and* $(S_2, \Gamma_2, \tau_2, k_2)$ *be a ditopological texture spaces and* (f, F) *be a difunction from* (S_1, Γ_1) *to* (S_2, Γ_2). *Then*

(1) (f, F) *is continuous if* $G \in \tau_2 \Rightarrow F^{\leftarrow}G \in \tau_1$.

(2) (f, F) *is cocontinuous if* $K \in k_2 \Rightarrow f^{\leftarrow}K \in k_1$.

(3) (f, F) *is bicontinuous if continuous and cocontinuous.*

or, (another definition) *[22] A difunction or an* $\omega-$*preserving point function between the ditopological texture spaces is called bicontinuous if the inverse image of every open set is open and the inverse image of every closed set is closed.*

Definition 2.13. *[11] Let* (S, Γ) *be a texture space and* $\emptyset \neq Z \subseteq S$. $\{A, B\} \subseteq P(S)$ *is said to be a partition of* Z *if* $A \cap Z \neq \emptyset$, $Z \nsubseteq B$ *and* $A \cap Z = B \cap Z$.

It can be noted that the roles of A *and* B *may interchanged. If* $\{A, B\}$ *is a partition of* Z, *then let* $B \cap Z \neq \emptyset$ *and* $Z \nsubseteq A$. *Let* $\Gamma = P(S)$, *then it can be verified that* $\{A, S \setminus B\}$ *and* $\{S \setminus A, B\}$ *are partition of* Z *in ordinary meaning. For example, we have* $Z \subseteq A \cup (S \setminus B)$, $Z \cap A \neq \emptyset$, $Z \cap (S \setminus B) \neq \emptyset$ *and* $Z \cap A \cap (S \setminus B) = \emptyset$.

Definition 2.14. *[11] Let* (S, Γ, τ, k) *be a ditopological texture space.* $Z \subseteq S$ *is said to be connected if there exists no partition* $\{G, F\}$ *with* $G \in \tau$ *and* $F \in k$.

Definition 2.15. *[11]* $Z \subseteq S$ *is a component if* Z *is a maximal connected set.*

3. Bicontinuitity

In this section, we establish some result on bicontinuity.

Theorem 3.1. *Let* $(X, \Gamma_1, \tau_1, k_1)$ *and* $(Y, \Gamma_2, \tau_2, k_2)$ *be ditopological texture spaces. Let* $\phi : X \to Y$ *be* $\omega-$*preserving point function. With the help of point function, define a difunction* $(f, F) : X \to Y$ *such that, if* (f, F) *is continuous then for each* $x \in X^{\flat}$ *and each neighbourhood* V *of* $f^{\rightarrow}(x)$, *there is a neighbourhood* U *of* x *such that* $f^{\rightarrow}(U) \subset V$ *and conversely.*

Proof. Let $x \in X^\flat$ and $f^\to(x) \in Y^\flat$. Let $V \in \Gamma_2$ be a neighbourhood of $f^\to(x)$ then the set $U = F^\leftarrow(V) \in \Gamma_1$ is a neighbourhood of x such that $U = F^\leftarrow(V) \Rightarrow f^\to(U) = f^\to(F^\leftarrow(V)) \subset V \Rightarrow f^\to(U) \subset V$. [4][theorem 2.24 (2,b)]

Conversely, let V be an open subset of Y. Let P_x be a subset of $F^\leftarrow(V)$ then $f^\to(P_x) \subset V$, so that by hypothesis there exists a neighbourhood of $U_x \in \Gamma_1$ of x such that $f^\to(U_x) \subset V$ then $U_x \subset F^\leftarrow(V)$. It follows that $F^\leftarrow(V)$ can be written as the join of open set U_x, so that it is open. □

We have the following result on the inclusion, composition, restriction of the domain, expanding the range and Local form of the difunctions.

Theorem 3.2. *Let* $(X_i, \Gamma_i, \tau_i, k_i)$; $i = 1, 2, 3$ *be ditopological texture spaces. Then*

(a) If A is a subspace of X_1, the inclusion difunction $(j, J) : A \to X_1$ is bicontinuous.

(b) If the difunction $(f, F) : X_1 \to X_2$ and $(g, G) : X_2 \to X_3$ are bicontinuous, then the difunction $(gof, GoF) : X_1 \to X_3$ is bicontinuous.

(c) If the difunction $(f, F) : X_1 \to X_2$ is bicontinuous and A is a subspace of X_1, then the restricted difunction $(f|A, F|A) : A \to X_2$ is bicontinuous.

