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On a Generalization of Prime Submodules of a Module over a Commutative Ring

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ABSTRACT: Let R be a commutative ring with identity, and $n \geq 1$ an integer. A proper submodule N of an R-module M is called an n-prime submodule if whenever $a_1 \cdots a_{n+1} m \in N$ for some non-units $a_1, \ldots, a_{n+1} \in R$ and $m \in M$, then $m \in N$ or there are n of the a_i's whose product is in $(N : M)$. In this paper, we study n-prime submodules as a generalization of prime submodules. Among other results, it is shown that if M is a finitely generated faithful multiplication module over a Dedekind domain R, then every n-prime submodule of M has the form $m_1 \cdots m_t M$ for some maximal ideals $m_1, \ldots, m_t$ of R with $1 \leq t \leq n$.

Key Words: n-prime submodule, n-absorbing ideal, AP n-module.

Contents

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we take R as a commutative ring with identity, $U(R)$ as the set of unit elements of R, M as an R-module, and $n \geq 1$ is a positive integer. A proper ideal I of a ring R is an n-absorbing ideal of R if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$, then there are n of the a_i's whose product is in I. It is evident that a 1-absobing ideal is just a prime ideal. This concept was firstly introduced for $n = 2$ by A. Badawi [3], and then it has been studied for any positive integer n by D. F. Anderson and A. Badawi [1]. The authors generalized this notion to (m, n)-absorbing ideals with $m > n$ [11]. In fact, these ideals absorb an n-subproduct of every m-product of elements which lies in I. In this case, $(n + 1, n)$-absorbing ideals are just n-absorbing ideals. Moreover, there are several generalizations of n-absorbing ideals of a ring to submodules of a module (see, for example, [8,10]). In this paper, we study the notion of an n-prime submodule of a module as a generalization of a prime submodule.

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Let M be an R-module. A proper submodule N of M is called a prime submodule if for $r \in R$, $m \in M$, $rm \in N$ implies that $r \in (N : M)$ or $m \in N$. Prime submodules have been introduced by J. Dauns in [4], and then this class of submodules has been extensively studied by several authors (see, for example, [5,7]).

Definition 1.1. *Let R be a ring, $U(R)$ the set of units of R, M an R-module and n a positive integer. A proper submodule N of M is called an n-prime submodule of M if whenever $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M$, then $m \in N$ or there are n of the a_i's whose product is in $(N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$. An ideal I of R is called an n-prime ideal of R if it is an n-prime submodule of the R-module R.*

By this definition, a 1-prime submodule is just a prime submodule. Moreover, every n-prime ideal is an n-absorbing ideal, but the converse is not true in general (Example 2.6). It is shown that if R is a non-local PID or a polynomial ring $S[X]$ over a domain S, then every n-prime ideal of R is just a prime ideal of R (Theorems 2.8 and 2.12). However, an example of an n-prime ideal of a ring is given which is not a prime ideal (Example 2.6).
It is shown that every n-prime submodule is primary. Also, if R is a Bézout ring and M is a faithful multiplication R-module, then every n-prime submodule contains the nth power of its radical (Theorem 2.4). Moreover it is proved that if M is a multiplication R-module, then N is an n-prime submodule of M if and only if $(N : M)$ is an n-prime ideal of R (Corollary 4.6).
It is shown that if $N \times N'$ is an n-prime submodule of $M \times M'$, then N and N' are respectively an n-prime submodule of M and M'. The converse is true if $(N : M) = (N' : M')$ (Theorem 3.10). Using this fact, an example of an n-prime submodule of a module is given which is not prime submodule (Example 3.11).

Finally, we introduce and study AP n-modules. Indeed, an AP n-module M has the property that for each n-absorbing ideal I of R, IM is an n-prime submodule of M. If R is an AP n-module over itself, then we call it AP n-ring. For example, every Artin local ring is an AP n-ring for some positive integer n (Theorem 4.8). Moreover, Noetherian valuation domains are AP n-rings for all positive integer n (Theorem 4.9). It is shown that every finitely generated faithful multiplication module over an AP n-ring is an AP n-module (Corollary 4.7).

2. On n-prime submodules

We start with several elementary results.

Theorem 2.1. *Let R be a ring, M a non-zero R-module and n be a positive integer.*

(1) *A proper submodule N of M is an n-prime submodule of M if and only if whenever $a_1 \cdots a_t m \in N$ for $a_1, \ldots, a_t \in R \setminus U(R)$ and $m \in M$ with $t > n$, then $m \in N$ or there are n of the a_i's whose product is in $(N : M)$.*

(2) *If N is an n-prime submodule of M, then N is a t-prime submodule of M for all $t \geq n$.*

Proof: The proof is routine, and thus it is omitted. □

Let N be a proper submodule of an R-module M. If N is an n-prime submodule of M for some positive integer n, then define $\nu(N) = \min\{n \mid N \text{ is an } n\text{-prime submodule of } M\}$; otherwise, set $\nu(N) = \infty$. It is convenient to define $\nu(M) = 0$. Thus for any submodule N of M, we have $\nu(N) \in \mathbb{N} \cup \{0, \infty\}$ with $\nu(N) = 1$ if and only if N is a prime submodule of M and $\nu(N) = 0$ if and only if $N = M$. So $\nu(N)$ measures, in some sense, how far N is from being a prime submodule of M. Clearly $\omega(I) \leq \nu(I)$, where $\omega(I) = \min\{n \mid I \text{ is an } n\text{-absorbing ideal of } R\}$.

