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The Maximal Subgroups of Sylow Subgroups and the Structure of Finite Groups*

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ABSTRACT: In this paper we investigate the influence of some subgroups of Sylow subgroups with semi cover-avoiding property and E-supplementation on the structure of finite groups. Some recent results are generalized and unified.

Key Words: Semi cover-avoiding property, E-supplemented, p-nilpotent.

Contents

1. Introduction

All groups considered in this paper will be finite.

A subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel. In 2007, Skiba (see [22]) introduced the concept of S-supplemented subgroup. A subgroup H of G is said to be S-supplemented in G if there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormal in G. As another generalization of the S-quasinormality, the concept of S-quasinormally embedded subgroup was given by Ballester-Bolinches and Pedraza-Aguilera (see [2]). A subgroup H is said to be S-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G. In 2012, Li (see [4]) proposed the definition of E-supplemented subgroup which covers properly both S-quasinormally embedding property and Skiba's weakly S-supplementation. A subgroup H is said to be E-supplemented in G if there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{eG}$, where H_{eG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormally embedded in G.

On the other hand, we say that a subgroup H of a group G covers G-chief factor A/B if $HA = HB$, and H avoids A/B if $H \cap A = H \cap B$. If H covers or

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avoids every chief factor of G, then H is said to have the cover-avoiding property in G. This conception was first studied by Gaschütz (see [2]) to study the solvable groups, later by Gillam (see [3]) and Ezquerro (see [5]), et al. As a generalization of the cover-avoiding property, Fan, Guo and Shum (see [8]) defined the semi cover-avoiding property. A subgroup H of a group G is said to have the semi cover-avoiding property in G, if there exists a chief series of G such that H either covers or avoids every G-chief factor of this series.

A subgroup that satisfies the cover-avoiding property does not necessary need to be E-supplemented and vice-versa. In this paper, we will focus on the two kinds of subgroups and establish the structure of groups under the assumption that all maximal subgroups of a Sylow subgroup either have the semi cover-avoiding property or are E-supplemented subgroups. A series of previously known results are generalized, such as in [6,9,11,13,15,16,17,18,19,21,23,24,25].

2. Preliminaries

In this section, we list some lemmas which will be useful for the proofs of our main results.

Lemma 2.1 ([11, Lemmas 2.5 and 2.6]). *Suppose that H has the semi cover-avoiding property in G.*

(1) *If $H \leq L \leq G$, then H has the semi cover-avoiding property in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N has the semi cover-avoiding property in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N has the semi cover-avoiding property in G/N.*

Lemma 2.2 ([4, Lemma 2.3]). *Let H be a E-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is E-supplemented in L.*

(2) *If $N \triangleleft G$ and $N \leq H \leq G$, then H/N is E-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is E-supplemented in G/N.*

Lemma 2.3 ([11, Lemma 3.1]). *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let P be a p-Sylow subgroup of G. If there is a maximal subgroup P_1 of P such that P_1 has the semi cover-avoiding property in G, then G is p-solvable.*

Lemma 2.4 ([18, Lemma 2.8]). *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.5 ([19, Lemma 2.7]). *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then $N \leq Z(G)$.*

(2) *If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*

(3) *If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

Lemma 2.6 ([20, Main Theorem]). *Suppose that G has a Hall π-subgroup where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.7 ([21, Lemma 2.6]). *Let $H \neq 1$ be a solvable normal subgroup of a group G. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.8 ([22, Lemma 2.16]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathfrak{F}$. If N is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.9 ([27, Lemma 2.3]). *Suppose that H is S-quasinormal in G, and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is S-quasinormal in G.*

Lemma 2.10 ([28, Lemma A]). *If P is an S-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.11 ([27, Lemma 2.4]). *Suppose that P is a p-subgroup of a group G contained in $O_p(G)$. If P is S-quasinormally embedded in G, then P is S-quasinormal in G.*

3. Main results

Theorem 3.1. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If G has a Sylow p-subgroup P such that every maximal subgroup of P either has the semi cover-avoiding property or is E-supplemented in G, then G is p-nilpotent.*

Proof: Assume that the assertion is false and let G be a minimal counterexample. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Suppose that $M/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then there exists a maximal subgroup P_1 of P such that $M = P_1O_{p'}(G)$. By the hypothesis of the theorem, P_1 either has the semi cover-avoiding property or is E-supplemented in G. Then $M/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ either has the semi cover-avoiding property or is E-supplemented in $G/O_{p'}(G)$ by Lemmas 2.1 and 2.2. It is clear that $(|G/O_{p'}(G)|, p-1) = 1$. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Therefore, we have $O_{p'}(G) = 1$.