(d) Let the difunction $(f, F) : X_1 \to X_2$ be bicontinuous. If X_3 is a subspace of X_2 containing the image set $f^\to(X_1)$ and coimage set $F^\to(X_1)$, then the difunction $(g, G) : X_1 \to X_3$ obtained by restricting the range of f and corange of F is bicontinuous. If X_3 is a space having X_2 as a subspace, then the difunction $(h, H) : X_1 \to X_3$ obtained by expanding the range of f and corange of F is bicontinuous.

(e) Let X_1 and X_2 be ditopological texture space. The difunction $(f, F) : X_1 \to X_2$ is bicontinuous if X_1 can be written as join of open set U_α and join of closed set V_α such that $f|U_\alpha$ and $F|V_\alpha$ is continuous and cocontinuous for each α.

Proof. (*a*) If $U \in \tau_1$ be open in X_1 and $V \in k_1$ be closed in X_1, then $J^\leftarrow(U) = U \cap A$, which is open in A and $j^\leftarrow(V) = V \cap A$, which is closed in A, (by definition of subspace ditopology texture space, one may refer to [ ]). Hence (j, J) is bicontinuous.

(*b*) Let $U \in \tau_3$ be open in X_3 and $V \in k_3$ be closed in X_3, then $G^\leftarrow(U) \in \tau_2$ be open in X_2 and $g^\leftarrow(V) \in k_2$ be closed in X_2. $F^\leftarrow(G^\leftarrow(U)) \in \tau_1$ be open in X_1 and $f^\leftarrow(g^\leftarrow(V)) \in k_1$ be closed in X_1.

Then we have $f^\leftarrow(g^\leftarrow(x)) = (gof)^\leftarrow(x)$ (by lattice theory). Hence $(GoF)^\leftarrow(U))$ $\in \tau_1$ is open in X_1 and $(gof)^\leftarrow(V)) \in k_1$ is closed in X_1. Thus (gof, GoF) is bicontinuous.

(*c*) The difunction $(f|A, F|A)$ equals to composite of the inclusion difunction $(j, J) : A \to X_1$ and the difunction $(f, F) : X_1 \to X_2$, both of which are bicontinuous.

(*d*) Let the difunction $(f, F) : X_1 \to X_2$ be bicontinuous. If $f^\to(X_1) \subset X_3 \subset X_2$ and $F^\to(X_1) \subset X_3 \subset X_2$, we show that the difunction $(g, G) : X_1 \to X_3$ obtained from (f, F) is bicontinuous. Let $B \in \tau_3$ and $C \in k_3$ open and closed set in X_3. Then $B = X_3 \cap U$ and $C = X_3 \cap V$ for some open set $U \in \tau_2$ and closed set

$V \in k_2$ in X_2, since X_3 contains the entire image set $f^{\to}(X_1)$ and and coimage set $F^{\to}(X_1)$.

$F^{\leftarrow}(U) = G^{\leftarrow}(B)$ and $f^{\leftarrow}(V) = g^{\leftarrow}(C)$ (by elementary lattice theory).

Since the difunction $F^{\leftarrow}(U)$ is open and $f^{\leftarrow}(V)$ is closed. So difunction $G^{\leftarrow}(B)$ is open and $g^{\leftarrow}(C)$ is closed.

Let the difunction $(h, H) : X_1 \to X_3$ be bicontinuous. Let X_2 be a subspace of X_3 and $h^{\to}(x) = (foj)^{\to}(x)$ be the composition of the map $f : X_1 \to X_2$ and $j : X_2 \to X_3$ and $H^{\to}(x) = (FoJ)^{\to}(x)$ is composition of the map $F : X_1 \to X_2$ and $J : X_2 \to X_3$.

(e) By hypothesis we can write X_1 as the join of the open set U_α and also join of closed set V_α such that $f|U_\alpha$ and $F|V_\alpha$ is continuous and cocontinuous for each α. Let U be open set in X_1 and V be closed set in X_1.Then

$F^{\leftarrow}(U) \cap U_\alpha = (f|U_\alpha)^{\to}(U)$ and $f^{\leftarrow}(V) \cap V_\alpha = (F|V_\alpha)^{\to}(V)$,

then both the expressions represent the set of those point x lying in U_α for which $f(x) \in U$ and y lying in V_α for which $f(y) \in V$. Since $f|U_\alpha$ is continuous, so $F^{\leftarrow}(U) \cap U_\alpha$ is open in U_α and $F|V_\alpha$ is cocontinuous, so $f^{\leftarrow}(V) \cap V_\alpha$ is closed in V_α. Hence $F^{\leftarrow}(U) \cap U_\alpha$ and $f^{\leftarrow}(V) \cap V_\alpha$ are open and closed in X.