Lemma 2.2. *Let R be a ring, M a non-zero R-module and n a positive integer. Then the following hold:*

(1) *A proper submodule N of M is an n-prime submodule if and only if whenever $a_1 \cdots a_{n+1}K \subseteq N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and submodule K of M, then $K \subseteq N$ or there are n of the a_i's whose product is in $(N : M)$.*

(2) *If a proper submodule N of M is an n-prime submodule of M, then $(N : M)$ is an n-prime ideal of R and so it is an n-absorbing ideal of R. Moreover $\omega(N : M) \leq \nu(N)$.*

Proof: (1) Let N be an n-prime submodule of M and $a_1 \cdots a_{n+1}K \subseteq N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and for a submodule K of M. Let $K \nsubseteq N$ and $m \in K \setminus N$. Since $a_1 \cdots a_{n+1}m \in N$ and N is an n-prime submodule of M, there are n of the a_i's whose product is in $(N : M)$. Conversely, if the given condition is true for a submodule N of M, and $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M$, then it suffices to take $K = Rm$.
(2) Let $a_1 \cdots a_{n+1}r \in (N : M)$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$, $r \in R$ and no proper subproduct of the a_i's is in $(N : M)$. Then $a_1 \cdots a_{n+1}rM \subseteq N$. Thus, by (1), $rM \subseteq N$. The "In particular" statement is clear. □

The converse of the Lemma 2.2(2) is not necessarily true, as the following example shows.

Example 2.3. (1) *Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 4\mathbb{Z} \oplus 4\mathbb{Z}$. Then, by [1, Theorem 2.1(d)], $(N : M) = 4\mathbb{Z}$ is a 2-absorbing ideal of R, but N is not an n-prime submodule of M for any positive integer n. In fact, if $a_1 = 2$ and $a_2, \ldots, a_{n+1}$ are odd prime numbers, then $a_1 \cdots a_{n+1}(2, 0) \in N$, but no proper subproduct of the a_i's is in $(N : M)$ and $(2, 0) \notin N$.*

(2) *Let $R = \mathbb{Z}$, $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$ and $N = 0 \oplus \mathbb{Z}_p$ for some prime integer p. Then $(N : M) = 0$ is 1-prime, but by [7, Example 3.7] N is not a 1-prime submodule of M.*

Let N be a submodule of an R-module M. By radical of N, denoted $\operatorname{rad} N$, we mean that the intersection of all prime submodules of M containing N. If there is no such prime exists, we define $\operatorname{rad} N = M$. For an ideal I of R, we denote the radical of I by $\sqrt{I}$.

An R-module M is called a multiplication module, if for each submodule N of M there exists an ideal I of R such that $N = IM$. In this case, we can take $I = (N : M)$. If $N_1 = I_1M$ and $N_2 = I_2M$ are two submodules of an R-module M for some ideals I_1 and I_2 of R, then N_1N_2 is used to denote I_1I_2M.

Theorem 2.4. *Let R be a ring, M an R-module and N a submodule of M. If N is an n-prime submodule of M for some positive integer n, then:*

(1) *N is a primary submodule of M, and so $(N : M)$ is a primary ideal of R and $\sqrt{N : M}$ is a prime ideal of R.*

(2) *If $(N : M)$ is a prime ideal of R, then N is a prime submodule of M.*

(3) *If M is finitely generated faithful multiplication, then* $\operatorname{rad} N$ *is a prime submodule of M.*

(4) *If R is a Bézout ring and M is multiplication, then* $(\operatorname{rad} N)^n \subseteq N$*. In particular, this holds if R is a valuation domain.*

Proof: (1) Let $am \in N$ for $a \in R$ and $m \in M \setminus N$. Clearly $a \in R \setminus U(R)$. Then $a^{n+1}m \in N$ implies that $a \in \sqrt{N : M}$.
(2) Since $(N : M)$ is a prime ideal of R, $\sqrt{N : M} = (N : M)$. Let $am \in N$ for $a \in R$ and $m \in M \setminus N$. By (1) N is primary and then $a \in \sqrt{N : M} = (N : M)$. Thus N is a prime submodule of M.
(3) Since N is proper and M is multiplication, by [5, Theorem 2.12], $\operatorname{rad} N = \sqrt{N : M}M$. By (1), $\sqrt{N : M}$ is prime. Now since M is finitely generated faithful, by [5, Theorem 3.1 and Lemma 2.10], $\operatorname{rad} N \neq M$ is a prime submodule of M.
(4) By (1) and Lemma 2.2(2), $\sqrt{N : M}$ is a prime ideal of R and $(N : M)$ is an n-absorbing ideal of R. Since R is Bézout, by [1, Lemma 5.4], $(\sqrt{N : M})^n \subseteq (N : M)$. Thus by using [5, Theorem 2.12], $(\operatorname{rad} N)^n = (\sqrt{N : M})^n M \subseteq (N : M)M = N$. □

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local ring, M an R-module and N a submodule of M such that $\mathfrak{m}^n \subseteq (N : M)$ for some positive integer n. Then N is an n-prime submoule of M.*

Proof: Let $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M \setminus N$. Since $R \setminus U(R) = \mathfrak{m}$ and $\mathfrak{m}^n \subseteq (N : M)$, every n-subproduct of the a_i's is in $(N : M)$. □

Example 2.6. *Let $R = \mathbb{Z}_{p^t}$ and $\mathfrak{m} = \bar{p}R$, where $p \in \mathbb{Z}$ is a positive integer. Then $(R, \mathfrak{m})$ is local. Every proper ideal of R has the form $I_n = \bar{p}^n R$ for $n < t$. Thus by Theorem 2.5, I_n is an n-prime ideal of R.*

Corollary 2.7. *Let R be a Noetherian ring, M an R-module and N a p-primary submodule of M for some prime ideal p of R. Then N_p is an n-prime submodule of M_p for some positive integer n.*

Proof: Let N be a p-primary submodule of M. Then $(N : M)$ is a p-primary ideal of R. Thus by [9, Theorem 5.37], $(N : M)_p$ is a pR_p-primary ideal of R_p. Since (R_p, pR_p) is Noetherian local ring, $p^n R_p \subseteq (N : M)_p \subseteq (N_p : M_p)$ for some positive integer n. Now by Theorem 2.5, N_p is an n-prime submodule of M_p. □

Theorem 2.8. *Let R be a PID and $n > 1$ an integer.*

(1) *If $(R, \mathfrak{m})$ is local, then every ideal of R is n-prime for some positive integer n.*

