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Some Generating Functions and Properties of Extended Second Appell Function

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abstract: Various families of generating functions have been established by a number of authors in many different ways. In this paper, we aim at establishing (presumably new) a generating function for the extended second Appell hypergeometric function $F_2(a, b, b'; c, c'; x, y; p)$. Further we derive a relation in terms of the Laguerre polynomials and differentiation formulas. We also present special cases of the main results of this paper.

Key Words: Extended Beta function, extended hypergeometric function, generating functions, second Appell function, extended second Appell function, Laguerre polynomials.

Contents

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}\,.$$

Extensions of a number of well-known special functions were investigated recently by several authors (see [7]-[9], [24]). In particular, Chaudhry *et al.* [7, p. 20, Eqn. (1.7)] presented the following extension of the Beta function as:

$$B(x,\, y\,;\, p) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t(1-t)}\right)\,dt, \tag{1.1}$$

$$(\Re(p) > 0;\ \text{For}\, p = 0,\ \Re(x) > 0,\ \Re(y) > 0).$$

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and obtained certain connections with Macdonald, error and Whittaker functions.

In 2004, Chaudhry *et al.* [8] used $B(x,\, y\,;\, p)$ to extend the hypergeometric and the confluent hypergeometric functions as follows:

$$F_p\,(a,\, b;\, c;\, z) := \sum_{n=0}^{\infty} (a)_n\, \frac{B(b+n,\, c-b\,;\, p)}{B(b,\, c-b)}\, \frac{z^n}{n!} \tag{1.2}$$

$$\big(p \geqq 0,\;\, |z| < 1\,;\, \Re(c) > \Re(b) > 0\big),$$

and

$$\Phi_p\,(b;\, c;\, z) := \sum_{n=0}^{\infty} \frac{B(b+n,\, c-b\,;\, p)}{B(b,\, c-b)}\, \frac{z^n}{n!}, \tag{1.3}$$

$$\big(p \geqq 0\,;\, \Re(c) > \Re(b) > 0\big).$$

They investigated above functions and gave their various integral representations, beta distribution, certain properties including differentiation formulas, Mellin transform, transformation formulas, recurrence relations, summation formula, asymptotic formulas and certain interesting connections with some well known special functions.

More recently, Özarslan and Özergin [16] defined the extended second Appell function as follows:

$$F_2(a, b, b'; c, c'; x, y; p) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\; B(b+m, c-b; p) B(b'+n, c'-b'; p)}{B(b, c-b) B(b', c'-b')}\, \frac{x^m}{m!}\, \frac{y^n}{n!} \tag{1.4}$$

$$\big(|x| + |y| < 1;\; \Re(p) \geqq 0\big),$$

or, equivalently, by means of an integral representation [16, p. 1826, Th.(2.2)] as follows:

$$F_2(a, b, b'; c, c'; x, y; p) = \frac{1}{B(b, c-b) B(b', c'-b')} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}\; s^{b'-1}(1-s)^{c'-b'-1}}{(1-xt-ys)^a} \times \exp\left(-\frac{p}{t(1-t)}\right) \exp\left(-\frac{p}{s(1-s)}\right) dt\, ds, \tag{1.5}$$

$$\big(\Re(p) > 0;\; p = 0 \quad \textit{and} \quad |x| + |y| < 1;\; \Re(c) > \Re(b) > 0;\; \Re(c') > \Re(b') > 0\big).$$

Clearly, the special cases of (1.1), (1.2), (1.3) and (1.4) when $p = 0$ reduce immediately to classical Euler Beta, Gauss's hypergeometric, confluent hypergeometric and second Appell functions, respectively.

Various types of special functions have become important tools for the scientists and engineers, in many areas of pure as well as applied mathematics. Integral transformations, generating and reduction (or summation) formulas, and fractional calculus images involving these functions of one and more variables have a wide range of applications to various fields of mathematical, physical and engineering sciences (see [1,6,10,12,15,17]). Most importantly, these functions provides solutions to certain problems formulated in terms of integral and differential equations (including fractional order differential equations), therefore, it has recently become a subject of interest for many authors in the field of fractional calculus and its applications. For detailed applications on the subject, one may refer [2]- [5], [11], [13], [18]- [20], [21] and the references cited therein.

In this paper, we aim at establishing (presumably new) a generating function for the extended second Appell hypergeometric function $F_2(a, b, b'; c, c'; x, y; p)$ defined by (1.4). Further we derive a relation in terms of the Laguerre polynomials and differentiation formulas. We also present special cases of the main result of this paper.