But $F^{\leftarrow}(U) = \bigvee_\alpha (F^{\leftarrow}(U) \cap U_\alpha)$ and $f^{\leftarrow}(V) = \bigvee_\alpha (f^{\leftarrow}(V) \cap V_\alpha)$. So that $F^{\leftarrow}(U)$ and $f^{\leftarrow}(V)$ are open and closed in X_1. Hence (f, F) is bicontinuous. □

Now we establish a result on Maps into product.

Theorem 3.3. *Let $(f, F) : A \to X_1 \times X_2$ be given by the equation $f^{\to}(a) = (f_1^{\to}(a)\ , f_2^{\to}(a))$ and $F^{\to}(a) = (F_1^{\to}(a)\ , F_2^{\to}(a))$ Then (f, F) is bicontinuous if and only if the maps $f_1^{\to} : A \to X_1$, $f_2^{\to} : A \to X_2$ are continuous and $F_1^{\to} : A \to X_1$, $F_2^{\to} : A \to X_2$ are cocontinuous.The maps f_1, f_2 are called the coordinate of image set f and F_1, F_2 are called the coordinate of coimage set F.*

Proof. Let the difunction $(\pi_1, \Pi_1) : X_1 \times X_2 \to X_1$ and $(\pi_2, \Pi_2) : X_1 \times X_2 \to X_2$ be projections onto the first and second factors respectively. These maps are bicontinuous. For $\Pi_1^{\leftarrow}(U) = U \times X_2$ and $\pi_1^{\leftarrow}(V) = V \times X_2$ are open and closed according to U and V are open and closed in X_1, $\Pi_2^{\leftarrow}(Y) = X_1 \times Y$ and $\pi_2^{\leftarrow}(Z) = X_1 \times Z$ are open and closed according to Y and Z are open and closed in X_2.

Note that for each $a \in A$,

$f_1(a) = \pi_1(f(a))$, $F_1(a) = \Pi_1(F(a))$ and $f_2(a) = \pi_2(f(a))$, $F_2(a) = \Pi_2(F(a))$

If the difunction (f, F) is bicontinuous, then the coordinate difunctions must be bicontinuous. Conversely, Suppose (f_1, F_1) and (f_2, F_2) are bicontinuous. We show that for each basis element $U \times E$ and cobasis element $V \times F$ for the ditopological texture space $X_1 \times X_2$, then their inverse images $F^{\leftarrow}(U \times E)$ and $f^{\leftarrow}(V \times F)$ are open and closed. A point $a \in F^{\leftarrow}(U \times E) = f^{\leftarrow}(U \times E)$ if and only if $f(a) \in U \times E$, that is if and only if $f_1(a) \in U$ and $f_2(a) \in E$.

Similarly for $b \in f^{\leftarrow}(V \times F) = F^{\leftarrow}(V \times F)$ if and only if $F(b) \in V \times F$, that is if and only if $F_1(b) \in V$ and $F_2(b) \in F$.

Therefore $F^{\leftarrow}(U \times E) = F_1^{\leftarrow}(U) \cap F_2^{\leftarrow}(E)$

and

$f^{\leftarrow}(V \times F) = f_1^{\leftarrow}(V) \cap f_2^{\leftarrow}(F)$,

where (f_1, F_1) and (f_2, F_2) are bicontinuous functions. So their intersection is also bicontinuous, which implies (f, F) is bicontinuous. □

4. Connectedness

Theorem 4.1. *Let $\{S,\ \Gamma, \tau, k\}$ be a ditopological texture space on S. P be a connected space and $ext(P) \cap \bar{P} = \emptyset$ then $\bar{P}$ is also connected.*

Proof. Let $\bar{P}$ be not connected then there exists $A \in \tau,\ B \in k$ such that $A \cap \bar{P} = B \cap \bar{P},\ A \cap \bar{P} \neq \emptyset,\ \bar{P} \nsubseteq B$. Since P is connected . It is obvious that $P \cap A = P \cap B$. So either $A \cap P = \emptyset$ *or* $P \subset B$.

