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Generalized Rough Lacunary Statistical Triple Difference Sequence Spaces in Probability of Fractional Order Defined by Musielak-Orlicz Function

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ABSTRACT: We generalized the concepts in probability of rough lacunary statistical by introducing the difference operator Δ_{γ}^{α} of fractional order, where α is a proper fraction and $\gamma = (\gamma_{mnk})$ is any fixed sequence of nonzero real or complex numbers. We study some properties of this operator involving lacunary sequence θ and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers and investigate the topological structures of related triple difference sequence spaces.

The main focus of the present paper is to generalized rough lacunary statistical of triple difference sequence spaces and investigate their topological structures as well as some inclusion concerning the operator Δ_{γ}^{α}.

Key Words: Analytic sequence, Musielak-Orlicz function, Triple sequences, Chi sequence, Lacunary statistical convergence, Rough convergence.

Contents

1. Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [11,12]*, *Esi et al. [1-4]*, *Datta et al. [5]*,*Subramanian et al. [13]*, *Debnath et al. [6]*, *Savas et al. [10]* and many others.
A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

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The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The difference triple sequence space was introduced by Debnath et al. (see [6]) and is defined as
$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \langle x_{mnk} \rangle$.

2. Definitions and Preliminaries

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively.
Subramanian et al. (see [13]) introduced triple entire sequence spaces, triple analytic sequences spaces and triple gai sequence spaces. The triple sequence spaces of $\chi^3(\Delta), \Lambda^3(\Delta)$ are defined as follows:

$$\chi^3(\Delta) = \left\{ x \in w^3 : ((m+n+k)!\,|\Delta x_{mnk}|)^{1/m+n+k} \to 0\, as\, m,n,k \to \infty \right\},$$

$$\Lambda^3(\Delta) = \left\{ x \in w^3 : sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty \right\}.$$

Definition 2.1. *An Orlicz function ([see [7]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function.*

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (f_{mnk})(u) : u \ge 0\}, m,n,k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [9]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0\, as\, m,n,k \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk} \right).$$

3. Some new difference triple sequence spaces with fractional order and rough lacunary statistical convergence

Let $\Gamma(\alpha)$ denote the Euler gamma function of a real number α. Using the definition $\Gamma(\alpha)$ can be expressed as an improper integral as follows: $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$, where α is a proper fraction. We have defined the generalized fractional triple sequence spaces of difference operator

$$\Delta_\gamma^\alpha(x_{mnk}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{u+v+w}\Gamma(\alpha+1)}{(u+v+w)!\Gamma(\alpha-(u+v+w)+1)}x_{m+u,n+v,k+w}. \tag{3.1}$$

In particular, we have

(i) $\Delta^{\frac{1}{2}}(x_{mnk}) = x_{mnk} - \frac{1}{16}x_{m+1,n+1,k+1} - \cdots$.

(ii) $\Delta^{-\frac{1}{2}}(x_{mnk}) = x_{mnk} + \frac{5}{16}x_{m+1,n+1,k+1} + \cdots$.

(iii) $\Delta^{\frac{2}{3}}(x_{mnk}) = x_{mnk} - \frac{4}{81}x_{m+1,n+1,k+1} - \cdots$.

Now we determine the new classes of triple difference sequence spaces $\Delta_\gamma^\alpha(x)$ as follows:

$$\Delta_\gamma^\alpha(x) = \left\{x : (x_{mnk}) \in w^3 : \left(\Delta_\gamma^\alpha x\right) \in X\right\}, \tag{3.2}$$

where $\Delta_\gamma^\alpha(x_{mnk}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{u+v+w}\Gamma(\alpha+1)}{(u+v+w)!\Gamma(\alpha-(u+v+w)+1)}x_{m+u,n+v,k+w}$ and

$$\begin{aligned} X \in \chi_f^{3\Delta}(x) =& \chi_f^3\left(\Delta_\gamma^\alpha x_{mnk}\right) \\ =& \mu_{mnk}\left(\Delta_\gamma^\alpha x\right) \\ =& \left[f_{mnk}\left(\left((m+n+k)!\left|\Delta_\gamma^\alpha\right|\right)^{\frac{1}{m+n+k}}, \bar{0}\right)\right]. \end{aligned}$$

Proposition 3.1. *(i) For a proper fraction α, $\Delta^\alpha : W \times W \times W \to W \times W \times W$ defined by equation of (2.1) is a linear operator.*
(ii) For $\alpha, \beta > 0$, $\Delta^\alpha\left(\Delta^\beta(x_{mnk})\right) = \Delta^{\alpha+\beta}(x_{mnk})$ and $\Delta^\alpha(\Delta^{-\alpha}(x_{mnk})) = x_{mnk}$.