2. Generating Functions of Extended Appell Function

In this section, we obtain certain generating functions for the extended second Appell hypergeometric function defined in (1.4).

Theorem 2.1. *The following generating function for* $F_2(a, b, b'; c, c'; x, y; p)$ *in* (1.4) *holds true:*

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_2(\lambda+k, b, b'; c, c'; x, y; p)\, t^k = (1-t)^{-\lambda}\, F_2\left(\lambda, b, b'; c, c'; \frac{x}{1-t}, \frac{y}{1-t}; p\right), \tag{2.1}$$

$$(\Re(p) \geqq 0,\ \lambda \in \mathbb{C} \text{ and } |t| < 1).$$

Proof: Suppose the left hand side the assertion (2.1) of Theorem 2.1 be denoted by S, then on using definition (1.4) in S, we obtain

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, F_2(a, b, b'; c, c'; x, y; p)\, t^k \\ &= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\sum_{m,n=0}^{\infty} (\lambda+k)_{m+n} \frac{B(b+m, c-b; p)B(b'+n, c'-b'; p)}{B(b, c-b)B(b', c'-b')} \frac{x^m}{m!} \frac{y^n}{n!} \right) t^k \\ &= \sum_{m,n=0}^{\infty} (\lambda)_{m+n} \frac{B(b+m, c-b; p)B(b'+n, c'-b'; p)}{B(b, c-b)B(b', c'-b')} \frac{x^m}{m!} \frac{y^n}{n!} \\ &\quad \times \left(\sum_{k=0}^{\infty} (\lambda+m+n)_k \frac{t^k}{k!} \right) \\ &= \sum_{m,n=0}^{\infty} (\lambda)_{m+n} \frac{B(b+m, c-b; p)B(b'+n, c'-b'; p)}{B(b, c-b)B(b', c'-b')} \frac{x^m}{m!} \frac{y^n}{n!} (1-t)^{-\lambda-m-n}, \end{aligned} \tag{2.2}$$

where we have reversal the order of summation and using the identity

$$(\lambda)_k(\lambda+k)_{m+n} = (\lambda)_{m+n}(\lambda+m+n)_k,$$

and the binomial theorem

$$(1-t)^{-\lambda-m-n} = \sum_{k=0}^{\infty} \frac{(\lambda+m+n)_k}{k!} t^k \qquad (|t|<1),$$

with identification of the series over k from (1.4) as $F_2\left(a,b,b';c,c';\frac{x}{1-t},\frac{y}{1-t};p\right)$ leads to the assertion in Theorem 2.1. □

Theorem 2.2. *Each of the following generating function for* $F_2(a,b,b';c,c';x,y;p)$ *in* (1.4) *holds true:*

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} F_2(-k,b,b';c,c';x,y;p)\, t^k = (1-t)^{-\lambda}\, F_2\left(\lambda,b,b';c,c';\frac{xt}{t-1},\frac{yt}{t-1};p\right), \tag{2.3}$$

$$(\Re(p) \geqq 0,\ \lambda\in\mathbb{C} \text{ and } |t|<1)$$

and

$$\sum_{k=0}^{\infty} F_2(-k,b,b';c,c';x,y;p)\,\frac{t^k}{k!} = e^t\ \Phi_p(b;c;-xt)\ \Phi_p(b';c';-yt), \tag{2.4}$$

$$(\Re(p) \geqq 0,\ \lambda\in\mathbb{C} \text{ and } |t|<1).$$

Proof: The proof of (2.3) in Theorem 2.2 is similar to that of Theorem 2.1. Next, by virtue of the following limit formula:

$$\lim_{|\lambda|\to\infty} \left\{(\lambda)_n \left(\frac{z}{\lambda}\right)^n\right\} = z^n \quad (n\in\mathbb{N}_0),$$

when t is replaced by $\frac{t}{\lambda}$ and $|\lambda| \to \infty$ in (2.3) and using (1.3), we get the desired exponential generating function asserted by (2.4). □

Remark 2.3. *The special cases of* (2.1) *when* $p=0$ *is easily seen to reduce to the known generating function of the second Appell hypergeometric function as:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_k}{k!} F_2(\lambda+k,b,b';c,c';x,y)\, t^k = (1-t)^{-\lambda}\, F_2\left(\lambda+k,b,b';c,c';\frac{x}{1-t},\frac{y}{1-t}\right), \tag{2.5}$$

$$(\lambda\in\mathbb{C} \text{ and } |t|<1).$$

3. Representation via Laguerre Polynomials and Derivative Formulas

Next, in terms of the simple Laguerre polynomials $L_n(x)$ given by (see, *e.g.*, [9, p. 238, Eqn. (5.152)])

$$L_n(x) := L_n^{(0)}(x) \qquad \text{and} \qquad L_n^{(\alpha)}(x) := \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!},$$

we derive the representations asserted by Theorem 3.1 below.

