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Nearly $\mathcal{I}$-Continuous Multifunctions

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ABSTRACT: The purpose of the present paper is to introduce and study upper and lower nearly $\mathcal{I}$-continuous multifunctions. Basic characterizations, several properties of upper and lower nearly $\mathcal{I}$-continuous multifunctions are investigated.

Key Words: Near $\mathcal{I}$-continuous multifunctions, $\mathcal{I}$-open set.

Contents

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2,9,11,12,13]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy [16]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [16] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [5] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Several characterizations and properties of $\mathcal{I}$-open sets were provided in [1,8]. Recently, Akdag [2] introduced and studied the concept of $\mathcal{I}$-continuous multifunctions in topological spaces. In this

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paper we introduce and study a new generalization of $\mathcal{I}$-continuous multifunctions called nearly $\mathcal{I}$-continuous multifunctions on ideal topological spaces.

2. Preliminaries

For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-open [5] if $S \subset \mathrm{Int}(S^\star)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). We set $\mathcal{I}O(X, x) = \{A \subset X : A \in \mathcal{I}O(X) \text{ and } x \in A\}$. A subset A of a topological space (X, τ) is said to be regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a regular open sets is called a regular closed set. The family of all regular open (resp. regular closed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$). By a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A point x of X is called a θ-cluster [15] point of $S \subset X$ if $\mathrm{Cl}(U) \cap S \neq \varnothing$ for every open subset of X containing x. The set of all θ-cluster points of S is called the θ-closure of S and is denoted by $\mathrm{Cl}_\theta(S)$. A subset S is said to be θ-closed if and only if $S = \mathrm{Cl}_\theta(S)$. The complement of a θ-closed set is said to be a θ-open set. The θ-interior [15] of A is defined as $\mathrm{Int}_\theta(A) = \{x \in X : \mathrm{Cl}(U) \subset A$ for some open set U containing $x\}$.

Definition 2.1. *[2] A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be*

1. *upper $\mathcal{I}$-continuous if $F^+(V) \in \mathcal{I}O(X)$ for each open set V of Y,*
2. *lower $\mathcal{I}$-continuous if $F^-(V) \in \mathcal{I}O(X)$ for each open set V of Y.*

Definition 2.2. *A subset A of a topological space (X, τ) is said to be N-closed [4] if every cover of A by regular open sets of X has a finite subcover.*

3. Upper (Lower) nearly $\mathcal{I}$-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:*

1. *upper nearly $\mathcal{I}$-continuous at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset V$.*
2. *lower nearly $\mathcal{I}$-continuous at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having N-closed complement, there exists an $\mathcal{I}$-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) nearly $\mathcal{I}$-continuous on X if it has this property at every point of X.*

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper nearly $\mathcal{I}$-continuous.*
2. *$F^+(V)$ is $\mathcal{I}$-open for each open set V of Y having N-closed complement.*
3. *$F^-(K)$ is $\mathcal{I}$-closed for every N-closed and closed subset K of Y.*
4. *$\mathcal{I}\,\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be any open set of Y having N-closed complement. From (1), there exists an $\mathcal{I}$-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \underset{x \in F^+(V)}{\cup} U_x$. Since any union of $\mathcal{I}$-open sets is $\mathcal{I}$-open, $F^+(V)$ is $\mathcal{I}$-open in $(X, \tau, \mathcal{I})$.
(2)⇒(3): Let K be any N-closed and closed subset of Y. Then by (2), $F^+(Y \setminus K) = X \setminus F^-(K)$ is an $\mathcal{I}$-open set. Then it is obtained that $F^-(K)$ is an $\mathcal{I}$-closed set.
(3)⇒(4): Let B be any subset of Y having N-closed closure. By (3), we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = \mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Cl}(B)))$. Hence $\mathcal{I}\,\mathrm{Cl}(F^-(B)) \subset \mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.
(4)⇒(1): Clear. □

Theorem 3.3. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is lower nearly $\mathcal{I}$-continuous.*
2. *$F^-(V)$ is $\mathcal{I}$-open for each open set V of Y having N-closed complement.*
3. *$F^+(K)$ is $\mathcal{I}$-closed for every N-closed and closed set K of Y.*
4. *$\mathcal{I}\,\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 3.4. *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *F is upper nearly $\mathcal{I}$-continuous;*
2. *$F^-(\mathrm{Cl}_\theta(B))$ is an $\mathcal{I}$-closed set in X for every subset B of Y such that $\mathrm{Cl}_\theta(B)$ is N-closed;*
3. *$F^-(K)$ is an $\mathcal{I}$-closed set in X for every θ-closed and N-closed set K of Y;*
4. *$F^+(V)$ is an $\mathcal{I}$-open set in X for every θ-open set V of Y having N-closed complement.*

