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Further Generalization of the Extended Hurwitz-Lerch Zeta Functions

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abstract: Recently various extensions of Hurwitz-Lerch Zeta functions have been investigated. Here, we first introduce a further generalization of the extended Hurwitz-Lerch Zeta functions. Then we investigate certain interesting and (potentially) useful properties, systematically, of the generalization of the extended Hurwitz-Lerch Zeta functions, for example, various integral representations, Mellin transform, generating functions and extended fractional derivatives formulas associated with these extended generalized Hurwitz-Lerch Zeta functions. An application to probability distributions is further considered. Some interesting special cases of our main results are also pointed out.

Key Words: Generalized Hurwitz-Lerch Zeta function, Extended beta function, Extended hypergeometric function, Extended Hurwitz-Lerch Zeta function, Mellin transform, Extended fractional derivative operator.

Contents

1. Introduction, Definitions and Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ denote the sets of complex numbers, nonpositive integers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ is defined by (see, *e.g.*, [5, p. 27, Eq. 1.11(1)]; see also [16, p. 121] and [17, p. 194]):

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.1}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\big).$$

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Various generalizations of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ have been considered by many authors (see, *e.g.*, [3,4,5,11,16,17,18]). For example, Goyal and Laddha [7, p. 100, Eq. (1.5)] and Garg *et al.* [6, p. 313, Eq. (1.7)] introduced to investigate certain interesting extensions of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ in (1.1) which are defined, respectively, by

$$\Phi_{\mu}^{*}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \tag{1.2}$$

$$\big(\mu \in \mathbb{C};\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1; \Re(s-\mu)>1 \text{ when } |z|=1\big)$$

and

$$\Phi_{\lambda,\mu;\nu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$\big(\lambda,\mu \in \mathbb{C};\ \nu,\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s+\nu-\lambda-\mu)>1 \text{ when } |z|=1\big).$$

The following known integral integral representations of (1.2) and (1.3) are given, respectively, by

$$\Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(s)} \int\limits_0^{\infty} \frac{t^{s-1}e^{-at}}{(1-ze^{-t})^{\mu}}\,dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{(e^t-z)^{\mu}}\,dt \tag{1.4}$$

$$\big(\Re(a)>0;\ \Re(s)>0 \text{ when } |z| \leqq 1\,(z\neq 1);\ \Re(s)>1 \text{ when } z=1\big)$$

and

$$\Phi_{\lambda,\mu;\nu}(z,s,a) := \frac{1}{\Gamma(s)} \int\limits_0^{\infty} t^{s-1}e^{-at}\ {}_2F_1(\lambda,\mu;\nu;ze^{-t})\,dt \tag{1.5}$$

$$\big(\Re(a)>0;\ \Re(s)>0 \text{ when } |z| \leqq 1\,(z\neq 1);\ \Re(s)>1 \text{ when } z=1\big).$$

Very recently, Parmar and Raina [14, p. 120, Eq. (2.1)] introduced the following extension of generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) := \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu+n,\ \nu-\mu;\ p)}{B(\mu,\ \nu-\mu)} \frac{z^n}{(n+a)^s} \tag{1.6}$$

$\big(p \geqq 0;\ \lambda,\mu \in \mathbb{C};\ \nu,\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s+\nu-\lambda-\mu)>1 \text{ when } |z|=1\big)$. and gave its integral representation:

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-at}\ F_p(\lambda,\mu;\nu;ze^{-t})dt \tag{1.7}$$

$$\big(\Re(p) \geqq 0;\ p=0,\ \Re(a)>0;\ \Re(s)>0 \text{ when } |z| \leqq 1\ (z\neq 1);\ \Re(s)>1 \text{ when } z=1\big),$$

where $B(x,\, y\,;\, p)$ and $F_p\,(a,\, b;\, c;\, z)$ are extended Beta function [1, p. 20, Eq. (1.7)] and extended hypergeometric function [2, p. 591, Eq. (2.2)]:

$$B(x,\, y\,;\, p) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t(1-t)}\right)\,dt \quad (\Re(p)>0) \tag{1.8}$$

and

$$F_p\,(a,\, b;\, c;\, z) = \sum_{n=0}^{\infty}(a)_n\,\frac{B(b+n,\, c-b\,;\, p)}{B(b,\, c-b)}\,\frac{z^n}{n!} \tag{1.9}$$

$$(p \geqq 0,\ |z|<1\,;\, \Re(c)>\Re(b)>0)\,.$$

Clearly, the special case of (1.6) when $p=0$ reduces immediately to (1.3).

Further, Özergin *et al.* [13] introduced the following generalizations of the extended Beta and hypergeometric functions which are defined, respectively, by

$$B_p^{(\rho,\sigma)}(\alpha,\, \beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\ {}_1F_1\left(\rho;\sigma;-\frac{p}{t(1-t)}\right)\,dt \tag{1.10}$$

$$\big(\min\{\Re(\alpha),\Re(\beta),\Re(\rho),\Re(\sigma)\}>0;\ \Re(p)\geqq 0\big)$$

and

$$F_p^{(\rho,\sigma)}\,(a,\, b;\, c;\, z) := \sum_{n=0}^{\infty}(a)_n\,\frac{B_p^{(\rho,\sigma)}(b+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!} \tag{1.11}$$

$$\big(|z|<1;\ \min\{\Re(\rho),\Re(\sigma)\}>0;\ \Re(c)>\Re(b)>0;\ \Re(p)\geqq 0\big).$$

Evidently, the special cases of (1.10) and (1.11) when $\rho=\sigma$ reduce, respectively, to the extended Beta and hypergeometric functions (1.8) and (1.9).

Motivated mainly by various recent interesting extensions of the Hurwitz-Lerch Zeta function, we introduce a further generalization of the extended Hurwitz-Lerch Zeta functions and investigate its certain interesting and (potentially) useful properties such as various integral representations, Mellin transform, generating functions, derivative formulas and relations associated with extended fractional derivative operator. An application to probability distributions is further considered. Some interesting special cases of our main results are also indicated.

2. Extension of Hurwitz-Lerch Zeta Functions

We consider the following generalization of the extended Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p) := \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,\frac{B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\,\frac{z^n}{(n+a)^s} \tag{2.1}$$

$$\big(p\geqq 0,\ \Re(\rho)>0,\,\Re(\sigma)>0;\ \lambda,\mu\in\mathbb{C};\ \nu,\,a\in\mathbb{C}\setminus\mathbb{Z}_0^-;$$
$$s\in\mathbb{C}\ \text{when}\ |z|<1;\ \Re(s+\nu-\lambda-\mu)>1\ \text{when}\ |z|=1\big).$$

Clearly, the special cases of (2.1) when $\rho=\sigma$ and $p=0$ reduce immediately to (1.6) and (1.3), respectively.

Remark 2.1. The generalized Hurwitz-Lerch function $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p)$ in (2.1) is seen to satisfy the following limit case:

$$\begin{aligned}\Phi^{*\,(\rho,\sigma)}_{\mu;\nu}(z,s,a;p) &:= \lim_{|\lambda|\to\infty}\left\{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}\left(\frac{z}{\lambda},s,a;p\right)\right\}\\ &= \sum_{n=0}^{\infty}\frac{B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\frac{z^n}{n!(n+a)^s}\end{aligned} \tag{2.2}$$

$$\big(p \geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ \mu\in\mathbb{C};\ \nu,\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\\ s\in\mathbb{C}\text{ when }|z|<1;\ \Re(s+\nu-\mu)>1\text{ when }|z|=1\big).$$

