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$W^{1,N}$ Versus C^1 Local Minimizer For a Singular Functional with Neumann Boundary Condition

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ABSTRACT: Let $\Omega \subset \mathbb{R}^N$, be a bounded domain with smooth boundary. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function on $(0, +\infty)$ non-increasing and satisfying

$$c_1 = \liminf_{t\to 0^+} g(t)t^\delta \leq \lim\sup_{t\to 0^+} g(t)t^\delta = c_2,$$

for some $c_1, c_2 > 0$ and $0 < \delta < 1$. Let $f(x, s) = h(x, s)e^{bs^{\frac{N}{N-1}}}$, $b > 0$ is a constant. Consider the singular functional $I : W^{1,N}(\Omega) \to \mathbb{R}$ defined as

$$I(u) \stackrel{\text{def}}{=} \frac{1}{N}\|u\|^N_{W^{1,N}(\Omega)} - \int_\Omega G(u^+)\,\mathrm{d}x - \int_\Omega F(x, u^+)\,\mathrm{d}x$$
$$-\frac{1}{q+1}||u||^{q+1}_{L^{q+1}(\partial\Omega)}$$

where $F(x, u) = \int_0^s f(x, s)\,\mathrm{d}s$, $G(u) = \int_0^s g(s)\,\mathrm{d}s$. We show that if $u_0 \in C^1(\overline{\Omega})$ satisfying $u_0 \geq \eta \mathrm{dist}(x, \partial\Omega)$, for some $0 < \eta$, is a local minimum of I in the $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology, then it is also a local minimum in $W^{1,N}(\Omega)$ topology. This result is useful to prove the multiplicity of positive solutions to critical growth problems with co-normal boundary conditions.

Key Words: N-Laplace operator, singular equations, Neumann boundary condition, Variational methods, Local minimizers.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain. Let $f(x, s) = h(x, s)e^{bs^{\frac{N}{N-1}}}$, $b > 0$ is a constant. Let $h : \overline{\Omega} \times \mathbb{R}^+ \to [0, \infty)$ be a C^1 function satisfying:

(h1) Nonnegative with $h(x, 0) = 0$. Moreover, $f(x, t) = h(x, t)e^{bu^{\frac{N}{N-1}}}$ is nondecreasing in respect to t for t large.

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(h2) $\forall\, \epsilon > 0$, $\liminf_{t\to\infty} h(x,t)e^{\epsilon|t|^{\frac{N}{N-1}}} = \infty$, $\liminf_{t\to\infty} h(x,t)e^{-\epsilon|t|^{\frac{N}{N-1}}} = 0$ uniformly in $x \in \overline{\Omega}$.

(h3) $\forall\, \epsilon > 0$, $\liminf_{t\to\infty} h(x,t)te^{\epsilon t^{\frac{1}{N-1}}} = \infty$,

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous on $(0, +\infty)$ satisfying

(g1) g is nonincreasing on $(0, +\infty)$,

(g2) $c_1 = \liminf_{t\to 0^+} g(t)t^\delta \leq \limsup_{t\to 0^+} g(t)t^\delta = c_2$ for some $c_1, c_2 > 0$ and $0 < \delta < 1$.

From **(g2)**, g is singular at the origin and $\lim_{t\to 0^+} g(t) = +\infty$. We Consider the singular functional $I : W^{1,N}(\Omega) \to \mathbb{R}$ defined as

$$I(u) \stackrel{\text{def}}{=} \frac{1}{N}\|u\|^N_{W^{1,N}(\Omega)} - \int_\Omega G(u^+)\,\mathrm{d}x - \int_\Omega F(x,u^+)\,\mathrm{d}x$$
$$-\frac{1}{q+1}||u||^{q+1}_{L^{q+1}(\partial\Omega)} \tag{1.1}$$

where $F(x,u) = \int_0^s f(x,s)\,\mathrm{d}s$, $G(u) = \int_0^s g(s)\,\mathrm{d}s$. Our aim in this paper is to show the following

Theorem 1.1. *Suppose that the conditions* **(h1)-(h2)** *and* **(g1)-(g2)** *are satisfied. Let* $u_0 \in C^1(\overline{\Omega})$ *satisfying*

$$u_0 \geq \eta d(x,\partial\Omega) \textit{ for some } \eta > 0 \tag{1.2}$$

be a local minimizer of I *in* $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ *topology; that is,*

$$\exists\, \epsilon > 0 \textit{ such that } u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\,, \|u - u_0\|_{C^1(\overline{\Omega})} < \epsilon \Rightarrow I(u_0) \leq I(u).$$

Then, u_0 *is a local minimum of* I *in* $W^{1,N}(\Omega)$ *also.*

This result is useful to prove multiplicity of positive solutions to critical growth problems with co-normal boundary conditions. From Lemma A.2 in Appendix A, we remark that the conditions on u_0 in the above theorem implies that u_0 satisfies in the distributions sense the Euler-Lagrange equation associated to I that is

$$(\mathrm{P}) \begin{cases} -\Delta_N u + |u|^{N-2}u & = g(u) + f(x,u) \quad u > 0 \quad \text{in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} & = |u|^{q-1}u \quad \text{on } \partial\Omega. \end{cases}$$

It means that $u_0 \in W^{1,N}(\Omega)$ is a weak solution to (P), i.e. satisfies $\operatorname{ess\,inf}_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{N-2}\nabla u_0 \cdot \nabla\phi\,\mathrm{d}x + \int_\Omega |u_0|^{N-2}u_0\phi\,\mathrm{d}x = \int_\Omega g(u_0)\phi\,\mathrm{d}x$$
$$+\int_\Omega f(x,u_0)\phi\,\mathrm{d}x + \int_{\partial\Omega} |u_0|^q\phi\,\mathrm{d}x \tag{1.3}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi\colon \Omega \to \mathbb{R}$ with compact support. We highlight that the condition (1.2) is natural. Indeed from Lemma A.4 in the Appendix A, any weak solution to (P) satisfies (1.2) for some $\eta > 0$. In particular, $u_0 \geq \underline{u}$ where $\underline{u}$ is the unique weak solution to the "pure singular" problem (PS):

$$\text{(PS)} \begin{cases} -\Delta_N u + |u|^{N-2}u & = g(u) \quad u > 0 \quad \text{in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} & = |u|^{q-1}u \quad \text{on } \partial\Omega. \end{cases}$$

given by Lemma A.3.