(2) *If R is not local, then every n-prime ideal of R is prime.*

Proof: (1) Let I be an ideal of R. Since every non-zero prime ideal of R is maximal and $(R, \mathfrak{m})$ is local, I is $\mathfrak{m}$-primary. Now since R is Noetherian, $\mathfrak{m}^n \subseteq I$ for some positive integer n. Then by Theorem 2.5, I is an n-prime ideal of R.
(2) Let R be a non-local PID. Then R has at least two distinct prime elements. Now if I is an n-prime ideal of R, then I is primary by Theorem 2.4(1). Thus $I = p^t R$ for some prime element p of R and positive integer $t \leq n$. Let $t \neq 1$ and $a_1 = \cdots = a_{t-1} = r = p$ and $a_t = \cdots = a_{n+1} = q$ which $q \neq p$ is a prime element of R. Then $a_1 \cdots a_{n+1} r \in I$. However, $r \notin I$ and no proper n-subproduct is in I, a contradiction. Therefore $t = 1$ and hence I is prime. □

Remark 2.9. *It is clear that every n-prime ideal of R is an n-absorbing ideal of R. However, the converse need not be true in general. For example, if $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$, then I is a 2-absorbing ideal of R which is not a 2-prime ideal of R by Theorem 2.8.*

Theorem 2.10. *Let R be a ring such that every proper ideal of R is an n-prime ideal for some positive integer n. Then R is a local ring.*

Proof: Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two maximal ideals of R. Then $I = \mathfrak{m}_1 \cap \mathfrak{m}_2$ is an n-prime ideal for some positive integer n. By Theorem 2.4(1), I is a primary ideal of R. Then $\mathfrak{m}_1 = \mathfrak{m}_2$. □

Corollary 2.11. *Let R be a ring and n a positive integer such that every proper ideal of R is an n-prime ideal of R. Then R is local and $dim R = 0$.*

Proof: By Theorem 2.10, R is local. Since every n-prime ideal is an n-absorbing ideal, by [1, Theorem 5.9], $dim R = 0$. □

Theorem 2.12. *Let $R = S[X]$ be a polynomial ring with coefficients in a domain S. Then every n-prime ideal of R is prime.*

Proof: Let I be a non-prime ideal of $R = S[X]$. Then there are $f, g \in R \setminus I$ such that $fg \in I$. Since S is domain, S has not non-zero nilpotent element. Then by [9, Exercise 1.36], $fg+1$ is non-unit. On the other hand, $f(fg+1)^n g = fg(fg+1)^n \in I$. However, $g \notin I$ and $(fg+1)^n \notin I$ and $f(fg+1)^{n-1} \notin I$. Then I is not an n-prime ideal of R. □

Theorem 2.13. *Let R be a Dedekind domain and M be a finitely generatsd faithful multiplication R-module. If N is an n-prime submodule of M, then $N = N_1 \cdots N_t$ for some maximal submodules $N_1, \ldots, N_t$ of M with $1 \leq t \leq n$.*

Proof: Suppose that N is an n-prime submodule of M. Then by Lemma 2.2(2), $(N : M)$ is an n-absorbing ideal of R. Now by [1, Theorem 5.1], $(N : M) = m_1 \cdots m_t$ for some maximal ideals $m_1, \ldots, m_t$ of R with $1 \leq t \leq n$. Thus $N = (N : M)M = m_1 \cdots m_t M = m_1 M \cdots m_t M$ and $m_1 M, \ldots, m_t M$ are maximal submodules of M by [5, Theorem 2.5 and Theorem 3.1]. □

3. Extensions of n-prime submodules

In this section, we investigate the stability of n-prime submodules in various module-theoretic constructions.

Let N be a proper submodule of an R-module M. For $x \in M$, $N_x = (N : x) = \{r \in R \mid rx \in N\}$ is an ideal of R and clearly $(N : M) \subseteq (N : x)$.

Proposition 3.1. *Let R be a ring and M an R-module. If N is an n-prime submodule of M, then N_x is an n-prime ideal of R and so is an n-absorbing ideal of R for all $x \in M \setminus N$. Moreover $\omega(N_x) \leq \nu(N)$ for all $x \in M$.*

Proof: Let N be an n-prime submodule of M and $a_1 \cdots a_{n+1} r \in N_x$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $r \in R \setminus N_x$. Then $a_1 \cdots a_{n+1} rx \in N$ and $rx \notin N$. Since N is an n-prime submodule of M, there are n of the a_i's whose product is in $(N : M) \subseteq (N : x) = N_x$. This implies that N_x is an n-prime ideal and so is an n-absorbing ideal of R.
The "moreover" statement is clear if $x \in M \setminus N$ by above argument. If $x \in N$, then $N_x = R$ and hence $\omega(N_x) = 0 \leq \nu(N)$. □

For each $r \in R$ and every submodule N of M, we consider $N_r = (N :_M r) = \{x \in M \mid rx \in N\}$.

Proposition 3.2. *Let R be a ring. If N is an n-prime submodule of an R-module M, then N_r is an n-prime submodule of M for any $r \in \sqrt{N : M} \setminus (N : M)$.*

Proof: Let $a_1 \cdots a_{n+1} m \in N_r$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$, $m \in M$. Then $a_1 \cdots a_{n+1} rm \in N$. Since N is an n-prime submodule of M, $rm \in N$ or there are n of the a_i's whose product is in $(N : M)$. Thus $m \in N_r$ or there are n of the a_i's whose product is in $(N_r : M)$, since $(N : M) \subseteq (N_r : M)$. □

Proposition 3.3. *Let R be a ring and M an R-module. If N_i is an n_i-prime submodule of M such that $(N_i : M) = (N_j : M)$ for all $1 \leq i, j \leq t$, then $\cap_{i=1}^{t} N_i$ is an n-prime submodule of M for $n = max\{n_i \mid 1 \leq i \leq t\}$.*

Proof: Let $t = 2$ and $n = \max\{n_1, n_2\}$. Suppose that $a_1 \cdots a_{n+1} m \in N_1 \cap N_2$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$, $m \in M$. Then $a_1 \cdots a_{n+1} m \in N_1$ and $a_1 \cdots a_{n+1} m \in N_2$.