Remark 2.2. If we set $\lambda=\nu=1$ in (2.1), we obtain another known extension of the Hurwitz-Lerch Zeta function $\Phi^*_\mu(z,s,a;p)$ given by Goyal and Laddha [7]:

$$\Phi^{*\,(\rho,\sigma)}_{\mu}(z,s,a;p) := \Phi^{(\rho,\sigma)}_{1,\mu;1}(z,s,a;p) = \sum_{n=0}^{\infty}\frac{B^{(\rho,\sigma)}(\mu+n,\,1-\mu;\,p)}{B(\mu,\,1-\mu)}\frac{z^n}{(n+a)^s} \tag{2.3}$$

$$\big(p \geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ \mu\in\mathbb{C};\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\\ s\in\mathbb{C}\text{ when }|z|<1;\ \Re(s+1-\mu)>1\text{ when }|z|=1\big).$$

3. Integral Representations

We present certain integral representations for the extended Hurwitz-Lerch Zeta function in (2.1).

Theorem 3.1. *The following integral representation holds true:*

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,F_p^{(\rho,\sigma)}(\lambda,\mu;\nu;ze^{-t})dt \tag{3.1}$$

$$\big(\Re(p)\geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ p=0,\ \Re(a)>0;\\ \Re(s)>0\ \ when\ \ |z|\leqq 1\,(z\neq 1);\ \Re(s)>1\ \ when\ \ z=1\big).$$

Proof: Using the Eulerian integral of the Gamma function $\Gamma(s)$ (see, *e.g.*, [17, p. 1, Eq. (1)]), it is easy to find the following identity:

$$\frac{1}{(n+a)^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(n+a)t}dt \quad \left(\min\{\Re(s),\,\Re(a)\}>0;\ n\in\mathbb{N}_0\right). \tag{3.2}$$

Applying (3.2) to (2.1) and interchanging the order of summation and integration which may be valid under the conditions stated in Theorem 3.1, we get

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu,}(z,s,a;p) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\left(\sum_{n=0}^{\infty}(\lambda)_n\frac{B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\frac{(ze^{-t})^n}{n!}\right)dt.$$

Finally the use of (1.11) to the last expression is seen to lead to the desired result. □

Theorem 3.2. *The following integral representations hold true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p) = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \times \int_0^\infty \frac{x^{\mu-1}}{(1+x)^\nu} \, {}_1F_1\left(\rho;\sigma;-2p-p(x+\frac{1}{x})\right) \Phi_\lambda^*\left(\frac{zx}{1+x},s,a\right) dx \tag{3.3}$$

$$\big(\Re(p) \geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ p=0,\ \Re(\nu)>\Re(\mu)>0\big)$$

and

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p) = \frac{\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{t^{s-1}e^{-at}x^{\mu-1}}{(1+x)^\nu} \times {}_1F_1\left(\rho;\sigma;-2p-p(x+\frac{1}{x})\right) \left(1-\frac{zxe^{-t}}{1+x}\right)^{-\lambda} dt\,dx \tag{3.4}$$

$$\big(\Re(p) \geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ p=0,\ \Re(\nu)>\Re(\mu)>0,\ \min\{\Re(s),\ \Re(a)\}>0\big),$$

provided the integrals in the right-hand sides of (3.3) *and* (3.4) *converge.*

Proof: By setting $\alpha=\mu+n$ and $\beta=\nu-\mu$ in the following integral representation of the extended Beta-function (see, *e.g.*, [13, p. 4603, Theorem (2.4)]):

$$B_p^{(\rho,\sigma)}(\alpha,\,\beta) = \int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \, {}_1F_1\left(\rho;\sigma;-2p-p(x+\frac{1}{x})\right) dx \tag{3.5}$$

$$(\Re(p)>0; p=0,\ \min\{\Re(\alpha),\Re(\beta),\Re(\rho),\Re(\sigma)\}>0),$$

we find

$$B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu) = \int_0^\infty \frac{x^{\mu+n-1}}{(1+x)^{\nu+n}} \, {}_1F_1\left(\rho;\sigma;-2p-p(x+\frac{1}{x})\right) dx, \tag{3.6}$$

which, by appealing to the (2.1) and using (1.2), immediately yields the first assertion (3.3) of Theorem 3.2.