For proving Theorem 1.1, we will need uniform L^∞-estimates for a family of solutions to (P$_\epsilon$). More precisely, we have the following result:

Theorem 1.2. *Let $\{u_\epsilon\}_{\epsilon\in(0,1)}$ be a family of solutions to* (P$_\epsilon$)*, where u_0 satisfies* (1.2) *and solves* (P)*. Let $\theta > 1$ be such that*

$$\sup_{\epsilon\in(0,1)} \left(||f(x,u_\epsilon)||_{L^\theta} + \|u_\epsilon\|_{W^{1,N}(\Omega)}\right) < \infty.$$

Then,

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} < \infty.$$

An important ingredient in our proof is the following Trudinger-Moser type inequality (see [5] and [6]):

$$\sup_{\|u\|_{1,N}\leq 1} \int_\Omega \exp^{\alpha_N |u|^{\frac{N}{N-1}}} \, dx < \infty, \tag{1.4}$$

where $\alpha_N = N w_N^{\frac{1}{n-1}}$, $w_N =$ volume of S^{N-1}. It follows immediately from (1.4) that the embedding $W^{1,N}(\Omega) \ni u \mapsto \exp^{|u|^\beta} \in L^1(\Omega)$ is compact for all $\beta = (0, \frac{N}{N-1})$ and is continuous for $\beta = \frac{N}{N-1}$. The fact that this imbedding is not compact for $\beta = \frac{N}{N-1}$ can be shown using a sequence of Moser functions that are suitable truncations and dilations of the fundamental solution of $-\Delta_N$ on $W^{1,N}(\Omega)$. Thus the growth given by the map $t \to \exp^{|t|^{\frac{N}{N-1}}}$ represents the critical growth for functions $u \in W^{1,N}(\Omega)$.

Theorem 1.1 was proved first in [1] for the case of critical growth functionals $I : H_0^1(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and later for critical growth functionals $I : W_0^{1,p}(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ in [2] and for critical and singular functionals in [9]. The sub-critical case of $p(x)-$Laplacian is studied in [7]. In contrast, in our approach we use the equations involving only one p-Laplacian.

Using constraints based on L^p- norms rather than Sobolev norms as in [2], the equations for which uniform estimates are required can be simplified to a standard type involving only one p-Laplace operator. This approach was followed in [3] in the subcitical case, in [4] in the critical case, adopted [8,10,11,12,13,14] and also adopted in this work to deal with the singular case and co-normal boundary conditions.

2. Proof of Theorem 1.1

We adapt the arguments in [8]. Assume that the conclusion of Theorem 1.1 is not true. Let $k:\mathbb{R}\to\mathbb{R}$ be defined as $k(s)=s^{p+1}e^{cs^{\frac{N}{N-1}}}$ for $p>1$ and for some constant $c>b$. We define the following constraint for each $\epsilon>0$:

$$\mathcal{S}_\epsilon \stackrel{\text{def}}{=} \{u\in W^{1,N}(\Omega):\rho(u)\stackrel{\text{def}}{=}\|k(u)\|_{L^1(\Omega)}+||u||^{\alpha+1}_{L^{\alpha+1}}(\partial\Omega)\le\epsilon\},\ \alpha\stackrel{\text{def}}{=}\max\{p,q\}. \tag{2.1}$$

We consider the following constraint minimization problem:

$$I_\epsilon=\inf_{u\in S_\epsilon} I(u).$$

Firstly, we have that $I_\epsilon>-\infty$. Indeed, since $F(x,s)=O(s^{\frac{N}{N-1}})$ as $s\longrightarrow 0$ (by **(h1)**) and from (1.4), we get for some constant C,K

$$\begin{aligned}
\frac{1}{N}||u||^N-\int_\Omega F(x,u)\mathrm{d}x &\ge \frac{1}{N}||u||^N-K\int_\Omega u^{\frac{N}{N-1}}e^{b(1+\epsilon)u^{\frac{N}{N-1}}}\,\mathrm{d}x\\
&\ge \frac{1}{N}||u||^N-K\left(\int_\Omega e^{b(1+\epsilon)qu^{\frac{N}{N-1}}}\,\mathrm{d}x\right)^{\frac1q}\left(\int_\Omega u^{\frac{N}{N-1}q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}\\
&\ge \frac{1}{N}||u||^N-KC|u^{\frac{N}{N-1}q'}|^{\frac{N}{N-1}}_{\frac{N}{N-1}q'}\\
&\ge \frac{1}{N}||u||^N-KC||u||^N,
\end{aligned}\tag{2.2}$$

where q,q' are conjugate. Therefore, from (2.2) and since $W^{1,N}(\Omega)\hookrightarrow L^{1-\delta}(\Omega)$, and the trace imbedding $W^{1,N}(\Omega)\hookrightarrow L^N(\partial\Omega)$ we get the result. Moreover, we note that $\mathcal{S}_\epsilon$ is a convex set. Using Trudinger-Moser and trace embeddings we see that $\mathcal{S}_\epsilon$ is also a closed set in $W^{1,N}(\Omega)$ which implies that $\mathcal{S}_\epsilon$ is weakly closed in $W^{1,N}(\Omega)$, the facts that, I is weakly lows semicontinuous in $W^{1,N}(\Omega)$, it follows that for $\epsilon\in(0,1)$ I_ϵ is achieved on some $u_\epsilon\in S_\epsilon$, that is

$$I(u_\epsilon)=I_\epsilon,\text{ and } I(u_\varepsilon)<I(u_0)\quad\forall\,\varepsilon\in(0,1). \tag{2.3}$$

Moreover, since $I(u_\epsilon^+)\le I(u_\epsilon)$ and $u_\epsilon^+\in\mathcal{S}_\epsilon$, we may assume that $u_\epsilon\ge 0$.
We now consider the following two cases:

1. Let $\rho(u_\epsilon)<\epsilon$.
Then u_ϵ is also a local minimizer of I in $W^{1,N}(\Omega)$. We now show that I admits a Gâteaux-derivatives on u_ϵ to derive that u_ϵ satisfies the Euler-Lagrange equation associated with I. For this, according to Lemma A.2, in Appendix A, we need to prove that $\exists\,\tilde\eta>0$ such that $u_\epsilon\ge\tilde\eta\,dist(x,\partial\Omega)$ or equivalently

$$\exists\,\eta>0\ \text{ such that } u_\epsilon\ge\eta\varphi_1; \tag{2.4}$$

[φ_1 is the eigenfunction corresponding to the principal eigenvalue of the problem

$$\begin{cases} -\Delta_N u + u^{N-1} &= 0, \quad u > 0 \quad \text{in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} &= \lambda u^{N-1} \quad \text{on } \partial\Omega.] \end{cases}$$