Since N_1 and N_2 are respectively n_1-prime and n_2-prime, either $m \in N_1 \cap N_2$ or there are n_1 of the a_i's whose product is in $(N_1 : M)$ or there are n_2 of the a_i's whose product is in $(N_2 : M)$. If $m \in N_1 \cap N_2$, then we are done. In otherwise, there are n of the a_i's whose product is in $(N_1 \cap N_2 : M) = (N_1 : M) = (N_2 : M)$ for $n = \max\{n_1, n_2\}$. This implies that $N_1 \cap N_2$ is an n-prime submodule of M. The proof for $t > 2$ is follows similarly by induction on t. □

The following example shows that Proposition 3.3 is not true in general.

Example 3.4. *Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, $N_1 = 2\mathbb{Z} \oplus \mathbb{Z}$ and $N_2 = 3\mathbb{Z} \oplus \mathbb{Z}$. Then N_1 and N_2 are 1-prime submodules, but $N = N_1 \cap N_2 = 6\mathbb{Z} \oplus \mathbb{Z}$ is not an n-prime submodule for all positive integer n. Since $(N : M) = 6\mathbb{Z}$ is not n-prime, by Theorem 2.8 and then by Lemma 2.2(2), N is not an n-prime submodule of M.*

Theorem 3.5. *Let R be a ring, M an R-module and N an n-prime submodule of M. Then for any submodule K of M either $K \subseteq N$ or $N \cap K$ is an n-prime submodule of K.*

Proof: Let K be a submodule of M such that $K \nsubseteq N$. Then $N \cap K \subset K$. Now if $a_1 \cdots a_{n+1}k \in N \cap K$ for $a_1, \ldots, a_{n+1} \in R \backslash U(R)$ and $k \in K$, then $a_1 \cdots a_{n+1}k \in N$. Since N is an n-prime submodule of M, $k \in N$ or there are n of the a_i's whose product is in $(N : M)$. Thus $k \in N \cap K$ or there are n of the a_i's whose product is in $(N \cap K : K)$, since $(N : M) \subseteq (N \cap K : K)$. □

Theorem 3.6. *Let R be a ring and $f : M \to M'$ be a homomorphism of R-modules. Then the following hold:*

(1) *If N' is an n-prime submodule of M' such that $f(M) \nsubseteq N'$, then $f^{-1}(N')$ is an n-prime submodule of M.*

(2) *If f is surjective and N is an n-prime submodule of M such that $\ker f \subseteq N$, then $f(N)$ is an n-prime submodule of M'.*

Proof: (1) Let N' be an n-prime submodule of M' and $a_1 \cdots a_{n+1}m \in f^{-1}(N')$ for non-unit elements $a_1, \ldots, a_{n+1} \in R$ and $m \in M$. Then $a_1 \cdots a_{n+1}f(m) = f(a_1 \cdots a_{n+1}m) \in N'$. Since N' is an n-prime submodule of M, $f(m) \in N'$ or there are n of the a_i's whose product is in $(N' : M')$. Hence $m \in f^{-1}(N')$ or there are n of the a_i's whose product is in $(f^{-1}(N') : M)$, since $(N' : M') \subseteq (f^{-1}(N') : M)$.
(2) Let N be an n-prime submodule of M and $a_1 \cdots a_{n+1}m' \in f(N)$ for $a_1, \ldots, a_{n+1} \in R \backslash U(R)$ and $m' \in M'$. Since f is surjective, $m' = f(m)$ for some $m \in M$. Then

$$a_1 \cdots a_{n+1}m' = a_1 \cdots a_{n+1}f(m) = f(a_1 \cdots a_{n+1}m) = f(n)$$

for some $n \in N$. Thus $a_1 \cdots a_{n+1}m - n \in \ker f \subseteq N$. Therefore $a_1 \cdots a_{n+1}m \in N$. Since N is an n-prime submodule of M, either $m \in N$ or there are n of the a_i's whose product is in $(N : M)$. Hence $m' \in f(N)$ or there are n of the a_i's whose product is in $(f(N) : M')$ (Note that, $(N : M) \subseteq (f(N) : M')$, since f is surjective). □

Corollary 3.7. *Let R be a ring, M an R-module and N, K proper submodules of M such that $N \subseteq K$. Then K is an n-prime submodule of M if and only if K/N is an n-prime submodule of M/N.*

Proof: Consider the natural projection $\pi : M \to M/N$ defined by $\pi(m) = m + N$ and use Theorem 3.6. □

Let R be a ring and M an R-module. Let N be a submodule of M. A submodule K of M maximal with respect to the property that $K \cap N = 0$ is called a complement of N in M. A submodule K of M will be called complement in M if there exists a submodule N of M such that K is a complement of N in M. A submodule N of M will be called essential if $N \cap K \neq 0$ for every non-zero submodule K of M. Also a submodule N of M will be called essential in a submodule L of M containing N, if N is essential as a submodule of L. It is not difficult to prove that if K is a complement in M, then K is not essential in any submodule L of M containing K.

Theorem 3.8. *Let R be a ring, M an R-module and N an n-prime submodule of M. If K is a submodule of M containing N such that K/N is a complement in M/N, then K is an n-prime submodule of M.*

Proof: Let $a_1 \cdots a_{n+1} m \in K$ for $a_1, \dots, a_{n+1} \in R \setminus U(R)$ and $m \notin K$. Then $L = K + Rm$ is a submodule of M which contains K properly and $a_1 a_2 \cdots a_{n+1} L \subseteq K$. K/N is not essential in L/N, since K/N is a complement in M/N. Thus there exists a submodule L' of L such that $N \subset L'$ and $K \cap L' = N$. Let $m' \in L' \setminus N$. Then $a_1 a_2 \cdots a_{n+1} m' \in a_1 a_2 \cdots a_{n+1} L' \subseteq (a_1 a_2 \cdots a_{n+1} L) \cap L' \subseteq K \cap L' = N$. Since N is an n-prime submodule of M, there are n of the a_i's whose product is in $(N : M) \subseteq (K : M)$. Hence K is an n-prime submodule of M. □

Let M be an R-module. By zero divisors of M, denoted $Z_R(M)$, we mean that the set of elements $r \in R$ such that $rm = 0$ for some non-zero element $m \in M$.