Also, using (1.4) in (3.3) is seen to lead to the second assertion (3.4) of Theorem 3.2. □

Theorem 3.3. *The following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p) := \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}e^{-t}\Phi_{\mu;\nu}^{*\,(\rho,\sigma)}(zt,s,a;p)dt \tag{3.7}$$

$$\big(\Re(p) \geqq 0,\ \Re(\rho)>0,\ \Re(\sigma)>0;\ p=0,\ \Re(\lambda)>0,\ \Re(a)>0;$$
$$\Re(s)>0 \ \ when\ \ |z| \leqq 1\,(z\neq 1);\ \Re(s)>1 \ \ when\ \ z=1\big),$$

where $\Phi_{\mu;\nu}^{*\,(\rho,\sigma)}(z,s,a;p)$ *is the limiting case in* (2.2).

Proof: Using the integral representation of the Pochhammer symbol $(\lambda)_n$:

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-t}\, dt$$

in (2.1) and inverting the order of summation and integration which may be permissible under the conditions stated Theorem 3.3, we get

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \sum_{n=0}^{\infty} \frac{B^{(\rho,\sigma)}_p(\mu+n,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{(zt)^n}{n!(n+a)^s}\, dt. \quad (3.8)$$

Applying (2.2) is seen to yield the desired integral representation. □

Remark 3.4. If we set $\rho = \sigma$ in (3.1),(3.3),(3.4) and (3.7), we obtain corresponding results earlier obtained by Parmar and Raina [14]. Further for $p = 0$, we obtain corresponding results in [6].

4. Mellin Transform and Generating Relations

The Mellin transform of a suitable integrable function $f(t)$ with index α is defined, as usual, by

$$\mathcal{M}\{f(\tau) : \tau \to \alpha\} := \int_0^\infty \tau^{\alpha-1}\, f(\tau)\, d\tau, \quad (4.1)$$

provided the improper integral in (4.1) exists.

Theorem 4.1. *The Mellin transform of the function* $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p)$ *in* (2.1) *is given as follows:*

$$\begin{aligned}\mathcal{M}&\left\{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p) : p \to \alpha\right\}\\ &= \frac{\Gamma(\sigma)\Gamma(\rho-\alpha)\Gamma^{(\rho,\sigma)}(\alpha)}{\Gamma(\rho)\Gamma(\sigma-\alpha)} \frac{B(\mu+\alpha,\, \nu-\mu+\alpha)}{B(\mu,\, \nu-\mu)}\, \Phi_{\lambda,\mu+\alpha,\nu+2\alpha}(z,s,a)\end{aligned} \quad (4.2)$$

$$(\Re(\alpha) > 0,\, \Re(\rho) > 0,\, \Re(\rho-\alpha) > 0,\, \Re(\sigma-\alpha) > 0 \quad \textit{and} \quad \Re(\nu-\mu+\alpha) > 0),$$

where $\Gamma^{(\rho,\sigma)}(\alpha)$ *is a special case* $p = 0$ *of the extended Gamma function* (*see* [13, p. 4602, Eq. (3)]):

$$\Gamma^{(\rho,\sigma)}_p(z) := \int_0^\infty t^{z-1}\, {}_1F_1\left(\rho; \sigma; -t - \frac{p}{t}\right) dt \quad (4.3)$$

$$\big(\min\{\Re(z), \Re(\rho), \Re(\sigma)\} > 0;\ \Re(p) \geqq 0\big).$$

Proof: Taking the Mellin transform (4.1) for (2.1), we find

$$\mathcal{M}\left\{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p):p\to\alpha\right\}:=\int_0^\infty p^{\alpha-1}\Phi_{\lambda,\mu;\nu}(z,s,a;p)dp$$
$$=\int_0^\infty p^{\alpha-1}\left(\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\frac{z^n}{(n+a)^s}\right)dp$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s}\frac{1}{B(\mu,\,\nu-\mu)}\int_0^\infty p^{\alpha-1}B_p^{(\rho,\sigma)}(\mu+n,\,\nu-\mu)\,dp.$$