To prove (2.4), we argue by contradiction: $\forall\ \eta > 0$ let $\Omega_\eta = Supp\{(\eta\varphi_1 - u_\epsilon)^+\}$ and suppose that Ω_η has a non zero measure.
Let $u_\eta = (\eta\varphi_1 - u_\epsilon)^+$ and for $0 < t \leq 1$ set $\xi(t) = I(u_\epsilon + tu_\eta)$. Then, there exists $c(t)$ satisfying $c(t) > \eta t$ such that $\inf \frac{u_\epsilon + tu_\eta}{\varphi_1} \geq c(t)$ for $t > 0$. Then, from Lemma A.3 ξ is differentiable for $0 < t \leq 1$ and $\xi'(t) = \langle I'(u_\epsilon + tu_\eta), u_\eta\rangle$. Thus,

$$\begin{aligned} \xi'(t) = &\int_\Omega |\nabla(u_\epsilon + tu_\eta)|^{N-2}\nabla(u_\epsilon + tu_\eta)\nabla u_\eta + \int_\Omega |u_\epsilon + tu_\eta|^{N-2}u_\eta \\ &- \int_\Omega g(u_\epsilon + tu_\eta)u_\eta - \int_\Omega f(x, u_\epsilon + tu_\eta)u_\eta \\ &- \int_\Omega |u_\epsilon + tu_\eta|^{q-1}(u_\epsilon + tu_\eta)u_\eta. \end{aligned}$$

From **(h1)** and **(g2)**, we see that

$$\begin{aligned} \xi'(1) = &\int_\Omega |\nabla\eta\varphi_1|^{N-2}\nabla(\eta\varphi_1)\nabla u_\eta + \int_\Omega |\eta\varphi_1|^{N-2}u_\eta \\ &- \int_\Omega g(\eta\varphi_1)u_\eta - \int_\Omega f(x, \eta\varphi_1)u_\eta - \int_\Omega |\eta\varphi_1|^{q-1}(\eta\varphi_1)u_\eta < 0. \end{aligned}$$

for $\eta > 0$ small enough.
Now, since $g(s) + f(x, s)$ is non increasing for $0 < s$ small enough uniformly to $x \in \Omega$ (by **(h1)**, **(g1)-(g2)**) and from the monotonicity of the operator $-\Delta_N u + |u|^{N-1}u$, we have that for $0 < \eta$ small enough $0 \leq \xi'(1) - \xi'(t)$. Therefore, from Taylor's expansion and since $\rho(u_\epsilon) \leq \epsilon$, there exists $0 < \theta < 1$ such that

$$0 \leq I(u_\epsilon + u_\eta) - I(u_\epsilon) = \langle I'(u_\epsilon + \theta u_\eta), u_\eta\rangle = \xi'(\theta). \tag{2.5}$$

Letting $t = \theta$ we have $\xi'(\theta) \leq \xi'(1) < 0$. We obtain a contradiction with (2.5) and then $u_\epsilon \geq \eta\varphi_1$ for some $\eta > 0$ (which depends a priori on ϵ). Since u_ϵ is a local minimizer of I, and I is Gâteaux differentiable in u_ϵ, we get $I'(u_\epsilon)$ is defined and $I'(u_\epsilon) = 0$. Recalling that $\underline{u}$ is the solution to (PS) given by Lemma A.4 and from the weak comparison principle, we have that $\eta\varphi_1 \leq \underline{u} \leq u_\epsilon$ for some $\eta > 0$ (independent of ϵ). Since $u_\epsilon \in S_\epsilon$ and from the fact that u_ϵ satisfies (P), we get that $\{u_\epsilon\}_{\epsilon \geq 0}$ is uniformly bounded in $W^{1,N}(\Omega)$. Now, using Theorem 1.2 and Theorem B.1 in [13], we get

$$|u_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leq C \quad \text{for some } \alpha \in (0, 1) \tag{2.6}$$

and as $\epsilon \to 0^+$

$$u_\epsilon \to u_0 \text{ in } C^1(\overline{\Omega})$$

which contradicts the fact that u_0 is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. Now, we deal with the second case:

2. $\rho(u_\epsilon) = \epsilon$:

We again show that $u_\epsilon \geq \eta\varphi_1$ in Ω for some $\eta > 0$. Taking $u_\eta = (\eta\varphi - u_\epsilon)^+$, $\xi(t) = I(u_\epsilon + tu_\eta)$, we obtain as above that $\xi'(t) \leq \xi'(1) < 0$ for $0 < t < 1$ and $0 < \eta$ small enough.
Then $\xi(t) = I(u_\epsilon + tu_\eta)$ is decreasing. This implies that $I(u_\epsilon) > I(u_\epsilon + tu_\eta)$ for $t > 0$ and using (1.2)

$$\rho(u_\epsilon + tu_\eta) < \rho(u_\epsilon) = \epsilon.$$

This yields a contradiction with the fact that u_ϵ is a global minimizer of I on S_ϵ.
In this case, using Lemma A.3 and from the Lagrange multiplier rule we have

$$I'(u_\epsilon) = \mu_\epsilon \rho'(u_\epsilon), \text{ for some } \mu_\epsilon \in \mathbb{R}. \tag{2.7}$$

We first show that $\mu_\epsilon \leq 0$. We argue by contradiction. Suppose that $\mu_\epsilon > 0$, then there exists $\varphi \in W^{1,N}(\Omega)$ such that

$$\langle I'(u_\epsilon), \varphi\rangle < 0 \text{ and } \langle \rho'(u_\epsilon), \varphi\rangle < 0$$

and then for t small we have

$$\begin{cases} I(u_\epsilon + t\varphi) < I(u_\epsilon), \\ \rho(u_\epsilon + t\varphi) < \rho(u_\epsilon) \leq \epsilon. \end{cases} \tag{2.8}$$

This contradicts the fact that u_ϵ is a global minimizer of I in S_ϵ.
We deal now with two following cases:
case (i): $\inf\limits_{\epsilon\in(0,1)} \mu_\epsilon \overset{\text{def}}{=} l > -\infty$. In this case, we write (2.7) in its P.D.E form as (with $K(s) = \rho'(s)$).

$$(\mathrm{P}_\epsilon) \qquad \begin{cases} -\Delta_N u_\varepsilon + u_\varepsilon^{N-1} &= g(u_\varepsilon) + f(x, u_\varepsilon) + \mu_\varepsilon K(u_\varepsilon), \quad u_\varepsilon > 0 \quad \text{in } \Omega, \\ |\nabla u_\varepsilon|^{N-2} \frac{\partial u_\varepsilon}{\partial \nu} &= u_\varepsilon^{q-1} u_\varepsilon + \mu_\varepsilon |u_\varepsilon|^{\alpha-1} u_\varepsilon \quad \text{on } \partial\Omega. \end{cases}$$

it is easy from the weak comparison principle to show that $\eta\varphi_1 \leq u_\epsilon$ with some $\eta > 0$, independent of $\epsilon >$. (Note that for η small enough and for all $l \leq \mu_\epsilon \leq 0$, we have that $\eta\varphi_1$ is a strict subsolution to (P_ϵ).)