Case (1) Let $A \cap P = \emptyset$, then

$A \subseteq ext(P),$ *since* $ext(P) \cap \bar{P} = \emptyset \Rightarrow A \cap \bar{P} = \emptyset$, which is a contradiction.

Case (2) Let $P \subseteq B \Rightarrow \bar{P} \subseteq B$ B is closed set, and we arrive at a contradiction.

Hence $\bar{P}$ is connected. □

Corollary 4.2. *Let $Z \subseteq S$ be a connected set, $Z \subseteq A \subseteq [Z]$ and $ext(Z) \cap A = \emptyset$. Then A is also connected.*

Definition 4.3. *Let X be an ordered set and (X, Γ, τ, k) be a ditopological texture space. Assume that X has more then one element. The collection of the sets of the form (a, b) or $[a_0, b)$, where a_0 is the smallest element of X if it exists or $(a, b_0]$, where b_0 is the largest element of X if it exists. Such type of set act as base element which belong to τ.*

Similarly the set of the form $[a, b]$ or $(a_0, b]$, where a_0 is the smallest element of X if such exists or $[a, b_0)$, where b_0 is the largest element of X if such exist. Such type of set act as cobase element which belong to k. It generates a ditopology texture space on X known as order ditopology texture space.

Definition 4.4. *In order ditopology texture space having $X = R$ and ordered set R has binary operation $<$ is referred as euclidean ditopology texture space.*

Example 4.5. *The order ditopology texture space on set N of natural number is discrete ditopology texture space where the elements of the base are given by*

$\{n\} = (n - \varepsilon, n + \varepsilon)$; $n \in N$ *and* ε *is taken howsoever small real number.*

Similiarly, cobase are

$\{n\} = [n - \varepsilon, n + \varepsilon]$.

Theorem 4.6. *The ditopological texure space (R, Γ, τ, k) with usual order ditopology texture space is connected.*

Proof. Let $\{A \in \tau, B \in k\} \subset \Gamma$ be a Partition of R. Take $a \in A,\ b \in B$. Suppose for convenience that $a < b \Rightarrow p_a \subset p_b$. The interval $[a, b]$ is contained in R. Hence $[a, b]$ is a partition of $A_0 = A \cap [a, b],\ B_0 = B \cap [a, b]$. It is obvious that $A_0 = B_0$ [by definition of partition]. But according to our assumption $[a, b]$ has partition. So either $[a, b] \cap A_0 \neq \emptyset$ or $[a, b] \nsubseteq B_0$.

Here we consider $A_0 \in \tau_1$ and $B_0 \in k_1$ in the subspace ditopology texture space $([a, b], \Gamma_1, \tau_1, k_1)$ which is same as order ditopology texture space.

The set A_0 and B_0 is non-empty because $a \in A_0 \subset B_0$.

Remark Let (X, Γ, τ, k), where X is any set. Suppose $\{A \in \tau, B \in k\} \subset \Gamma$ be a partition of $Z \subset X$ then it must satisfy the condition $A \cap (X - B) = \emptyset$ (where $(X - B) \subset X) \Rightarrow A \subseteq B$.

Let $c = Sup\{A_0\}$.

We show that $P_c \nsubseteq A_0 \subset B_0$, which contradicts to the fact that $\{A_0, B_0\}$ be the partition of $[a, b]$.

Case 1. Suppose $P_c \subseteq B_0$. Then $P_c \neq P_b$. So neither $P_c = P_a$ nor $P_a \subset P_c \subset P_b$ because B_0 is closed in $[a, b]$. Hence the must be some interval of the form $[c, e]$ contained in B_0. If $P_c = P_a$ we have a contradiction according to our assumption point $c = sup\{A_0\}$.If $P_a \subset P_c \subset P_b$ then there exists a point z such that $P_c \subset P_z \subset P_e$ which is again a contradiction according to our assumption point $c = sup\{A_0\}$.

Case 2. Suppose $P_c \subseteq A_0 \Rightarrow P_c \subseteq A_0 \subset B_0$. So $P_c \subseteq B_0$ which is Case 1, which is a contradiction.