Applying the following known result (see [1, p. 21, Eq.(2.1)]):

$$\int_0^\infty p^{\alpha-1}B_p^{(\rho,\sigma)}(x,\,y)\,dp=\Gamma^{(\rho,\sigma)}(\alpha)B(x+\alpha,\,y+\alpha)\qquad(\Re(\alpha)>0),\tag{4.4}$$

we obtain

$$\mathcal{M}\left\{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p):p\to\alpha\right\}$$
$$=\Gamma^{(\rho,\sigma)}(\alpha)\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s}\frac{B(\mu+\alpha+n,\,\nu-\mu+\alpha)}{B(\mu,\,\nu-\mu)}$$
$$=\frac{\Gamma^{(\rho,\sigma)}(\alpha)\,B(\mu+\alpha,\,\nu-\mu+\alpha)}{B(\mu,\,\nu-\mu)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu+\alpha)_n}{(\nu+2\alpha)_n}\frac{z^n}{n!(n+a)^s},\tag{4.5}$$

Using the known result [13, p. 4603, Eq.(5)]:

$$\Gamma^{(\rho,\sigma)}(\alpha)=\frac{\Gamma(\sigma)\Gamma(\rho-\alpha)\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\sigma-\alpha)}$$

considering (2.1) gives the desired Mellin transform. □

Remark 4.2. The generalized Hurwitz-Lerch Zeta function can be expressed in terms of $\overline{H}$-function (see [8,9]) as follows (see [6, p. 316, Eq.(3.2)]; see also [20, p. 499, Eq.(3.7)]):

$$\Phi_{\lambda,\mu;\nu}(z,s,a)=\frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\mu-\nu)}\overline{H}^{1,3}_{3,3}\left[-z\;\middle|\;\begin{matrix}(1-\lambda,1;1),(1-\mu,1;1),(1-a,1;s)\\(0,1),(1-\nu,1;1),(-a,1;s)\end{matrix}\right].\tag{4.6}$$

Applying the relationship (4.6) to (4.2), we can deduce Mellin representations in terms of $\overline{H}$-function asserted by Corollary 4.3 below, whose proof is omitted.

Corollary 4.3. *The following Mellin representation holds true:*

$$\mathcal{M}\left\{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p):p\to\alpha\right\}=\frac{\Gamma(\sigma)\Gamma(\rho-\alpha)\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\sigma-\alpha)}\,\frac{\Gamma(\nu-\mu+\alpha)}{\Gamma(\lambda)B(\mu,\,\nu-\mu)}$$
$$\times\,\overline{H}^{1,3}_{3,3}\left[-z\;\middle|\;\begin{matrix}(1-\lambda,1;1),(1-\mu-\alpha,1;1),(1-a,1;s)\\(0,1),(1-\nu-2\alpha,1;1),(-a,1;s)\end{matrix}\right]. \tag{4.7}$$

Theorem 4.4. *The following generating function for $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p)$ in (1.6) holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n\;\Phi^{(\rho,\sigma)}_{\lambda+n,\mu;\nu}(z,s,a;p)\,\frac{t^n}{n!}=(1-t)^{-\lambda}\;\Phi^{(\rho,\sigma)}_{\lambda+n,\mu;\nu}\left(\frac{z}{1-t},s,a;p\right) \tag{4.8}$$

$$(p\geqq 0,\ \lambda\in\mathbb{C}\ \ and\ \ |t|<1).$$

Proof: Let the left-hand side of the assertion (4.8) be denoted by S. Then we find from (2.1) that