Theorem 3.1. *The following Laguerre polynomial representation for the extended second Appell function in* (1.4) *holds true:*

$$\begin{aligned} F_2(a,b,b';c,c';x,y;p) &= \frac{e^{-4p}}{B(b,c-b)B(b',c'-b')} \\ &\times \sum_{j,k,m,n=0}^{\infty} B(b+m+1,c-b+n+1)B(b'+j+1,c'-b'+k+1) \\ &\qquad \times F_2(a,b+m+1,b'+j+1;c+m+n+2,c'+j+k+2;x,y) \\ &\qquad\qquad \times L_m(p)\,L_n(p)\,L_j(p)\,L_k(p), \end{aligned} \tag{3.1}$$

where $\Re(p) > 0$.

Proof: We start by recalling the following known identity due to Miller [14, p. 30, Eqn. (3.5)]:

$$\exp\left(-\frac{p}{t(1-t)}\right) = e^{-2p}\left\{\sum_{m,\,n=0}^{\infty} L_m(p)\,L_n(p)\,t^{m+1}\,(1-t)^{n+1}\right\} \tag{3.2}$$

in (1.5), we have

$$\begin{aligned} F_2(a,b,b';c,c';x,y;p) &= \frac{e^{-4p}}{B(b,c-b)B(b',c'-b')} \int_0^1 \frac{t^{b-1}\,s^{b'-1}}{(1-xt-ys)^a} \\ &\times (1-t)^{c-b-1}\,(1-s)^{c'-b'-1}\left\{\sum_{m,\,n=0}^{\infty} L_m(p)\,L_n(p)\,t^{m+1}\,(1-t)^{n+1}\right\} \\ &\times \left\{\sum_{j,\,k=0}^{\infty} L_j(p)\,L_k(p)\,s^{j+1}\,(1-s)^{k+1}\right\} dt\,ds. \end{aligned} \tag{3.3}$$

Now, changing summation and integral and using the latter together with definition (1.5), in the representation (3.3), we are led to the desired result. □

Theorem 3.2. *Let the extended second Appell function* $F_2(a,b,b';c,c';x,y;p)$ *defined by* (1.4), *then the following differentiation formulas holds true:*

$$\frac{d^m}{dx^m} F_2(a,b,b';c,c';x,y;p) = \frac{(a)_m\,(b)_m}{(c)_m} F_2(a+m,b+m,b';c+m,c';x,y;p), \tag{3.4}$$

$$\frac{d^n}{dy^n} F_2(a,b,b';c,c';x,y;p) = \frac{(a)_n\,(b')_n}{(c')_n} F_2(a+n,b,b'+n;c,c'+n;x,y;p) \quad (3.5)$$

and

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} F_2(a,b,b';c,c';x,y;p) = \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m (c')_n} \times F_2(a+m+n,b+m,b'+n;c+m,c'+n;x,y;p) \quad (3.6)$$

$$(m,\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Proof: On differentiating both sides of (1.4) with respect to x, we have

$$\frac{d}{dx} F_2(a,b,b';c,c';x,y;p) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(a)_{m+n} \frac{B(b+m,c-b;p)B(b'+n,c'-b';p)}{B(b,c-b)B(b',c'-b')} \times \frac{x^{m-1}}{(m-1)!}\,\frac{y^n}{n!}$$

which, upon replacing m by $m+1$ along with the following formulas

$$B(b,\,c-b) = \frac{c}{b}\,B(b+1,\,c-b) \quad \text{and} \quad (\lambda)_{m+1} = \lambda\,(\lambda+1)_m,$$

gives

$$\frac{d^m}{dx^m} F_2(a,b,b';c,c';x,y;p) = \frac{a\,b}{c} F_2(a+1,b+1,b';c+1,c';x,y;p).$$

A repeated application of this process gives the general form (3.4). Similarly we can prove (3.5) and (3.6). □

4. Concluding Remarks and Observations

In our present investigation, with the help of the extended second Appell hypergeometric function $F_2(a,b,b';c,c';x,y;p)$, we investigated their diverse properties such mainly as generating function, exponential generating function and differentiation formulas. Also we obtained a Laguerre polynomial representation. The special cases of Theorems 2.1, 2.2 and 3.2 presented here when $p = 0$ would reduce to the corresponding well-known results for the second Appell hypergeometric function $F_2(a,b,b';c,c';x,y)$ (see, for details, [22], [23]).

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