Hence R with usual order ditopology texture space is connected. □

Definition 4.7. *$\{S,\ \Gamma\}$ be texture space non empty set $S \subset X$ and S be connected space. A point p of S is called cut point of S means point p separates S provided $\{A, B\} \subseteq \Gamma$ is a partition [11] of $T = S - p$. Otherwise p is a non-cut point of S.*

Definition 4.8. *let (S, Γ, τ, k) be a ditopological texture space. Let p, q be points of the connected space S. We denote $E(p, q)$, the subset of S consisting of the points p and q together with all cut points of S that separates p and q.*

The separation order in $E(p, q)$ is defined as follows:

Let x, y be two points in $E(p, q)$, then x precedes y, $x < y$ in $E(p, q)$ if either $x = p$ or if x separates p and y in S.

Theorem 4.9. *Let (S, Γ, τ, k) be a ditopological texture space. If p, q are two points of connected space S. The separation order in $E(p, q)$ is a simple order.*

Proof. For each point x in $E(p, q)$, $x \neq p$ or q then there exist a separation in $S - x = T$ (say), such that $\{A_x \in \tau, B_x \in k\} \subseteq \Gamma$ is a partition of T, where $p \in \{A_x,\ B_x\}$, $q \notin \{A_x,\ B_x\}$. It is easy to check that $ext(\mathrm{A}) \cap \{A_x \cup \{x\}\} = \emptyset$. So by Corollary 4.1.1, we have $A_x \cup \{x\}$ is connected.

In the above partition, A_r and A_s do not contain point r and s respectively but B_r and B_s contains the point r and s respectively. Let r and s be two points in $E(p, q) - p - q$. If s is not in B_r, then A_s contain $A_r \cup r$ and B_s contain B_r. To see this note that in first case the connected set $A_r \cup r$ contain p but not contain s and so lies entirely in A_s. The set $B_s \cap \{A_r \cup r\} \neq \emptyset$. So B_r must lies in B_s. The second case is similar.

Let r and s be two points of $E(p, q) - p - q$. If neither $s \in A_r$ nor $s \in B_r$, then $P_r \subset P_s$ in $E(p, q)$. If neither $r \in A_s$ nor $r \in B_s$, then $P_s \subset P_r$ in $E(p, q)$. Hence any two element in $E(p, q)$ are ordered.

No element of $E(p,q)$ precedes itself.

If $P_r \subset P_s$, then $\{A_r \cup r\} \subset A_s$. If $P_s \subset P_t$, then $\{A_s \cup s\} \subset A_t$ means $\{A_r \cup r\} \subset A_s \subset \{A_s \cup s\} \subset A_t$.

Which implies $\{A_r \cup r\} \subset A_t$, it follows that $P_r \subset P_t$. Hence any two element in $E(p,q)$ are simple ordered.

The case $E(p,q) = p \cup q$ is trivial. □

5. H(i) Connected Space

In this section we introduce the notion of $H(i)$ connected sets in ditopological texure space and study its different properties.

Definition 5.1. *A ditopological texture space is $H(i)$, if every open cover of X has finite subcollection such that closure of the member of that subcollection cocovers X.*

We assume ditopological texture space to be non-degenerate, which means that the space contains at least two points. ctX will be used to denote the set of all cut point of a space X. For $x \in ctX$, a separation A and B of $X - \{x\}$ will be denoted by A_x and B_x. A^*_x will be used to denote $A_x \cup \{x\}$. We denote $[A]_X$ means the closure of A in the space X.

Lemma 5.2. *Let X be an $H(i)$ space, $\{P_x, A\} \subseteq X$.*

(i) If A is $H(i)$, then $A \cup \{x\}$ is $H(i)$.

(ii) If A , $X - [A] = B$ are open in X and $A \cap [B] = \emptyset$, then $[A]$ is $H(i)$.

Proof. (i) Let $Y = A \cup \{x\}$. Since for an open cover $\xi = \{G_\lambda$ such that $\lambda \in \Delta$ $\}$ of Y in Y, $\{H_\lambda = G_\lambda \cap A$ such that $\lambda \in \Delta\}$ is an open cover of A in A and A is $H(i)$. Therefore there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4.....\lambda_n \in \Delta$ such that $A = \bigcup\{[H_{\lambda_i}]_A$ such that $i = 1, 2, 3, ..., n\}$. For each $\lambda, [H_\lambda]_A \subset [H_\lambda]_Y \subset [G_\lambda]_Y$. So $A \subset \bigcup\{[G_{\lambda_i}]_Y$ such that $i = 1, 2, 3...n\}$. Now choose $G_{\lambda_0} \in \xi$ containing x, $A \cup \{x\} = \bigcup\{[G_{\lambda_i}]_Y$ such that $i = 0, 1, 2, 3...n\}$.