Theorem 3.9. *Let R be a ring, M an R-module and N a submodule of M. Let S be a multiplicatively closed subset of R such that $S \cap Z_R(M/N) = \emptyset$. If N is an n-prime submodule of M, then $S^{-1}N$ is an n-prime submodule of $S^{-1}M$.*

Proof: Let N be an n-prime submoule of M. Since $S \cap Z_R(M/N) = \emptyset$, it is easily seen that $S^{-1}N \neq S^{-1}M$. Suppose that $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{m}{s} \in S^{-1}N$ for $\frac{a_1}{s_1}, \dots, \frac{a_{n+1}}{s_{n+1}} \in S^{-1}R \setminus U(S^{-1}R)$ and $\frac{m}{s} \in S^{-1}M$. Then $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} \frac{m}{s} = \frac{n}{t}$ for some $n \in N$ and $t \in S$. Thus $a_1 \cdots a_{n+1} tum = s_1 \cdots s_{n+1} sun \in N$ for some $u \in S$. Clearly a_i's are non-unit in R. Thus, since N is an n-prime submodule of M, there are n of the a_i's whose product is in $(N : M)$ or there are $n-1$ of the a_i's whose product with tu is in $(N : M)$ or $m \in N$. If $m \in N$, then $\frac{m}{s} \in S^{-1}N$. If there are n of the a_i's whose product is in $(N : M)$, then there are n of the $\frac{a_i}{s_i}$'s whose product is in $S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$. If there are $n-1$ of the a_i's whose product with tu is in $(N:M)$, for example $a_1\cdots a_{n-1}(tu)\in(N:M)$, then $a_1\cdots a_n(tu)\in(N:M)$.Thus

$$\begin{aligned}\frac{a_1}{s_1}\cdots\frac{a_{n-1}}{s_{n-1}}\frac{a_n}{s_n}&=\frac{a_1}{s_1}\cdots\frac{a_{n-1}}{s_{n-1}}\frac{a_n(tu)}{s_n(tu)}\\&=\frac{a_1\cdots a_n(tu)}{s_1\cdots s_n(tu)}\in S^{-1}(N:M)\subseteq(S^{-1}N:S^{-1}M).\end{aligned}$$

This implies that $S^{-1}N$ is an n-prime submodule of $S^{-1}M$. □

Theorem 3.10. *Let R be a ring, M, M' R-modules, N a submodule of M, N' a submodule of M' and I, I' two ideals of R. Then the following hold:*

1. *If $N\times N'$ is an n-prime submodule of $M\times M'$, then N and N' are respectively an n-prime submodule of M and M'. The converse is true if $(N:M)=(N':M')$.*
2. *N (resp. N') is an n-prime submodule of M (resp. M') if and only if $N\times M'$ (resp. $M\times N'$) is an n-prime submodule of $M\times M'$.*
3. *If $I\times I'$ is an n-prime submodule of the R-module $R\times R$, then I and I' are n-prime ideals of R. The converse is true if $I=I'$.*
4. *If $I\times I'$ is an n-prime ideal of $R\times R$, then I and I' are n-prime ideals of R. The converse is true if $I=I'$.*
5. *I (resp. I') is an n-prime ideal of R (resp. R') if and only if $I\times R'$ (resp. $R\times I'$) is an n-prime submodule of the R-module $R\times R'$.*
6. *I (resp. I') is an n-prime ideal of R (resp. R') if and only if $I\times R'$ (resp. $R\times I'$) is an n-prime ideal of $R\times R'$.*

Proof: (1) Let $N\times N'$ be an n-prime submodule of $M\times M'$ and let $a_1\cdots a_{n+1}m\in N$ for $a_1,\ldots,a_{n+1}\in R\backslash U(R)$ and $m\in M$. Then $a_1\cdots a_{n+1}(m,0)\in N\times N'$. Thus $(m,0)\in N\times N'$ or there are n of the a_i's whose product is in $(N\times N':M\times M')=(N:M)\cap(N':M')$. Hence N is an n-prime submodule of M. By a similar argument, N' is an n-prime submodule of M'. Convesely let $a_1\cdots a_{n+1}(m,m')\in N\times N'$ for $a_1,\ldots,a_{n+1}\in R\setminus U(R)$ and $(m,m')\in M\times M'\setminus N\times N'$. Then $m\notin N$ or $m'\notin N'$. Let $m\notin N$. Then $a_1\cdots a_{n+1}m\in N$, implies that there are n of the a_i's whose product is in $(N:M)=(N\times N':M\times M')$.
(2) Let N be an n-prime submodule of M and $a_1\cdots a_{n+1}(m,m')\in N\times M'$ for $a_1,\ldots,a_{n+1}\in R\setminus U(R)$ and $(m,m')\in M\times M'$. Then $a_1\cdots a_{n+1}m\in N$. Since N is an n-prime submodule of M, either $m\in N$ or there are n of the a_i's whose product is in $(N:M)=(N\times M':M\times M')$. This implies that $N\times M'$ is an n-prime submodule of $M\times M'$. The converse is similar to (1). By a similar argument, N' is an n-prime submodule of M' if and only if $M\times N'$ is an n-prime submodule of $M\times M'$.

(3) By (1).
(4) The proof is similar to the proof of (1).
(5) By (2).
(6) The proof is similar to the proof of (2). □

Example 3.11. *Let $R = \mathbb{Z}_{P^t}$, $M = R \oplus R$, $N_n = I_n \oplus I_n$, $L_n = R \oplus I_n$ and $K_n = I_n \oplus R (n < t)$. Then by Example 2.6 and Theorem 3.10(3), N_n, L_n and K_n are n-prime submodules of M.*

Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. We view R as a subring of $R(+)M$ via $r \mapsto (r, 0)$.