(2) $O_p(G) \neq 1$.

If not, suppose that $O_p(G) = 1$. If there is a maximal subgroup of P which has the semi cover-avoiding property in G, then G is p-solvable by Lemma 2.3. Since $O_{p'}(G) = 1$ by step (1), we have $O_p(G) \neq 1$, a contradiction. Thus we may assume that all maximal subgroups of P are E-supplemented in G. If $p \neq 2$, then G is odd from the assumption that $(|G|, p-1) = 1$. By the Feit-Thompson Theorem, G is solvable. It follows that $O_p(G) \neq 1$ by step (1), a contradiction. If $p = 2$, then we get also G is solvable by [4, Lemma 3.1], the same contradiction.

(3) If $N \leq O_p(G)$, then G/N is p-nilpotent. Consequently, G is solvable.

Suppose that M/N is a maximal subgroup of P/N. Then M is a maximal subgroup of P. By the hypothesis of the theorem, M either has the semi cover-avoiding property or is E-supplemented in G. Then M/N either has the semi cover-avoiding property or is E-supplemented in G/N by Lemmas 2.1 and 2.2. Therefore G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. If p is odd, then G is solvable. If $p = 2$, then G/N is solvable, and so G is solvable.

(4) $O_p(G)$ is the unique minimal normal subgroup of G.

Let N be a minimal normal subgroup of G. Since G is solvable by step (3), N is an elementary abelian subgroup. Note that $O_{p'}(G) = 1$, then we have N is a p-subgroup and so $N \leq O_p(G)$. Step (3) implies that $G/O_p(G)$ is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Choose M to be a maximal subgroup of G such that $G = NM$. Obviously, $G = O_p(G)M$ and so $O_p(G) \cap M$ is normal in G by Lemma 2.4. The uniqueness of N yields $N = O_p(G)$.

(5) The final contradiction.

By the proof in step (4), G has a maximal subgroup M such that $G = MO_p(G)$ and $G/O_p(G) \cong M$ is p-nilpotent. Clearly, $P = O_p(G)(P \cap M)$. Furthermore, $P \cap M < P$. Thus, there exists a maximal subgroup V of P such that $P \cap M \leq V$. Hence, $P = O_p(G)V$. By the hypothesis, V either has the semi cover-avoiding property or is E-supplemented in G.

Case I: V has the semi cover-avoiding property in G. Since $O_p(G)$ is the unique minimal normal subgroup of G, V covers or avoids $O_p(G)/1$. If V covers $O_p(G)/1$, then $VO_p(G) = V$, i.e., $O_p(G) \leq V$. It follows that $P = O_p(G)V = V$, a contradiction. If V avoids $O_p(G)/1$, then $V \cap O_p(G) = 1$. Since $V \cap O_p(G)$ is a maximal subgroup of $O_p(G)$, we have that $O_p(G)$ is of order p and so $O_p(G)$ lies in $Z(G)$ by Lemma 2.5. By step (3), we have $G/O_p(G)$ is p-nilpotent. Then $G/Z(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

Case II: V is E-supplemented in G. Then there is a subgroup T of G such that $G = VT$ and $V \cap T \leq V_{eG}$. Assume that T is p-nilpotent. Let $T_{p'}$ be the normal Hall p'-subgroup of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$ and $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence, we may assume $M = N_G(M_{p'})$. By applying Lemma 2.6 and the Feit-Thompson Theorem, there exists $g \in G$ such that $T^g_{p'} = M_{p'}$. Hence, $T^g \leq N_G(T^g_{p'}) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of V. Thus, $G = VT^g = VM$ and $P = V(P \cap M) = V$, a contradiction. Hence T is not p-nilpotent. If $V_{eG} = 1$, then $|T|_p = p$. By Lemma 2.5, T is p-nilpotent, a contradiction. Thus we may assume that $V_{eG} \neq 1$. Let $U_1, U_2, ..., U_s$ be all the nontrivial subgroups of V which are S-quasinormally embedded in G. For every $i \in \{1, 2, ..., s\}$, then there is an S-quasinormal subgroup K_i of G such that U_i is a Sylow p-subgroup of K_i. Suppose that for some $i \in \{1, 2, ..., s\}$, we have $(K_i)_G \neq 1$. Then $O_p(G) \leq (K_i)_G \leq K_i$ by step (4). It follows that $O_p(G) \leq U_i \leq V$, and so