(ii) To show that $[A]$ is $H(i)$, let $\{G_\lambda$ such that $\lambda \in \Delta\}$ be an open cover of $[A]$ in $[A]$. Then $[A] = \bigvee\{G_\lambda$ such that $\lambda \in \Delta\}$. As each G_λ is open in $[A]$, we can write $G_\lambda = H_\lambda \cap [A]$ for some H_λ open in X. Therefore $[A] \subset \bigvee\{H_\lambda$ such that $\lambda \in \Delta\}$. Given that $X - [A] = B$ is open set in X and $X = \bigvee\{H_\lambda$ such that $\lambda \in \Delta\} \cup B$. Let $H_t = B$ and $\Delta' = \{t\} \cup \Delta$. Since X is $H(i)$, $X = \bigcup\{[H_{\lambda_i}]$ such that $i = 1, 2...n\}$ for some $\lambda_1, \lambda_2, \lambda_3.....\lambda_n \in \Delta'$. Therefore $A = \bigcup\{[H_{\lambda_i}] \cap A$; $i = 1, 2, 3, ..., n\}$. So $A \subset \bigcup\{[H_{\lambda_i} \cap A]$ such that $i = 1, 2...n\}$ as A is open in X. This implies that $A \subset \bigcup\{[H_{\lambda_i} \cap [A]]$ such that $i = 1, 2...n\}$ and therefore $A \subset \bigcup\{[G_{\lambda_i}]$ such that $i = 1, 2...n\}$. Given that $X - [A] = B$, so obvious $[A] \cap B = \emptyset \Rightarrow A \cap B = \emptyset$. Also we have by hypothesis, $A \cap [B] = \emptyset$ and A is open. Therefore we can suppose that $t \notin \{\lambda_1, \lambda_2, \lambda_3....\lambda_n\}$. This implies $[A] = \bigcup\{[G_{\lambda_i}] \cap [A]$; $i = 1, 2, 3, ..., n\}$ and so $[A] = \bigcup\{[G_{\lambda_i}]_A$ such that $i = 1, 2...n\}$. This establishes, $[A]$ is $H(i)$. □

Lemma 5.3. *For a connected space X, if ξ is a chain of member of the form $\{A_x^*$ such that $x \in ct(X)\}$ covering X, then for each $A_x^* \in \xi$, there exists $A_y^* \in \xi$ such that $P_y \neq P_x$ and $P_x \subseteq A_y^*$.*

Proof. Suppose that there is some $A_x^* \in \xi$ such that $P_x \nsubseteq A_y^*$ for all $A_y^* \in \xi,\ P_y \neq P_x$. Then $A_x^* \nsubseteq A_y^*$ for all $A_y^* \in \xi,\ P_y \neq P_x$. Since ξ is a chain, $A_y^* \subset A_x^*$ for all $A_y^* \in \xi,\ P_y \neq P_x$. Thus $X = A_x^*$. This implies that $X - \{x\} = A_x$, which is not possible. Hence the result. □

Theorem 5.4. *Let X be a connected space and $x \in ctX$. Let y be a non-cut point of A_x^* in A_x^* and $P_y \neq P_x$. Then y is a non cut point of X.*

Proof. Since y is a non-cut point of A_x^* in A_x^*, $A_x^* - \{y\}$ is connected. Since $B_x^* = B \cap \{x\}$ is connected because B is connected and x is a cut-point. Therefore $(A_x^* - \{y\}) \cap B_x^*$ is connected. But $X - \{y\} = (A_x^* - \{y\}) \cap B_x^*$, so $X - \{y\}$ is connected. Thus y is a non-cut point of X. □

Theorem 5.5. *Let H be a subset of a connected space X. Let $P_a \subseteq H$ be such that $H - \{a\} \subset ctX$. If $A_x^*(a) \subset H$ for every $P_x \subseteq H - \{a\}$, then H is connected.*

Proof. Let $W = \bigvee\{A_x^*(a)$ such that $P_x \subseteq H - \{a\}\}$. Since W is connected, as join of connected set whose meet is non-empty. For each $P_x \subseteq H - \{a\}$, $P_x \subseteq A_x^* \subset W$. Thus $H \subset W$. On the other hand, if $P_x \subseteq W$, then $P_x \subseteq A_y^*(a)$ for some $P_y \subseteq H - \{a\}$ and by assumption $A_y^*(a) \subset H$, so $P_x \subseteq H$. Thus $H = W$ and thus is connected. □

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