Theorem 3.12. *Let R be a ring, M an R-module, I an n_1-absorbing ideal of R and N an n_2-prime submodule of M with $IM \subseteq N$. Then $I(+)N$ is an n-absorbing ideal of $R(+)M$ for $n = n_1 + n_2$. Conversly if $I(+)N$ is an n-absorbing ideal of $R(+)M$, then I is an n-absorbing ideal of R.*

Proof: Let $n = n_1 + n_2$. Assume that $(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)N$ for $(a_1, m_1), \ldots, (a_{n+1}, m_{n+1}) \in R(+)M$. Without loss of generality suppose that these elements are not in $U(R(+)M)$. Then $a_1 \cdots a_{n+1} \in I$ and

$$\sum_{i=1}^{n+1} a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} m_i \in N \tag{3.1}$$

Since I is an n_1-absorbing ideal of R, there are n_1 of the a_i's whose product is in I. For example, let $a_1 \cdots a_{n_1} \in I$. The terms of (3.1) that contain $a_1 \cdots a_{n_1}$, are in $IM \subseteq N$. Thus $\sum_{i=1}^{n_1} a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} m_i \in N$, where a_0 is assumed that to be 1. But

$$\sum_{i=1}^{n_1} a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} m_i = a_{n_1+1} \cdots a_{n+1} \sum_{i=1}^{n_1} a_1 \cdots a_{i-1} a_{i+1} m_i \in N$$

Since $U(R(+)M) = U(R)(+)M$ by [2, Theorem 3.7], a_i's $(1 \leq i \leq n+1)$ are non-unit. Now, since N is an n_2-prime submodule of M, $\sum_{i=1}^{n_1} a_1 \cdots a_{i-1} a_{i+1} m_i \in N$ or there are n_2 of the a_i's $(n_1 + 1 \leq i \leq n+1)$ whose product is in $(N : M)$.
If $\sum_{i=1}^{n_1} a_1 \cdots a_{i-1} a_{i+1} m_i \in N$, then $(a_1, m_1) \cdots (a_{n_1}, m_{n_1}) \in I(+)N$, and if there are n_2 of the a_i's $(n_1 + 1 \leq i \leq n+1)$ whose product is in $(N : M)$, for example $a_{n_1+1} \cdots a_n \in (N : M)$, then

$$(a_1, m_1) \cdots (a_{n_1}, m_{n_1})(a_{n_1+1}, m_{n_1+1}) \cdots (a_n, m_n) \in I(+)N.$$

Hence $I(+)N$ is an $n = n_1 + n_2$-absorbing ideal of $R(+)M$.
Now let $I(+)N$ be an n-absorbing ideal of $R(+)M$, and let $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$. Then $(a_1, 0) \cdots (a_{n+1}, 0) \in I(+)N$. Thus there are n of $(a_i, 0)$'s whose product is in $I(+)N$. Hence there are n of the a_i's whose product is in I and so I is an n-absorbing ideal of R. □

4. AP n-modules

Let R be a ring, M an R-module and I a proper ideal of R. Let $M_n(I)$ denote a submodule of M generated by the following set:
$\{m \mid a_1 \cdots a_{n+1}m \in IM$ for some $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ such that $a_1 \cdots a_{n+1} \notin I\}$.

Lemma 4.1. *Let R be a ring, M an R-module and I an n-absorbing ideal of R. If $M_n(I) \subseteq IM \neq M$, then IM is an n-prime submodule of M.*

Proof: Let $a_1 \cdots a_{n+1}m \in IM$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ such that no proper subproduct of the a_i's is in $(IM : M)$. Since $I \subseteq (IM : M)$ and I is an n-absorbing ideal of R, $a_1 \cdots a_{n+1} \notin I$. Thus $m \in M_n(I) \subseteq IM \neq M$, and hence IM is an n-prime submodule of M. □

The following example shows that Lemma 4.1 fails if the condition that $M_n(I) \subseteq IM$ is removed.

Example 4.2. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Then $I = 4\mathbb{Z}$ is a 2-absorbing ideal of R, but $IM = 4\mathbb{Z} \oplus 4\mathbb{Z}$ is not a 2-prime submodule of M. It is easily seen that $2\mathbb{Z} \oplus 2\mathbb{Z} \subseteq M_2(I)$, and thus $M_2(I) \nsubseteq IM$.*

Definition 4.3. *Let R be a ring, M an R-module and n a positive integer. We say that M is an AP n-module if $M_n(I) \subseteq IM \neq M$ for any n-absorbing ideal I of R. Also, R is called an AP n-ring if R is an AP n-module as R-module.*

Remark 4.4. *We say that M is an AP n-module because for any n-Absorbing ideal I of R, IM is an n-Prime submodule of M by Lemma 4.1.*

Lemma 4.5. *Let R be a ring and n a positive integer. Then R is an AP n-ring if and only if every n-absorbing ideal of R is an n-prime ideal of R.*

Proof: Let R be an AP n-ring, I an n-absorbing ideal of R and let $a_1 \cdots a_{n+1}r \in I$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $r \in R$ such that no proper subproduct of the a_i's is in I. Since I is n-absorbing, $a_1 \cdots a_{n+1} \notin I$. Thus $r \in M_n(I) \subseteq I$. Hence I is n-prime. Conversely suppose that every n-absorbing ideal of R is an n-prime ideal of R. Let I be an n-absorbing ideal of R and $r \in R$ be a generator of $M_n(I)$. Then $a_1 \cdots a_{n+1}r \in IR = I$ for some $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ such that $a_1 \cdots a_{n+1} \notin I$. Thus no proper subproduct of the a_i's is in I and hence $r \in I$, since I is an n-prime ideal of R. Therefore R is an AP n-ring. □

Corollary 4.6. *Let M be a multiplication R-module, N a proper submodule of M and n a positive integer. Consider the following statements:*

(1) *N is an n-prime submodule of M.*

(2) *$(N : M)$ is an n-prime ideal of R.*

(3) *$N = IM$ for some n-prime ideal I of R.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Moreover, if* M *is a finitely generated faithful module, then* (3) $\Rightarrow$ (2).