In this case, we show that (up to a subsequence) $u_\varepsilon \to u_0$ in $W^{1,N}(\Omega)$. To see this, we define a new functional $J_\varepsilon : W^{1,N}(\Omega) \to \mathbb{R}$ by

$$J_\varepsilon(u) \overset{\text{def}}{=} I(u) - \mu_\epsilon \rho(u), \quad u \in W^{1,N}(\Omega), \ \varepsilon \in (0,1). \tag{2.9}$$

Then, we see that using (2.7), $J'_\varepsilon(u_\varepsilon) = 0, \quad \epsilon \in (0,1)$. Since $\{J(u_\varepsilon)\}_{\varepsilon\in(0,1)}$ is a bounded sequence (thanks to (2.2) and (2.3)) in $\mathbb{R}$, we may choose a subsequence such that $J_\varepsilon(u_\varepsilon) \to \tau$ as $\varepsilon \to 0$. Now, using **(h2)** and Moser-Trudinger embedding, we get that

$$\int_\Omega F(x,u_\epsilon)\,\mathrm{d}x \to \int_\Omega F(x,u_0)\,\mathrm{d}x. \tag{2.10}$$

Indeed, for q^* and q conjugate for some $C_1, C_2 > 0$ independent of $u \in W^{1,N}(\Omega)$,

$$\begin{aligned}
&\int_\Omega |F(x,u_\epsilon) - F(x,u_0)|\,\mathrm{d}x \le \int_\Omega e^{bu_0^{\frac{N}{N-1}}}|h(x,u_\epsilon) - h(x,u_0)|\mathrm{d}x\\
&+\int_\Omega |e^{bu_\epsilon^{\frac{N}{N-1}}} - e^{bu_0^{\frac{N}{N-1}}}|h(x,u_\epsilon)\mathrm{d}x\\
&\le o(1) + \left(\int_\Omega \left|e^{bu_\epsilon^{\frac{N}{N-1}}} - e^{bu_0^{\frac{N}{N-1}}}\right|^{q^*}\right)^{\frac{1}{q^*}}\left(\int_\Omega Ce^{b\epsilon u_\epsilon^{\frac{N}{N-1}q}}\mathrm{d}x\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}$$

The last quantity in (2.11) is bounded from the Moser-Trudinger inequality (1.4) and ϵ small enough whereas

$$\begin{aligned}
&\int_\Omega \left|e^{bu_\epsilon^{\frac{N}{N-1}}} - e^{bu_0^{\frac{N}{N-1}}}\right|^{q^*} \le \int_{u_\epsilon\le A}\left|e^{bu_\epsilon^{\frac{N}{N-1}}} - e^{bu_0^{\frac{N}{N-1}}}\right|^{q^*}\\
&+K\left(\int_{u_\epsilon>A} e^{bq^*u_\epsilon^{\frac{N}{N-1}}}\mathrm{d}x + \int_{u_\epsilon>A} e^{bq^*u_0^{\frac{N}{N-1}}}\mathrm{d}x\right)\\
&= I + II
\end{aligned} \tag{2.12}$$

I in (2.12) goes to 0 when $\epsilon \longrightarrow 0$ by dominated convergence. II can be estimated as

$$II \le Ke^{-bA^{\frac{N}{N-1}}}\int_{u_\epsilon>A} e^{b(q^*+1)u_\varepsilon^{\frac{N}{N-1}}}\mathrm{d}x + K\int_{u_\epsilon>A} e^{bq^*u_0^{\frac{N}{N-1}}}\mathrm{d}x \tag{2.13}$$

From (2.11), (2.12), (2.13) and taking q^* such that $(q^*+1)r_0^{\frac{N}{N-1}} \le 1$ and letting $A \longrightarrow \infty$, (2.10) follows.

On the other hand, using the uniform estimate $\eta\phi_1 \le u_\epsilon \le k\phi_1$, we have

$$\int_\Omega g(u_\varepsilon)\,\mathrm{d}x \to \int_\Omega g(u_0)\,\mathrm{d}x \quad \text{when}\ \epsilon \to 0. \tag{2.14}$$

Then, since $u_\varepsilon \rightharpoonup u_0$ in $W^{1,N}(\Omega)$, by Fatou?s Lemma $I(u_0) \le \tau$. Since $\tau = \lim\limits_{\epsilon\to 0} J_\epsilon(u_\varepsilon) \le I(u_0)$ (from (2.3)), we obtain that $\tau = I(u_0)$. From (2.10)-(2.14), and the fact that $\int_{\partial\Omega}|u_\varepsilon|^{q+1} \to \int_{\partial\Omega}|u_0|^{q+1}$ we obtain that $||u_\varepsilon||_{W^{1,N}(\Omega)} \to 0$ as claimed before.

Hence, using the Trudinger-Moser type inequality in (1.4) we can apply Theorem 1.2 to conclude that $\sup_{\epsilon\in(0,1)}||v_\epsilon||_{L^\infty(\Omega)} \leq C$. Using Lemma A.6 in [13], we deduce that $v_\epsilon \leq k\phi_1$ for some $k > 0$ independent of ϵ. From the uniform estimate $\eta\phi_1 \leq v_\epsilon \leq k\phi_1$, we can apply Theorem B.1 in [13] and get $|v_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ for some constant $C > 0$ independent of ϵ. Then we conclude as above.
Let us consider **the case (ii):** $\inf_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$. From above, we can assume that $\mu_\epsilon \leq -1$ for $0 < \varepsilon$ small enough. As above, we have that $v_\epsilon \geq \eta\varphi_1$ for $\eta > 0$ small enough and independent of ϵ. Furthermore, since k is odd, we can find a number $M > 0$ independent of $\epsilon > 0$ and $x \in \overline{\Omega}$, such that $(g(s) + f(x, s) + \mu_\varepsilon)s$ and $(|s|^{q-1}s+\mu_\varepsilon|s|^{\alpha-1}s)s$ are negative for all $|s| \geq M$. Then, from the weak comparison principle we have that $v_\epsilon \leq M$ for $\epsilon > 0$ small enough. From Lemma A.2, since $u_0 \in W^{1,N}(\Omega)$ satisfies (1.2) and is a C^1 local minimizer, u_0 is a weak solution to (P), i.e. satisfies $\operatorname{ess\,inf}_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{N-2}\nabla u_0\nabla\phi\,\mathrm{d}x \quad + \quad \int_\Omega |u_0|^{N-2}\phi\,\mathrm{d}x - \int_\Omega g(u_0)\phi\,\mathrm{d}x - \int_\Omega f(x, u_0)\phi\,\mathrm{d}x \\ - \int_\Omega |u_0|^{q-1}u_0\phi\,\mathrm{d}x.$$