$$S=\sum_{n=0}^{\infty}(\lambda)_n\left\{\sum_{k=0}^{\infty}(\lambda+n)_k\,\frac{B^{(\rho,\sigma)}_p(\mu+k,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\,\frac{z^k}{k!(k+a)^s}\right\}\frac{t^n}{n!}$$
$$=\sum_{k=0}^{\infty}(\lambda)_k\,\frac{B^{(\rho,\sigma)}_p(\mu+k,\,\nu-\mu)}{B(\mu,\,\nu-\mu)}\left\{\sum_{n=0}^{\infty}(\lambda+k)_n\,\frac{t^n}{n!}\right\}\frac{z^k}{k!(k+a)^s},$$

where the second equality follows by reversing the order of summations and using the identity $(\lambda)_n(\lambda+n)_k=(\lambda)_k(\lambda+k)_n$.

Now, applying the binomial expansion

$$(1-t)^{-\lambda-k}=\sum_{n=0}^{\infty}(\lambda+k)_n\,\frac{t^n}{n!}\qquad(|t|<1)$$

and considering (2.1) as a function of the form $\Phi^{(\rho,\sigma)}_{\lambda+n,\mu;\nu}\left(\frac{z}{1-t},s,a;p\right)$, we are led to the assertion (4.8). □

Theorem 4.5. *Each of the following generating functions for $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p)$ in (2.1) holds true:*

$$\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\;\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s+n,a;p)\,t^n=\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a-t;p) \tag{4.9}$$

$$(p\geqq 0,\ \lambda\in\mathbb{C}\ \ and\ \ |t|<|a|;\ \ s\neq 1).$$

More generally

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\delta)_n}{n!} \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z, s+n, a; p)\, t^n \\ = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{(k+a)^{s-\delta}(k+a-t)^{\delta}}\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z, s+n, a; p)\, \frac{t^n}{n!} \\ = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{(k+a)^{s}} \exp\left(\frac{t}{k+a}\right).\end{aligned} \tag{4.11}$$

Proof: Using (2.1) in the right-hand side of the assertion (4.9), we have

$$\begin{aligned}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z, s, a-t; p) &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{k!(k+a-t)^{s}} \\ &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{k!(k+a)^{s}} \left(1-\frac{t}{k+a}\right)^{-s} \\ &= \sum_{k=0}^{\infty} (\lambda)_k \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{k!(k+a)^{s}} \left\{\sum_{n=0}^{\infty} \frac{(s)_n}{n!} \frac{t^n}{(k+a)^n}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left\{\sum_{k=0}^{\infty} (\lambda)_k \frac{B_p^{(\rho,\sigma)}(\mu+k,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \frac{z^k}{k!(k+a)^{s+n}}\right\} t^n.\end{aligned}$$

Then, by making use of (2.1), we are led to the desired assertion (4.9).

Similarly, it is not difficult to show the assertion in (4.10).

The generating function (4.10) would reduce immediately to the expansion formula (4.9) in its special case when $\delta = s$.

Next, by virtue of the following limit formula:

$$\lim_{|\lambda|\to\infty} \left\{(\lambda)_n \left(\frac{z}{\lambda}\right)^n\right\} = z^n \quad (n \in \mathbb{N}_0)$$

when t is replaced by $\frac{t}{\lambda}$ and $|\lambda| \to \infty$ in (4.10), we get the desired exponential generating function asserted by (4.11). □

5. Extended Fractional Derivative Operator

For the Riemann–Liouville fractional derivative operator $\mathcal{D}_z^\mu$ defined by (see, *e.g.*, [15] and [10, p. 70 *et seq.*]):

$$\mathcal{D}_z^\mu \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \; f(t)\,dt & \big(\Re(\mu) < 0\big) \\ \dfrac{d^m}{dz^m}\Big\{\mathcal{D}_z^{\mu-m}\{f(z)\}\Big\} & \big(m-1 \leqq \Re(\mu) < m \;\; (m \in \mathbb{N})\big), \end{cases} \tag{5.1}$$

it is known that

$$\mathcal{D}_z^\mu \left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} \, z^{\lambda-\mu} \qquad \big(\Re(\lambda) > -1\big). \tag{5.2}$$

It is noted that the path of integration in the definition (5.1) is a line in the complex t-plane from $t = 0$ to $t = z$.