Proof: (1) $\Rightarrow$ (2) By Lemma 2.2(2).
(2) $\Rightarrow$ (1) Let $I = (N : M)$. We show that $(M_n(I) : M) \subseteq I$ and use Lemma 4.1. Let $r \in (M_n(I) : M)$ and $m \in M \setminus M_n(I)$. Then $rm \in M_n(I)$. Thus $rm = s_1m_1 + \cdots + s_tm_t$ for some $s_i \in R$ and $m_i \in M_n(I)$ $(1 \leq i \leq t)$ by definition of $M_n(I)$. Since $m_i \in M_n(I)$, there are $a_{i_1}, \ldots, a_{i_{n+1}} \in R \setminus U(R)$ such that $a_{i_1} \cdots a_{i_{n+1}} m_i \in IM$ and $a_{i_1} \cdots a_{i_{n+1}} \notin I$. Then $\prod_{i=1}^{t} \prod_{j=1}^{n+1} a_{i_j} rm \in IM$. Since $m \notin M_n(I)$,

$$a_{1_1} \cdots a_{1_{n+1}} (\prod_{i=2}^{t} \prod_{j=1}^{n+1} a_{i_j} r) \in I.$$

Thus we have $\prod_{i=2}^{t} \prod_{j=1}^{n+1} a_{i_j} r \in I$, since I is an n-prime and no proper subproduct of a_{1_j}'s $(1 \leq j \leq n+1)$ is in I. Repeating this process follows that $r \in I$. Hence $(M_n(I) : M) \subseteq I$. Since M is multiplication, $M_n(I) \subseteq IM = N$.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (2) By [5, Theorem 3.1], $(N : M) = I$. □

Corollary 4.7. *Let* R *be a ring and* M *a finitely generated faithful multiplication* R*-module. Then* R *is an AP* n*-ring if and only if* M *is an AP* n*-module.*

Proof: Let R be an AP n-ring and I an n-absorbing ideal of R. Then I is an n-prime ideal of R, by Lemma 4.5. Since M is a multiplication module, by the proof of Corollary 4.6((2) $\Rightarrow$ (1)), $M_n(I) \subseteq IM$. Now since M is a finitely generated faithful multiplication module, by [5, Theorem 3.1], $IM \neq M$. Hence M is an AP n-module. Conversely suppose that M is an AP n-module and I is an n-absorbing ideal of R. Then by Lemma 4.1, IM is an n-prime submodule of M. Since M is a finitely generated faithful multiplication module, by Lemma 2.2(2) and [5, Theorem 3.1], $(IM : M) = I$ is an n-prime ideal of R. Hence by Lemma 4.5, R is an AP n-ring. □

Theorem 4.8. *Let* $(R, \mathfrak{m})$ *be an Artinian local ring and* n *a positive integer such that* $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$. *Then every ideal of* R *is an* n*-prime ideal. In particular,* R *is an AP* n*-ring.*

Proof: Note that R is Noetherian and $dim R = 0$, by [9, Corollary 8.45]. Let I be an ideal of R. Then I is an $\mathfrak{m}$-primary. Thus $\mathfrak{m}^n \subseteq \mathfrak{m}^t \subseteq I$ for some positive integer $t \leq n$. Hence by Theorem 2.5, I is an n-prime ideal of R. The "in particular" statement is clear. □

Theorem 4.9. *Let* R *be a Noetherian valuation domain and* n *a positive integer. Then* R *is an AP* n*-ring.*

Proof: Note that $(R, \mathfrak{m})$ is a local PID and then $dimR = 1$. Let I be an n-absorbing ideal of R. By [9, Theorem 15.42], $I = \mathfrak{m}^t$ for some positive integer t. Let $\mathfrak{m} = Rp$ for some prime element $p \in R$ and $t > n$. Then $p^n = rp^t$ for some $r \in R$, since I is an n-absorbing ideal of R. Thus $rp^{t-n} = 1$ and hence p is unit, which is a contradiction. Therefore $t \leq n$. Then by Theorem 2.5, I is a t-prime ideal and so it is an n-prime ideal of R. Hence by Lemma 4.5, R is an AP n-ring. □

Corollary 4.10. *Let R be a DVR and n a positive integer. Then R is an AP n-ring.*

Example 4.11. *Let $R = \mathbb{Z}[\sqrt{-5}]$, $M = 2R + (\sqrt{-5} - 1)R$ and n a positive integer. Since R is a Dedekind domain, M is a finitely generated faithful multiplication R-module. Then by Corollary 4.10, R_P is an AP n-ring for all non-zero prime ideal P of R. Since M_P is a finitely generated faithful multiplication R_P-module, by Corollary 4.7, M_P is an AP n-module.*

Theorem 4.12. *Let R be a zero-dimensional Bézout ring and n a positive integer. Then for each prime ideal p of R, R_p is an AP n-ring.*

Proof: Without loss of generality, we may assume that R is a local ring. Let I be an n-absorbing ideal of R. Since $(R, \mathfrak{m})$ is local and $dimR = 0$, $\sqrt{I} = \mathfrak{m}$. By [1, Lemma 5.4], $\mathfrak{m}^n \subseteq I$. Thus by Theorem 2.5, I is an n-prime ideal of R. □

Proposition 4.13. *Let R be a ring, M an R-module and N_i an n_i-prime submodule of M for all $1 \leq i \leq t$. Then $(\cap_{i=1}^t N_i : M)$ is an n-absorbing ideal of R for $n = n_1 + \cdots + n_t$. Moreover if M is a multiplication AP n-module, then $\cap_{i=1}^t N_i$ is an n-prime submodule of M.*

Proof: By Theorem 4.6, $(N_i : M)$ is an n_i-absorbing ideal of R. Hence by [1, Theorem 2.1(c)], $(\cap_{i=1}^t N_i : M) = \cap_{i=1}^t (N_i : M)$ is an n-absorbing ideal of R for $n = n_1 + \cdots + n_t$.

For "moreover" part, since M is multiplication, $\cap_{i=1}^t N_i = (\cap_{i=1}^t N_i : M)M$. Therefore $\cap_{i=1}^t N_i$ is an n-prime submodule of M, by Lemma 4.1. □

Let R be a ring and M an R-module. If I is an n_1-absorbing ideal of a ring R and N is an n_2-prime submodule of an R-module M, then IN is not necessarily an n-prime submodule of M for some positive integer n, as the following example shows.