for all $\phi \in C_c^\infty(\Omega)$. From Lemma A.3, for every function $w \in W^{1,N}(\Omega)$, u_0 satisfies

$$\int_\Omega |\nabla u_0|^{N-2}\nabla u_0\nabla w\,\mathrm{d}x \quad + \quad \int_\Omega |u_0|^{N-2}w\,\mathrm{d}x - \int_\Omega g(u_0)w\,\mathrm{d}x - \int_\Omega f(x, u_0)w\,\mathrm{d}x \\ - \int_\Omega |u_0|^{q-1}u_0 w\,\mathrm{d}x.$$

Similarly,

$$\int_\Omega |\nabla u_\epsilon|^{N-2}\nabla u_\epsilon\nabla w\,\mathrm{d}x \quad + \quad \int_\Omega |u_\epsilon|^{N-2}w\,\mathrm{d}x - \int_\Omega g(u_\epsilon)w\,\mathrm{d}x - \int_\Omega f(x, u_\epsilon)w\,\mathrm{d}x \\ - \int_\Omega |u_\epsilon|^{q-1}u_\epsilon w\,\mathrm{d}x.$$

Now, substracting the above relations with $w = (u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}$, with $\beta \geq 1$, as a test function in (P_ε), integrate by parts and use the fact that $u \mapsto -\Delta_N u + |u|^{N-1}u$ is a monotone operator to obtain,

$$\begin{aligned} - \quad & \mu_\varepsilon \left[\int_\Omega k(u_\varepsilon - u_0)|u_\epsilon - u_0|^{\beta-1}(u_\varepsilon - u_0)\,\mathrm{d}x + \int_{\partial\Omega} |u_\varepsilon - u_0|^{\alpha+\beta}\,\mathrm{d}x\right] \\ \leq \quad & \int_\Omega (g(u_\varepsilon) - g(u_0))(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}\mathrm{d}x \\ & + \int_\Omega (f(x, u_\varepsilon) - f(x, u_0))(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}\mathrm{d}x \\ & + \int_{\partial\Omega} (|u_\varepsilon|^{q-1}u_\varepsilon - |u_0|^{q-1}u_0)(u_\varepsilon - u_0)|u_\epsilon - u_0|^{\beta-1}\mathrm{d}x. \end{aligned}$$

Using the bounds of u_ϵ, u_0 we get

$$-\mu_\varepsilon\left[\int_\Omega k(u_\varepsilon-u_0)|u_\epsilon-u_0|^{\beta-1}(u_\varepsilon-u_0)\,\mathrm{d}x+\int_{\partial\Omega}|u_\varepsilon-u_0|^{\alpha+\beta}\,\mathrm{d}x\right]$$
$$\leq C\left[\int_\Omega|u_\epsilon-u_0|^\beta\mathrm{d}x+\int_{\partial\Omega}|u_\epsilon-u_0|^\beta\mathrm{d}x\right]$$

where C does not depend on β and ε. Now, using the inequality $k(s)s\geq c|s|^{p+1}\ \forall\, s\in\mathbb{R}$, $\alpha\geq p$ and the Hölder inequality we obtain

$$-\mu_\varepsilon\left[\int_\Omega|u_\epsilon-u_0|^{p+\beta}\,\mathrm{d}x+\int_{\partial\Omega}|u_\varepsilon-u_0|^{p+\beta}\,\mathrm{d}x\right]$$
$$\leq C(|\Omega|)\left[\int_\Omega|u_\epsilon-u_0|^{p+\beta}\mathrm{d}x+\int_{\partial\Omega}|u_\epsilon-u_0|^{p+\beta}\mathrm{d}x\right]^{\frac{\beta}{p+\beta}}.$$

Therefore, for any $\beta>1$

$$-\mu_\epsilon\left[\|u_\epsilon-u_0\|^p_{L^{\beta+p}(\Omega)}+\|u_\epsilon-u_0\|^p_{L^{\beta+p}(\partial\Omega)}\right]\leq C(|\Omega|). \tag{2.15}$$

Passing to the limit in (2.15) $\beta\to+\infty$ we get

$$\mu_\epsilon\left[\|v_\epsilon-u_0\|^p_{L^\infty(\Omega)}+\|v_\epsilon-u_0\|^p_{L^\infty(\partial\Omega)}\right]\leq C. \tag{2.16}$$

Thus, using (2.16), the uniform L^∞ bounds for $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ in Ω as well as $\partial\Omega$ and the fact that $k(s)|s|^{-p}$ is function bounded below in $\mathbb{R}$, we get that the right-hand side terms in (P_ε) are uniformly bounded in $L^\infty(\Omega)$ and in $L^\infty(\partial\Omega)$ from which as in the first case, we obtain that u_ϵ, $(0<\epsilon\leq 1)$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ independently of ϵ and we conclude as above. □

3. Uniform estimates

Consider the problems

$$(\mathrm{P}_\epsilon)\qquad\begin{cases}-\Delta_N u_\varepsilon+u_\varepsilon^{N-1} &= g(u_\varepsilon)+f(x,u_\varepsilon)+\mu_\varepsilon k(u_\varepsilon),\quad u_\varepsilon>0\quad\text{in }\Omega,\\ |\nabla u_\varepsilon|^{N-2}\frac{\partial u_\varepsilon}{\partial\nu} &= u_\varepsilon^{q-1}u_\varepsilon+\mu_\varepsilon|u_\varepsilon|^{\alpha-1}u_\varepsilon\quad\text{on }\partial\Omega.\end{cases}$$

where ν is the unit normal on $\partial\Omega$. In this section, we obtain the uniform L^∞-estimates for a family of solutions to (P_ϵ). More precisely, we prove Theorem 1.2.