Proof. (1)⇒(2): Let B be any subset of Y such that $\mathrm{Cl}_\theta(V)$ is N-closed. Then $\mathrm{Cl}_\theta(B)$ is closed and N-closed. By Theorem 3.2, $F^-(\mathrm{Cl}_\theta(B))$ is an $\mathfrak{I}$-closed set in X.
(2)⇒(3): Let K be any N-closed and θ-closed set of Y. Then $K = \mathrm{Cl}_\theta(K)$ is N-closed. By (2), it follows that $F^-(K)$ is an $\mathfrak{I}$-closed set in X
(3)⇒(4): Let V be any θ-open set of Y having N-closed complement. Then $Y\backslash V$ is θ-closed and N-closed and by (3) $F^-(Y\backslash V) = \mathfrak{I}\,\mathrm{Cl}(F^-(Y\backslash V))$. Then $X\backslash F^+(V) = \mathfrak{I}\,\mathrm{Cl}(X\backslash F^+(V)) = X\backslash \mathfrak{I}\,\mathrm{Int}(F^+(V))$. Then $F^+(V)$ is an $\mathfrak{I}$-open set in X.
(4)⇒(1): Let V be any open set of Y having N-closed complement. Since Y is regular, V is a θ-open set in Y having N-closed complement and by (4) we have $F^+(V)$ is an $\mathfrak{I}$-open set in X. By Theorem 3.2, F is upper nearly $\mathfrak{I}$-continuous. □

Theorem 3.5. *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *F is lower nearly $\mathfrak{I}$-continuous;*
2. *$F^-(\mathrm{Cl}_\theta(B))$ is an $\mathfrak{I}$-closed set in X for every subset B of Y such that $\mathrm{Cl}_\theta(B)$ is N-closed;*
3. *$F^-(K)$ is an $\mathfrak{I}$-closed set in X for every θ-closed and N-closed set K of Y;*
4. *$F^+(V)$ is an $\mathfrak{I}$-open set in X for every θ-open set V of Y having N-closed complement.*

Proof. The proof is similar to that of Theorem 3.4. □

Theorem 3.6. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction such that (Y, σ) has a base of open sets having N-closed complements. If F is lower nearly $\mathfrak{I}$-continuous, then F is lower $\mathfrak{I}$-continuous.*

Proof. Let V be any open set of Y. By the hypothesis, $V = \bigcup_{i \in I} V_i$, where V_i is an open set having N-closed complement for each $i \in I$. By Theorem 3.2 $F^-(V_i)$ is $\mathfrak{I}$-open in X for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V)$ is $\mathfrak{I}$-open in X. Hence F is lower $\mathfrak{I}$-continuous. □

Definition 3.7. *A topological space (Y, σ) is said to be N-normal [?] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U, H \subset V$ and $U \cap V = \emptyset$.*

Definition 3.8. *An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be $\mathfrak{I}$-T_2 [8] if for each distinct points $x, y \in X$, there exist $\mathfrak{I}$-open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Theorem 3.9. *If $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ is an upper nearly $\mathfrak{I}$-continuous multifunction satisfying the following conditions:*

1. *$F(x)$ is closed in Y for each $x \in X$,*

2. *$F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,*

3. *(Y, σ) is an N-normal space,*

then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2.

Proof. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Theorem 3.2, we obtain, an $\mathcal{I}$-open set $F^+(U)$ in X containing x and an $\mathcal{I}$-open set $F^+(V)$ in X containing y and $F^+(U) \cap F^+(V) = \emptyset$. This shows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2. □

Theorem 3.10. *If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F form $(X, \tau, \mathcal{I})$ into an N-normal space (Y, σ) satisfying the following conditions:*

1. *$F(x_1)$ and $F(x_2)$ are closed in Y,*

2. *F is upper nearly $\mathcal{I}$-continuous at x_1 and x_2, and*

3. *$F(x_1) \cap F(x_2) = \emptyset$,*

then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2.

Proof. Let x_1 and x_2 be distinct points of X. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(x_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is upper nearly $\mathcal{I}$-continuous at x_1 and x_2, there exist U_1 and U_2 which are $\mathcal{I}$-open sets in X containing x_1 and x_2 respectively, such that $F(U_1) \subset V_1$ and $F(U_2) \subset V_2$. This implies that $U_1 \cap U_2 = \emptyset$. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-T_2-space. □

Theorem 3.11. *Let F and G be upper nearly $\mathcal{I}$-continuous and point closed multifunctions from an ideal topological space $(X, \tau, \mathcal{I})$ to a N-normal topological space Y. Then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is $\mathcal{I}$-closed in X.*

Proof. Let $x \in X \backslash A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a N-normal space, it follows that there exist disjoint open sets U and V having N-closed complements containing $F(x)$ and $G(x)$, respectively. Since F and G are upper nearly $\mathcal{I}$-continuous, then $F^+(U)$ and $G^+(V)$ are $\mathcal{I}$-open sets containing x. Let $H = F^+(U) \cup G^+(V)$. Then H is an $\mathcal{I}$-open set containing x and $H \backslash A = \emptyset$. Hence, A is $\mathcal{I}$-closed in X. □

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