Example 4.14. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Then $I = 4\mathbb{Z}$ is a 2-absorbing ideal of R and $N = 3\mathbb{Z} \oplus \mathbb{Z}$ is a 1-prime (prime) submodule of M but $IN = 12\mathbb{Z} \oplus 4\mathbb{Z}$ is not an n-prime submodule of M for any positive integer n. Since $(IN : M) = 12\mathbb{Z}$ is not an n-prime ideal of R, by Theorem 2.8.*

Theorem 4.15. *Let R be a ring, M a finitely generated faithful multiplication R-module, I an n_1-absorbing ideal of R and N an n_2-prime submodule of M. If R is an AP n-ring for $n = n_1 + n_2$ and two ideals I and $(N : M)$ are comaximal, then IN is an n-prime submodule of M.*

Proof: Since M is a multiplication R-module, $(IN : M)M = IN = I(N : M)M$. By [5, Theorem 3.1], hypoteses and [1, Theorem 2.1(c)], $(IN : M) = I(N : M) = I \cap (N : M)$ is an n-absorbing ideal of R for $n = n_1 + n_2$. Hence, IN is an n-prime submodule of M, by Corollary 4.7 and Lemma 4.1. □

Lemma 4.16. *Let M be a finitely generated faithful multiplication R-module. Then the ideals $I_1, \dots, I_t$ are pairwise comaximal ideals of R if and only if $N_1 = I_1M, \dots, N_t = I_tM$ are pairwise comaximal submodules of M. In this case, $N_1 \cdots N_t = N_1 \cap \cdots \cap N_t$.*

Proof: The necessity is clear. To prove the sufficiency, we observe that $(I_1 + I_2)M = I_1M + I_2M = N_1 + N_2 = M$. Since M is finitely generated faithful multiplication, by [5, Theorem 3.1], $I_1 + I_2 = R$. In this case, by [5, Corollary 1.7], $N_1N_2 = I_1I_2M = (I_1 \cap I_2)M = I_1M \cap I_2M = N_1 \cap N_2$. Now the assertion follows by induction on t. □

Theorem 4.17. *Let R be a ring and M a finitely generated faithful multiplication R-module. If R is an AP n-ring and $P_1, \dots, P_n$ are prime submodules of M that are pairwise comaximal, then $N = P_1 \cdots P_n$ is an n-prime submodule of M.*

Proof: By [5, Corollary 2.11], $P_i = p_iM$ for some prime ideal p_i of R and by Lemma 4.16, p_i's are pairwise comaximal . Then $N = P_1 \cdots P_n = p_1 \cdots p_nM$. By [1, Theorem 2.6], $p_1 \cdots p_n$ is an n-absorbing ideal of R. Therefore N is an n-prime submodule of M, by Corollary 4.7 and Lemma 4.1.

□

Lemma 4.18. *Let R be a ring, M a multiplication R-module and N a maximal submodule of M. If M is an AP n-module, then N^n is an n-prime submodule of M. Moreover, $\nu(N^n) \leq n$, and $\nu(N^n) = n$ if $N^{n+1} \subset N^n$.*

Proof: By [5, Theorem 2.5], $N = mM$ for some maximal ideal m of R. Then m^n is an n-absorbing ideal of R, by [1, Lemma 2.8]. Hence, $N^n = m^nM$ is an n-prime submodule of M, by Lemma 4.1.
The first part of the "moreover" statement is clear. Now if $N^{n+1} \subset N^n$, then $m^{n+1} \subset m^n$. Thus by [1, Lemma 2.8], $\omega(m^n) = n$. On the other hand, $(N^n : M) = m^n$ and by Lemma 2.2(2), $\omega(N^n : M) \leq \nu(N^n)$. Hence $\nu(N^n) = n$. □

Theorem 4.19. *Let R be a ring, M a multiplication R-module and $N_1, \dots, N_n$ are maximal submodules of M. If M is an AP n-module, then $N = N_1 \cdots N_n$ is an n-prime submodule of M. Moreover, $\nu(N) \leq n$.*

Proof: By [5, Theorem 2.5], $N_i = m_i M$ for some maximal ideal m_i of R. Then $N = m_1 M \cdots m_n M = m_1 \cdots m_n M$ and $m_1 \cdots m_n$ is an n-absorbing ideal of R by [1, Theorem 2.9]. Since M is an AP n-module, N is an n-prime submodule of M, by Lemma 4.1. The "moreover" statement is clear. □

We call a submodule N is a minimal n-prime submodule of M if N is minimal among all n-prime submodules of M with respect to inclusion.

Proposition 4.20. *Let R be a ring, M a finitely generated faithful multiplication R-module and n a positive integer. If M is an AP n-module, then the set of minimal n-prime submodules of M is equal to*

$$\{IM \mid I \text{ is a minimal } n\text{-absorbing ideal of } R\}.$$

Proof: Let I be a minimal n-absorbing ideal of R. Since M is an AP n-module, IM is an n-prime submodule of M. Assume that N is an n-prime submodule of M such that $N \subseteq IM$. Then by [5, Theorem 3.1],

$$(N : M) \subseteq (IM : M) = I.$$

Since I is a minimal n-absorbing ideal of R and $(N : M)$ is an n-absorbing ideal of R by Lemma 2.2(2), $(N : M) = I$. Hence $N = (N : M)M = IM$ and thus IM is a minimal n-prime submodule of M. Now, assume that N is a minimal n-prime submodule of M. Then $(N : M)$ is an n-absorbing ideal of R and $N = (N : M)M$. Assume that I is an n-absorbing ideal of R such that $I \subseteq (N : M)$. Then $IM \subseteq N$ and IM is an n-prime submodule of M. Thus minimality of N implies that $IM = N$. Therefore $I = (IM : M) = (N : M)$. Hence $(N : M)$ is a minimal n-absorbing ideal of R. □

At the end of this paper should be noted that every n-absorbing ideal of a ring R contains a minimal n-absorbing ideal [6, Corollary 2.2]. Now if M is a finitely generated faithful multiplication AP n-module, then by Proposition 4.20 every n-prime submodule N of M contains a minimal n-prime submodule of M.

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