Proof: For $k>0$ we consider the test function

$$T_k(s)\overset{\text{def}}{=}(s+k)\chi_{(-\infty,-k]}+(s-k)\chi_{[k,\infty)} \tag{3.1}$$

and define the two following set

$$\Omega_k=\{x\in\Omega_k\setminus|u_\epsilon|\geq k\},\quad \partial\Omega_k=\{x\in\partial\Omega_k\setminus|u_\epsilon|\geq k\}.$$

Using $T_k(u_\epsilon)$ as test function in (P_ϵ), and the fact that $\mu_\epsilon \leq 0$ we get

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\epsilon|^{N-2}\nabla u_\epsilon\right) \nabla(T_k(u_\epsilon)) + \int_\Omega \left(|u_\epsilon|^{N-2}u_\epsilon\right) T_k(u_\epsilon) \\
&\leq \int_\Omega g(u_\epsilon)T_k(u_\epsilon) + \int_\Omega f(x,u_\epsilon)T_k(u_\epsilon) + \int_\Omega |u_\epsilon|^{q-1}u_\epsilon T_k(u_\epsilon).
\end{aligned} \tag{3.2}$$

The term on the right-hand side of (3.2) may be estimated using the Hölder inequality

$$\begin{aligned}
&\int_\Omega g(u_\epsilon)T_k(u_\epsilon) + \int_\Omega f(x,u_\epsilon)T_k(u_\epsilon) + \int_\Omega |u_\epsilon|^{q-1}u_\epsilon T_k(u_\epsilon). \\
\leq\ & (\int_\Omega (g(u_\epsilon))^\theta)^{\frac{1}{\theta}}(\int_\Omega |T_k(u_\epsilon)|^r)^{\frac{1}{r}}|\Omega_k|^{1-\frac{1}{\theta}-\frac{1}{r}} \\
& +(\int_\Omega (f(u_\epsilon))^\theta)^{\frac{1}{\theta}}(\int_\Omega |T_k(u_\epsilon)|^r)^{\frac{1}{r}}|\Omega_k|^{1-\frac{1}{\theta}-\frac{1}{r}} \\
& +(\int_{\partial\Omega} (|u_\epsilon|^q)^\theta)^{\frac{1}{\theta}}(\int_{\partial\Omega} |T_k(u_\epsilon)|^r)^{\frac{1}{r}}|\partial\Omega_k|^{1-\frac{1}{\theta}-\frac{1}{r}}. \\
\leq\ & C(\int_\Omega |T_k(u_\epsilon)|^r + \int_{\partial\Omega} |T_k(u_\epsilon)|^r)^{\frac{1}{r}}(|\partial\Omega_k| + |\partial\Omega_k|)^{\frac{r-1-\eta}{r}},
\end{aligned} \tag{3.3}$$

where, $\eta = \frac{N+1}{\theta-1}$ and $r = \theta\eta$, here, $|C|$ denotes the Lebesgue measure of the measurable set C. Now, using Sobolev and trace imbeddings we can estimate from below the term on the left-hand side of (3.2) as follows:

$$\begin{aligned}
&\int_\Omega |\nabla u_\epsilon|^{N-2}\nabla u_\epsilon \nabla(T_k(u_\epsilon)) + \int_\Omega |u_\epsilon|^{N-2}u_\epsilon T_k(u_\epsilon) \\
&\geq C(\int_\Omega |\nabla(T_k(u_\epsilon))|^N + \int_\Omega |T_k(u_\epsilon)|^N) \\
&\geq C(\int_\Omega |T_k(u_\epsilon)|^r + \int_{\partial\Omega} |T_k(u_\epsilon)|^r)^{\frac{N}{r}}.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.2), we get

$$\int_\Omega |T_k(u_\epsilon)|^r + \int_{\partial\Omega} |T_k(u_\epsilon)|^r \leq (|\partial\Omega_k| + |\partial\Omega_k|)^{\frac{N}{N-1}} \tag{3.5}$$

Notice that for $0 < k < h$, $\Omega(h) \subset \Omega(k)$ since $T_k(s) = (|s|-k)(1-\chi_{[-k,k]}(s))\ \forall\, s \in \mathbb{R}$ and then

$$|\Omega_h|(h-k)^r = \int_{\Omega_h} (h-k)^r \leq \int_{\Omega_h} (|u_\epsilon|-k)^r \leq \int_{\Omega_k} (|u_\epsilon|-k)^r = \int_\Omega |T_k(u_\epsilon)|^r. \tag{3.6}$$

In the same way we have

$$|\partial\Omega_h|(h-k)^r \leq \int_{\partial\Omega} |T_k(u_\epsilon)|^r. \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5), we obtain

$$\Phi(h) \le C(h-k)^{-r}(\Phi(k))^{\frac{N}{N-1}} \qquad 0<k<h \tag{3.8}$$

where $\Phi(k) \stackrel{\text{def}}{=} |\Omega_k| + |\partial\Omega|\ k>0$.
Now we have the following
Claim: *Assume* $\Phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a non-increasing function such that if* $h>k>k_0$

$$\Phi(h) \le C(h-k)^{-r}(\Phi(k))^{\frac{N}{N-1}} \qquad 0<k<h.$$

Then $\Phi(d+k_0)=0$ *where* $d \stackrel{\text{def}}{=} 2^N C^{\frac{1}{r}} \Phi(k_0)^{\frac{1}{(N-1)r}}$.

By the Claim we get that $\Phi(d) = |\Omega_d| + |\partial\Omega_d| = 0$ namely

$$\sup_{\epsilon\in(0,1)} ||u_\epsilon||_{L^\infty} \le d.$$

To finish we need to prove the Claim.

Proof of the claim: Given d as above, define the sequence $\{k_n\}$ by $k_0=0$ and $k_n = k_{n-1} + \frac{d}{2^n}$ for $n = 1,2,.....$ By recurrence we have that

$$\Phi(k_n) \le \frac{\Phi(k_0)}{2^{nr(N-1)}} \longrightarrow 0 \text{ as } n\to\infty.$$

Then

$$0 \le \Phi(d+k_0) \le \lim_{n\to\infty}\Phi(k_n) = 0.$$

This gives the proof of the Claim and the proof of Theorem 1.2. □

4. appendix

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional $I\colon W^{1,N}(\Omega)\to\mathbb{R}$ defined in (1.1).