Proof: Omitted.

Proposition 3.2. *For a proper fraction α and Musielak-Orlicz function f, if $\chi_f^3(x)$ is a linear space, then $\chi_f^{3\Delta_\gamma^\alpha}(x)$ is also a linear space.*

Proof: Omitted.

Definition 3.3. *The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that*

$$\begin{gathered} m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty \text{ and} \\ n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty. \\ k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty. \end{gathered}$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i\overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by $I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}$, $q_i = \frac{m_i}{m_{i-1}}$, $\overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

Definition 3.4. *Let α be a proper fraction, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be the triple difference lacunary sequence spaces of $\left(\Delta^{\alpha}_{\gamma} X_{mnk}\right)$ is said to be $\Delta^{\alpha}_{\gamma}-$ lacunary statistically convergent to a number $\bar{0}$ if for any $\epsilon > 0$,*
$lim_{\frac{1}{h_{rst}}} \left|\left\{(m,n,k) \in I_{rst} : f_{mnk}\left[\left|\Delta^{\alpha}_{\gamma} X_{mnk}, \bar{0}\right|\right] \geq \epsilon\right\}\right| = 0$ *, where*
$I_{r,s,t} = \{(m,n,k) : m_{r-1} < m < m_r \, and \, n_{s-1} < n \leq n_s \, and \, k_{t-1} < k \leq k_t\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}$*. In this case write* $\Delta^{\alpha}_{\gamma} X \rightarrow^{S_\theta} \Delta^{\alpha}_{\gamma} x$.

Definition 3.5. *If α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be the triple difference sequence spaces of lacunary. A number X is said to be $\Delta^{\alpha}_{\gamma} - N_\theta-$ convergent to a real number $\bar{0}$ if for every $\epsilon > 0$,*
$lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} f_{mnk}\left[\left|\Delta^{\alpha}_{\gamma} X_{mnk}, \bar{0}\right|\right] = 0$*. In this case we write* $\Delta^{\alpha}_{\gamma} X_{mnk} \rightarrow^{N_\theta} \bar{0}$.

Definition 3.6. *Let α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be $\Delta^{\alpha}_{\gamma}-$ rough lacunary statistically convergent in probability to $\Delta^{\alpha}_{\gamma} X : W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for any $\epsilon, \delta > 0$, $lim_{rst\to\infty}$ $\frac{1}{h_{rst}} \left|\left\{(m,n,k) \in I_{rst} : P\left(\left[f_{mnk}\left(\left|\Delta^{\alpha}_{\gamma}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right| = 0$ and we write $\Delta^{\alpha}_{\gamma} X_{mnk} \rightarrow^{S^P}_{\beta} \bar{0}$. It will be denoted by βS^P_θ.*

Definition 3.7. *Let α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be $\Delta^{\alpha}_{\gamma}-$ rough $N_\theta-$ convergent in probability to $\Delta^{\alpha}_{\gamma} X : W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for any $\epsilon > 0$, $lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t}$ $\left|\left\{P\left(\left[f_{mnk}\left(\left|\Delta^{\alpha}_{\gamma} X_{mnk}\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right)\right\}\right| = 0$, and we write $\Delta^{\alpha}_{\gamma} X_{mnk} \rightarrow^{N^P_\theta}_{\beta} \Delta^{\alpha}_{\gamma} X$. The class of all $\beta - N_\theta-$ convergent triple difference sequence spaces of random variables in probability will be denoted by βN^P_θ.*

Definition 3.8. *Let $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be lacunary triple difference sequence spaces of lacunary refinement of θ is a triple difference lacunary sequence spaces of $\theta' = \left\{m'_r n'_s k'_t\right\}_{(rst)\in\mathbb{N}\bigcup 0}$ satisfying $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0} \subset \left\{m'_r n'_s k'_t\right\}_{(rst)\in\mathbb{N}\bigcup 0}$.*

Remark 3.9. *Let f be an Musielak-Orlicz function and triple sequence spaces of $\left\|\chi^3_f, d(x)\right\|_p = \left[f_{mnk}\left(\left\|\mu_{mnk}(X), d(x)\right\|_p\right)\right]$, where*

$$\mu_{mnk}(X) = \left(\left((m+n+k)! X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)$$

and

$$d(x) = \left(d(x_1), d(x_2), \cdots, d(x_{n-1})\right).$$

4. Main Results

In this section by using the operator Δ_γ^α, we introduce some new triple difference sequence spaces involving rough lacunary statistical and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers, α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function, the following theorems are obtained:

Theorem 4.1. *Let $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be a triple difference rough lacunary statistical sequence spaces. Then the followings are equivalent:*
(i) $\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}\right), d(x)\right\|_p$ is $\beta-$ triple lacunary statistically convergent in probability to $\bar{0}$.
(ii) $\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}\right), d(x)\right\|_p$ is $\beta - N_\theta$ convergent in probability to $\bar{0}$.