Further, Srivastava *et al.* [19, p. 243, Eq. (2.1)] introduced an *extended* Riemann–Liouville fractional derivative operator as follows:

$$\mathcal{D}_{z,(\rho,\sigma)}^{\mu,p} \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \; {}_1F_1\left(\rho;\sigma;-\frac{pz^2}{(z-t)t}\right) f(t)\,dt & \big(\Re(\mu) < 0\big) \\ \dfrac{d^m}{dz^m}\Big\{\mathcal{D}_{z,(\rho,\sigma)}^{\mu-m,p}\{f(z)\}\Big\} & \big(m-1 \leqq \Re(\mu) < m \;\; (m \in \mathbb{N})\big), \end{cases} \tag{5.3}$$

where, as before, $\Re(p) \geqq 0$. The path of integration in the Definition (5.3), which immediately yields the definition (5.1) when $p = 0$, is also a line in the complex t-plane from $t = 0$ to $t = z$.

Also when $\rho = \sigma$, (5.3) reduces to the extended Riemann–Liouville fractional derivative operator $D_z^{\mu,p}$ introduced by Özarslan and Özergin [12]:

$$\mathcal{D}_z^{\mu,p} \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \; \exp\left(-\frac{pz^2}{(z-t)t}\right) f(t)\,dt & \big(\Re(\mu) < 0\big) \\ \dfrac{d^m}{dz^m}\Big\{\mathcal{D}_z^{\mu-m}\{f(z)\}\Big\} & \big(m-1 \leqq \Re(\mu) < m \;\; (m \in \mathbb{N})\big). \end{cases} \tag{5.4}$$

Making use of the Definition (5.3), we can easily derive the following analogue of the familiar fractional derivative formula (5.2)

$$\mathcal{D}_{z,(\rho,\sigma)}^{\mu,p} \left\{z^\lambda\right\} = \frac{B_p^{(\rho,\sigma)}(\lambda+1,-\mu;p)}{\Gamma(-\mu)} \, z^{\lambda-\mu} \qquad \big(\Re(\lambda) > -1;\; \Re(\mu) < 0\big), \tag{5.5}$$

which would readily yield an extension of the fractional derivative formula in Theorem 7 and Theorem 8 below.

Theorem 5.1. *The following fractional derivative formulas holds true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\, z^{1-\nu}\, \mathcal{D}_{z,(\rho,\sigma)}^{\mu-\nu,p}\{z^{\mu-1}\,\Phi_{\lambda}^{*}(z,s,a)\}$$
$$= \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\, z^{1-\nu}\, \mathcal{D}_{z,(\rho,\sigma)}^{\mu-\nu,p}\{z^{\mu-1}\,\mathcal{D}_{z}^{\lambda-1}\{z^{\lambda-1}\,\Phi(z,s,a)\}\} \quad (5.6)$$

$$(p \geq 0, \Re(\nu) > \Re(\mu) > 0),$$

$$\Phi_{\mu;\nu}^{*\,(\rho,\sigma)}(z,s,a;p) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\, z^{1-\nu}\, \mathcal{D}_{z,(\rho,\sigma)}^{\mu-\nu,p}\{z^{\mu-1}\,\Phi(z,s,a)\} \quad (5.7)$$

$$(p \geq 0, \Re(\nu) > \Re(\mu) > 0)$$

and

$$\Phi_{\mu}^{*\,(\rho,\sigma)}(z,s,a;p) = \frac{1}{\Gamma(\mu)}\, \mathcal{D}_{z,(\rho,\sigma)}^{\mu-1,p}\{z^{\mu-1}\,\Phi(z,s,a)\} \quad (5.8)$$

$$(p \geq 0,\ \Re(\mu) > 0).$$

Proof: Using extended fractional-calculus result (5.5), we can easily deduce extended fractional derivative formula for (2.1), (2.2) and (2.3) which are given in (5.6), (5.7) and (5.8), respectively. □