Lemma 4.1. *Let* $0<\delta<1$. *Then there exists a constant* $C_\delta>0$ *such that the inequality*

$$\int_0^1 |\mathbf{a}+s\mathbf{b}|^{-\delta}\,\mathrm{d}s \le C_\delta \left(\max_{0\le s\le 1}|\mathbf{a}+s\mathbf{b}|\right)^{-\delta} \tag{4.1}$$

holds true for all $\mathbf{a},\mathbf{b}\in\mathbb{R}^N$ *with* $|\mathbf{a}|+|\mathbf{b}|>0$.

An elementary proof of this lemma can be found in Takáč [15, Lemma A.1, p. 233].

We continue by showing the Gâteaux-differentiability of the energy functional I at a point $u\in W_0^{1,N}(\Omega)$ satisfying $u\ge\varepsilon\varphi_1$ in Ω with a constant $\varepsilon>0$.

Lemma 4.2. *Let the assumptions* **(h1)-(h2)** *and* **(g1)-(g2)** *be satisfied. Assume that* $u, v \in W^{1,N}(\Omega)$ *and* u *satisfies* $u \geq \varepsilon\varphi_1$ *in* Ω *with a constant* $\varepsilon > 0$. *Then we have*

$$\lim_{t\to 0} \frac{1}{t}\left(I(u+tv) - I(u)\right) = \int_\Omega |\nabla u|^{N-2}\nabla u \nabla v \,\mathrm{d}x - \int_\Omega |u|^{N-1} v \,\mathrm{d}x$$
$$- \int_\Omega g(u)v \,\mathrm{d}x - \int_\Omega f(x,u)v \,\mathrm{d}x - \int_{\partial\Omega} |u|^{q-1}uv \,\mathrm{d}x \tag{4.2}$$

Proof: We show the result only for the singular term $\int_\Omega g(u)v \,\mathrm{d}x$; the other two terms are treated in a standard way. So let

$$H(u) = \int_\Omega G(u(x)^+) \,\mathrm{d}x \qquad \text{for } u \in W^{1,N}(\Omega).$$

For $\xi \in \mathbb{R}\setminus\{0\}$ we define

$$z(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} G(\xi^+) = \begin{cases} g(\xi) & \text{if } \xi > 0; \\ 0 & \text{if } \xi < 0. \end{cases}$$

Consequently,

$$\frac{1}{t}\left(H(u+tv) - H(u)\right) = \int_\Omega \left(\int_0^1 z(u+stv)\,\mathrm{d}s\right) v \,\mathrm{d}x\,. \tag{4.3}$$

Notice that for almost every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 z(u(x)+stv(x))\,\mathrm{d}s \longrightarrow z(u(x)) = g(u(x)) \qquad \text{as } t \to 0\,.$$

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

$$\int_0^1 z(u(x)+stv(x))\,\mathrm{d}s \leq C \int_0^1 |u(x)+stv(x)|^{-\delta}\,\mathrm{d}s$$
$$\leq C_\delta \left(\max_{0\leq s\leq 1} |u(x)+stv(x)|\right)^{-\delta}$$
$$\leq C_\delta\, u(x)^{-\delta} \leq C_\delta\, (\varepsilon\varphi_1(x))^{-\delta} = C_{\delta,\varepsilon}\, \varphi_1(x)^{-\delta}$$

with constants $C, C_{\delta,\varepsilon} > 0$ independent of $x \in \Omega$. Here, we have used the estimate (4.1) from Lemma 4.1 above. Finally, we have $v\varphi_1^{-\delta} \in L^1(\Omega)$, by $v \in W^{1,N}(\Omega)$ and Hardy's inequality. That's

$$\left(\int_0^1 z(u(x)+stv(x))\,\mathrm{d}s\right) v \leq C_{\delta,\varepsilon}\, \varphi_1(x)^{-\delta} v < \infty$$

Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in (4.3) from which the lemma follows by letting $t \to 0$. □

Corollary 4.3. *Let the assumptions* **(h1)-(h2)** *and* **(g1)-(g2)** *be satisfied. Then the energy functional* $I\colon W^{1,N}(\Omega) \to \mathbb{R}$ *is Gâteaux-differentiable at every point* $u \in W^{1,N}(\Omega)$ *that satisfies* $u \geq \varepsilon\varphi_1$ *in* Ω *with a constant* $\varepsilon > 0$. *Its Gâteaux derivative* $I'(u)$ *at* u *is given by*

$$\begin{aligned}\langle I'(u),\, v\rangle &= \int_\Omega |\nabla u|^{N-2}\nabla u\nabla v\,\mathrm{d}x - \int_\Omega |u|^{N-1}v\,\mathrm{d}x - \int_\Omega g(u)v\,\mathrm{d}x \\ &- \int_\Omega f(x,u)v\,\mathrm{d}x - \int_{\partial\Omega} |u|^{q-1}uv\,\mathrm{d}x \end{aligned} \tag{4.4}$$

for $v \in W^{1,N}(\Omega)$.

We continue by proving the C^1-differentiability of the cut off energy functional $\overline{I}$ defined below:

Lemma 4.4. *Let the assumptions* **(h1)-(h2)** *and* **(g1)-(g2)** *be satisfied, and* $w \in W^{1,N}(\Omega)$ *such that* $w \geq \epsilon\varphi_1$ *with some* $\epsilon > 0$.

Define $\tilde{g}_\lambda : \Omega \to \mathbb{R}$ *by*

$$\tilde{g}(s) = \begin{cases} g(s) & s \geq w(x), \\ g(w(x)) & s < w(x). \end{cases}$$

$\tilde{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ *by*

$$\tilde{f}_\lambda(x,s) = \begin{cases} f(x,s) & s \geq w(x), \\ f(x,w(x)) & s < w(x), \end{cases}$$

and $\tilde{h}_\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *by*

$$\tilde{h}_\lambda(x,s) = \begin{cases} |s|^q & s \geq w(x), \\ |w(x)|^q & s < w(x). \end{cases}$$

Let $\tilde{G}_\lambda(s) = \int_0^s \tilde{g}(t)dt$, $\tilde{F}(x,s) = \int_0^s \tilde{f}(x,t)dt$ *and* $\tilde{H}(x,s) = \int_0^s \tilde{h}(x,t)dt$. *Consider the functional* $\tilde{I} : W^{1,N}(\Omega) \to \mathbb{R}$ *defined by*

$$\tilde{I}(u) = \frac{1}{N}\int_\Omega (|\nabla u|^N + |u|^N) - \int_\Omega \tilde{G}(u)dx - \int_\Omega \tilde{F}(x,u)dx - \int_{\partial\Omega} \tilde{H}_\lambda(x,u)dx. \tag{4.5}$$

We have that $\tilde{I}$ *belongs to* $C^1(W^{1,N}(\Omega),\mathbb{R})$.