Proof: (i)$\Longrightarrow$ (ii) First suppose that $\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}\right), d(x)\right\|_p \to_\beta^{S_\theta^P} \bar{0}$. Then we can write

$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t} \left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| =$$

$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t, P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}} \left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| +$$

$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t, P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)<\frac{\delta}{2}} \left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| \leq$$

$$\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}\right\}\right| + \frac{\delta}{2}.$$

(ii)$\Longrightarrow$ (i) Next suppose that condition (ii) holds. Then $\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}$

$$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| \geq$$

$$\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t} \left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|$$

$$\geq\delta\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$$ Therefore

$$\frac{1}{\delta}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t} \left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| \geq$$

$$\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta_\gamma^\alpha X\right)-\mu\left(\Delta_\gamma^\alpha X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$$ Hence $\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}\right), d(x)\right\|_p \to_\beta^{S_\theta^P} \bar{0}$.

Theorem 4.2. *If $\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}\right), d(x)\right\|_p \to_\beta^{S_\theta^P} \bar{0}$ and $\left\|\chi_f^3\left(\Delta_\gamma^\alpha Y_{mnk}\right), d(x)\right\|_p \to_\beta^{S_\theta^P} \bar{0}$ then*

$$P\left(\left|\left\{\left\|\chi_f^3\left(\Delta_\gamma^\alpha X_{mnk}-\Delta_\gamma^\alpha Y_{mnk}\right), d(x)\right\|_p\right|\geq\beta+\epsilon\right)\right\} = \bar{0}.$$

Proof: Consider $\left\|\chi^3_f\left(\Delta^\alpha_\gamma X_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_\theta}_\beta \bar{0}$. and $\left\|\chi^3_f\left(\Delta^\alpha_\gamma Y_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_\theta}_\beta \bar{0}$. Then we can write

$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| =$
$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t, P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| +$
$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t, P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)<\frac{\delta}{2}}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| \leq$
$\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}\right\}\right| + \frac{\delta}{2}.$
Therefore
$\frac{1}{\delta}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right| \geq$
$\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma Y\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$
Hence $\left\|\chi^3_f\left(\Delta^\alpha_\gamma X_{mnk}-\Delta^\alpha_\gamma Y_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_\theta}_\beta \bar{0}$.

Theorem 4.3. *Let $\theta^{'}=\left\{m^{'}_r n^{'}_s k^{'}_t\right\}_{(rst)\in\mathbb{N}\bigcup 0}$ be a triple lacunary refinement of the triple lacunary sequence spaces of $\theta=\{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$. Let $h_r=(m_{r-1},m_r]$, $h_s=(n_{s-1},n_s], h_t=(k_{t-1},h_r], r,s,t=1,2,3\cdots$. If there exists a $\eta>0$ such that $\frac{|h_{rst}|}{|I_{rst}|}>\eta$ for every $h_{rst}\subseteq I_{rst}$. Then $\left\|\chi^3_f\left(\Delta^\alpha_\gamma X_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_\theta}_\beta \bar{0}\Longrightarrow$ $\left\|\chi^3_f\left(\Delta^\alpha_\gamma X_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_{\theta'}}_\beta \bar{0}$.*

Proof: Let $\left\|\chi^3_f\left(\Delta^\alpha_\gamma X_{mnk}\right), d(x)\right\|_p \rightarrow^{S^P_\theta}_\beta \bar{0}$ and $\epsilon,\delta>0$. Therefore
$\lim^{rst\to\infty}\frac{1}{|I_{rst}|}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right| = 0$. For every h_{rst} we can find I_{rst} such that $h_{rst}\subseteq I_{rst}$. We obtain
$\frac{1}{|h_{rst}|}\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|$
$=\frac{|I_{rst}|}{|h_{rst}|}\frac{1}{|I_{rst}|}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right| \leq \frac{1}{\eta}\frac{1}{|I_{rst}|}$
$\left|\left\{P\left(\left\|\left[f_{mnk}\left(\left(\mu_{mnk}\left(\Delta^\alpha_\gamma X\right)-\mu\left(\Delta^\alpha_\gamma X\right)\right), d(x)\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$