Theorem 5.2. *The following fractional derivative formula holds true:*

$$\mathcal{D}_{z,(\rho,\sigma)}^{\mu-\tau,p}\left\{z^{\mu-1}\Phi_{\lambda,\mu;\nu}(z,s,a)\right\} = \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{\tau-1}\Phi_{\lambda,\mu;\tau}^{(\rho,\sigma)}(z,s,a;p) \qquad (\Re(\nu) > 0). \quad (5.9)$$

Proof: It is easy to use (1.3) and (5.5) to derive (5.9). □

6. Application to the Probability Distributions

Here we consider a general probability distribution involving the extended generalized Hurwitz-Lerch Zeta function (2.1) defined as follows:

A continuous random variable ξ is said to be generalized Hurwitz distributed if its probability density function is given by

$$f_{\xi}(a) =: \begin{cases} \dfrac{s\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s+1,a;p)}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)} & (a \geqq 1) \\ 0 & \text{otherwise,} \end{cases} \quad (6.1)$$

where it is *tacitly* assumed that the arguments z, s, p and the parameters λ, μ and ν are fixed and suitably constrained so that the probability density function $f_{\xi}(a)$ remains nonnegative.

Theorem 6.1. *The moment generating function $M(t)$ of the continuous random variable ξ of probability density function $f_\xi(a)$ in (6.1) is given as follows:*

$$M(t) = E_s[e^{\xi t}] = \sum_{n=0}^{\infty} E_s[\xi^n]\frac{t^n}{n!}, \tag{6.2}$$

where the moments $E_s[\xi^n]$ of order n are given by

$$E_s[\xi^n] = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{\Gamma(s-k)}{\Gamma(s)}\frac{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s-k,1;p)}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}. \tag{6.3}$$

Proof: The assertion in (6.2) can be derived easily by using the series expansion of $e^{\xi t}$. To establish (6.3), we observe that

$$\frac{d}{da}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)\right\} = -s\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s+1,a;p), \tag{6.4}$$

which follows readily from (2.1), and thus from the definition of $E_s[\xi^n]$, we have

$$\begin{aligned}
E_s[\xi^n] &= \int_1^\infty a^n f_\xi(a)da \\
&= \frac{s}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\int_1^\infty a^n \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s+1,a;p)da \\
&= -\frac{1}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\int_1^\infty a^n \frac{d}{da}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)\right\}da \\
&= \left[-\frac{a^n\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\right]_{a=1}^{\infty} + \frac{n}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\int_1^\infty a^{n-1}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)da \\
&= 1 - \lim_{a\to\infty}\left\{\frac{a^n\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\right\} \\
&\quad + \frac{n}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\int_1^\infty a^{n-1}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)da \\
&= 1 + \frac{n}{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,1;p)}\int_1^\infty a^{n-1}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p)\,da,
\end{aligned} \tag{6.5}$$

where, in addition to the derivative property (6.4), we have used the following limit formula:

$$\lim_{a\to\infty}\left\{a^n\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p)\right\} = \lim_{a\to\infty}\left\{\frac{a^n}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,F_p^{(\rho,\sigma)}(\lambda,\mu;\nu;ze^{-t})dt\right\}$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\lim_{a\to\infty}\left\{a^n e^{-at}\right\}\,F_p^{(\rho,\sigma)}(\lambda,\mu;\nu;ze^{-t})dt$$
$$= 0. \tag{6.6}$$

Consequently, we have the following reduction formula for $E_s[\xi^n]$:

$$E_s[\xi^n] = 1 + \frac{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s-1,1;p)}{\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,1;p)}\frac{n}{s-1}E_{s-1}[\xi^{n-1}], \tag{6.7}$$

and by iterating the recurrence (6.5), we arrive at the desired result (6.3). □

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