Proof:

As in Lemma 4.2, we concentrate on the singular term, the others being standard. Let

$$\tilde{g}(s) = \begin{cases} g(s) & \text{if } s \geq w(x), \\ g(w(x)) & \text{if } s < w(x), \end{cases}$$

$\tilde{G}(s) = \int_0^s g(t)\,\mathrm{d}t$, and $S(u) = \int_\Omega G(u)\mathrm{d}x$. Proceeding as in Lemma 4.2, we obtain that for all $u \in W^{1,N}(\Omega)$, $S(u)$ has a Gâteaux derivative $S'(u)$ given by

$$\langle S'(u), v\rangle = \int_\Omega g((\max\{u(x), w(x)\})v(x)\,\mathrm{d}x.$$

Let $u_k \in W^{1,N}(\Omega)$, $u_k \to u_0$. Then

$$\begin{aligned} |\langle S'(u_k) - S'(u_0), v\rangle| &= \Big|\int_\Omega (g(\max\{u_k(x), w(x)\})v(x) \\ &\quad -g(\max\{u(x), w(x)\})v(x))\,\mathrm{d}x| \\ &\leq 2C\int_\Omega w^{-\delta}|v|\,\mathrm{d}x \\ &\leq 2C\epsilon^{-\delta}\int_\Omega \varphi_1^{-\delta}|v|\,\mathrm{d}x \end{aligned}$$

for all $v \in W^{1,N}(\Omega)$. Again, as in Lemma 4.2, we use Hardy's inequality to deduce that $\varphi_1^{-\delta}v \in L^1(\Omega)$, so that by Lesbegue's dominated convergence theorem we conclude that the Gâteaux derivative of S is continuous which implies that $S \in C^1(W^{1,N}(\Omega), \mathbb{R})$. □

We give now the existence of a subsolution to (P):

Lemma 4.5. *Assume assumptions* **(g1)?(g2).** *Then problem* (PS) *possesses a weak solution in* $W^{1,N}(\Omega)$ *in the sense of distributions. This solution, denoted by* $\underline{u}$, *is the unique global minimizer to the energy functional* $\tilde{E}$ *given by*

$$\tilde{E} \stackrel{\text{def}}{=} \frac{1}{N}\left(\int_\Omega |\nabla u|^N \mathrm{d}x - \int_\Omega |u|^N\,\mathrm{d}x\right) - \int_\Omega G(u^+)\mathrm{d}x - \frac{1}{q+1}\int_{\partial\Omega}|u|^{q+1}\,\mathrm{d}x$$

$\forall\, u \in W^{1,N}(\Omega)$. *In addition,* $\underline{u}$ *is the unique solution to (PS) in* $\underline{\Omega} \stackrel{\text{def}}{=} \{u \in W^{1,N}(\Omega)$ *such that* $u \geq \eta\varphi_1$ *for some* $\eta > 0\}$.

Proof: First, by Hölder's inequality and Sobolev embedding and trace embedding $W^{1,N}(\Omega) \hookrightarrow L^q(\partial\Omega)$ we get for some $C_2 > 0$,

$$\int_{\partial\Omega}|u|^{q+1} \leq C_2\|u\|^{q+1}_{L^{q+1}(\partial\Omega)} \leq C_2\|u\|^{q+1}. \tag{4.6}$$

Thus, from (4.6) and owing to the Poincaré inequality, assumption **(g2)** and $0 < 1-\delta < 1 < N < \infty$, the functional $\tilde{E}$ is coercive.
Now, we prove that $\tilde{E}$ is weakly lower semicontinuous. To do this it is sufficient to show that for $u_j \rightharpoonup u$ weakly in $W^{1,N}(\Omega)$ we have

$$\int_\Omega g(u_j)\,\mathrm{d}x \to \int_\Omega g(u)\,\mathrm{d}x \text{ when } j \to +\infty. \tag{4.7}$$

and

$$\int_{\partial\Omega} |u_j|^{q-1} u_j \, dx \to \int_{\partial\Omega} |u|^{q-1} u \, dx \text{ when } j \to +\infty. \tag{4.8}$$

(4.7) follows from the definition of weak convergence and using assumption **(g2)**. Finally, (4.8) follows from the trace embedding. It follows that $\tilde{E}$ possesses a global minimizer $\tilde{u} \in W^{1,N}(\Omega)$. We have $\tilde{u} \neq 0$ owing to $\tilde{E}(0) = 0 > \tilde{E}(\epsilon\varphi_1)$ for $\epsilon > 0$ small enough.

Second, the polar decomposition $u = u^+ - u^-$ of any function $u \in W^{1,N}(\Omega)$ gives $\nabla u = \nabla u^+ - \nabla u^-$. Thus, if $\tilde{u}$ is a global minimizer for $\tilde{E}$, then so is its absolute value $|\tilde{u}|$, by $\tilde{E}(|\tilde{u}|) \leq \tilde{E}(\tilde{u})$ holds if and only if $\tilde{u}^- = 0$ a.e. in Ω, that is, if and only if $\tilde{u} \geq 0$ a.e. in Ω. Thus, any global minimizer $\tilde{u}$ for $\tilde{E}$, must satisfy $\tilde{u} \geq 0$ a.e. in Ω. Equivalently, $\tilde{u} \in W^{1,N}(\Omega)_+$ where

$$W^{1,N}(\Omega)_+ \stackrel{\text{def}}{=} \{u \in W^{1,N}(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}$$

stands for the positive cone in $W^{1,N}(\Omega)$.

From the fact $-\Delta_N u - |u|^{N-1}u$ is a monotone operator in the cone $\underline{\Omega} \stackrel{\text{def}}{=} \{u \in W^{1,N}(\Omega)$ such that $u \geq \eta\varphi_1$ for some $\eta > 0\}$ and the weak comparison principle, we conclude that $\tilde{E}$ has a unique global minimizer denoted by $\underline{u}$ in $W^{1,N}(\Omega)$ with the property $\operatorname{essinf}_K \underline{u} > 0$ for any compact set $K \subset \Omega$. $\underline{u}$ is then the unique weak solution to (PS) in $\underline{\Omega}$ and satisfies (1.2). □

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