Remark 4.4. *In this refinements $\theta^{'}$ of θ exists. The following example, let $(u,v,w)\in\mathbb{N}\setminus\{1,1,1\}$ and introducing $(u-1,v-1,w-1)$ points in the interval $h_r=(m_{r-1},m_r]$, $h_s=(n_{s-1},n_s], h_t=(k_{t-1},h_r], r,s,t=1,2,3\cdots$. Then*

$h_1 = \Big((m_0, n_0, k_0), (m_0, n_0, k_0) + \frac{j_1}{(uvw)}\Big]$
$h_2 = \Big((m_0, n_0, k_0) + \frac{j_1}{(uvw)}, (m_0, n_0, k_0) + \frac{2j_1}{(uvw)}\Big]$
$\vdots$

$h_{(uvw)} = \Big((m_0, n_0, k_0) + \frac{(u-1)(v-1)(w-1)j_1}{(uvw)}, (m, n, k)\Big]$
$h_{(u+1,v+1,w+1)} = \Big((m_1, n_1, k_1), (m_1, n_1, k_1) + \frac{j_2}{(uvw)}\Big]$
$h_{(u+2,v+2,w+2)} = \Big((m_1, n_1, k_1) + \frac{j_2}{(uvw)}, (m_1, n_1, k_1) + \frac{2j_2}{(uvw)}\Big]$
$\vdots$

$h_{(2u,2v,2w)} = \Big((m_1, n_1, k_1) + \frac{(u-1)(v-1)(w-1)j_2}{(uvw)}, (m_2, n_2, k_2)\Big]$
$\vdots$

$h_{r-1(u+1,v+1,w+1)} = \Big((m_{r-1}, n_{r-1}, k_{r-1}), (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{h_r}{(uvw)}\Big]$
$h_{r-1(u+2,v+2,w+2)} = \Big((m_{r-1}, n_{r-1}, k_{r-1}) + \frac{h_r}{(uvw)}, (m_{r-1}, n_{r-1}, k_{r-1}) + \frac{2h_r}{(uvw)}\Big]$
$\vdots$

$h_{r-1(2u,2v,2w)} = \Big((m_{r-1}, n_{r-1}, k_{r-1}) + \frac{(u-1)(v-1)(w-1)h_{r-1}}{(uvw)}, (m_r, n_r, k_r)\Big].$
$\vdots$

Then $|h_{rst}| \to \infty$ *as* $r, s, t \to \infty$ *and* $\frac{|h_{rst}|}{|I_{rst}|} \geq \frac{1}{(uvw)}$ *for every* $h_{rst} \subseteq I_{rst}$.

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References

1. A. Esi, Statistical convergence of triple sequences in topological groups, *Annals of the University of Craiova, Mathematics and Computer Science Series;* 40(1), (2013), 29-33.
2. A. Esi , On some triple almost lacunary sequence spaces defined by Orlicz functions, *Research and Reviews:Discrete Mathematical Structures*, 1(2), (2014), 16-25.
3. A. Esi and M. Necdet Catalbas,Almost convergence of triple sequences, *Global J. Math. Anal.*, 2(1), (2014), 6-10.
4. A. Esi and E. Savas, On lacunary statistically convergent triple sequences in probabilistic normed space,*Appl.Math.and Inf.Sci.*, 9 (5) , (2015), 2529-2534.
5. A. J. Datta A. Esi and B.C. Tripathy,Statistically convergent triple sequence spaces defined by Orlicz function ,*J. Math. Anal.*, 4(2), (2013), 16-22.
6. S. Debnath, B. Sarma and B.C. Das ,Some generalized triple sequence spaces of real numbers , *J. Nonlinear Aanal. Optimi.*, Vol. 6, No. 1 (2015), 71-79.
7. P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c. New York* , 1981.
8. J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, 10 (1971), 379-390.
9. J. Musielak, Orlicz Spaces,Lectures Notes in Math.,1034, *Springer-Verlag*, 1983.
10. E. Savas and A. Esi, Statistical convergence of triple sequences on probabilistic normed space ,*of the University of Craiova, Mathematics and Computer Science Series*, 39 (2) , (2012), 226-236.
11. A. Sahiner, M. Gurdal and F.K. Duden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.* , 8 No. (2)(2007), 49-55.
12. A. Sahiner, B.C. Tripathy , Some I related properties of triple sequences, *Selcuk J. Appl. Math. ,* 9 No. (2)(2008), 9-18.
13. N. Subramanian and A. Esi, The generalized tripled difference of χ^3 sequence spaces, *Global J. Math. Anal.*, 3 (2) (2015), 54-60.

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