$P = O_p(G)V = V$. This contradiction shows that for all $i \in \{1, 2, ..., s\}$ we have $(K_i)_G = 1$. By Lemma 2.9, U_i is S-quasinormal in G. Hence V_{eG} is S-quasinormal in G. From Lemma 2.10 we have $O^p(G) \le N_G(V_{eG})$. Since V_{eG} is subnormal in G, we have $V_{eG} \le O_p(G)$. Thus, $V_{eG} \le V \cap O_p(G)$ and $1 < V_{eG} \le (V_{eG})^G = (V_{eG})^{O^p(G)P} = (V_{eG})^P \le (V \cap O_p(G))^P = V \cap O_p(G) \le O_p(G)$. It follows that $(V_{eG})^G = V \cap O_p(G) = O_p(G)$. Then $O_p(G) \le V$ and so $P = V$, a contradiction.□

Corollary 3.2. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P either has the semi cover-avoiding property or is E-supplemented in G, then G is p-nilpotent.*

Proof: In view of Lemmas 2.1 and 2.2, every maximal subgroup of P has the semi cover-avoiding property or is E-supplemented in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal Hall p'-subgroup of H. Obviously, $H_{p'} \trianglelefteq G$.

Case I: $H_{p'} \neq 1$. We consider $G/H_{p'}$. Applying Lemmas 2.1 and 2.2, it is easy to see that $G/H_{p'}$ satisfies the hypotheses for the normal subgroup $H/H_{p'}$. Therefore, $G/H_{p'}$ is p-nilpotent by induction. It follows that G is p-nilpotent.

Case II: $H_{p'} = 1$, i.e., $H = P$ is a p-group. Since G/P is p-nilpotent, we can let K/P be the normal Hall p'-subgroup of G/P. By the Schur-Zassenhaus Theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = PK_{p'}$. A new application of Theorem 3.1 yields that K is p-nilpotent and so $K = P \times K_{p'}$. It is easy to see that $K_{p'}$ is a normal p-complement of G. Consequently, G is p-nilpotent. □

Corollary 3.3. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P either has the semi cover-avoiding property or is E-supplemented in G, then G is p-nilpotent.*

Corollary 3.4. *Suppose that every maximal subgroup of any Sylow subgroup of a group G either has the semi cover-avoiding property or is E-supplemented in G. Then G is a Sylow tower group of supersolvable type.*

Proof: Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Theorem 3.1, G is p-nilpotent. Let T be the normal Hall p'-subgroup of G. In view of Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of T has the semi cover-avoiding property or is E-supplemented in T. Thus T satisfies the hypothesis of the corollary. It follows by induction that T, and hence G is a Sylow tower group of supersolvable type. □

Corollary 3.5 ([13, Theorem 3.3])**.** *Let G be a group, p a prime dividing the order of G, and P a Sylow p-subgroup of G. If $(|G|, p-1) = 1$ and every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Corollary 3.6 ([11, Theorem 3.2])**.** *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If P is cyclic or every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Proof: If P is cyclic, by Lemma 2.5, we have that G is p-nilpotent. Thus we may assume that every maximal subgroup of P has the semi cover-avoiding property in G. By Theorem 3.1, G is p-nilpotent. □

Corollary 3.7 ([15, Theorem 3.4]). *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

Corollary 3.8 ([16, Theorem 3.2]). *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

Corollary 3.9 ([29, Theorem 3.1]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented in G, then G is p-nilpotent.*

Corollary 3.10 ([6, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and $G \in C_{p'}$, then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.*

Corollary 3.11 ([19, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.*

Corollary 3.12 ([27, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is S-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 3.13 ([29, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly S-permutably embedded in G, then G is p-nilpotent.*

Corollary 3.14 ([30, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly S-permutable in G, then G is p-nilpotent.*

Corollary 3.15 ([31, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly S-permutable in G, then G is p-nilpotent.*

Corollary 3.16 ([32, Theorem 3.1]). *Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is S-permutably embedded in G, then G is p-nilpotent.*

Corollary 3.17 ([14, Theorem 3.1]). *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c-normal or S-permutably embedded in G, then G is p-nilpotent.*

Theorem 3.18. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, where $\mathfrak{U}$ is the class of all supersolvable groups. A group $G \in \mathfrak{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of H either has the semi cover-avoiding property or is E-supplemented in G.*

Proof: The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

(1) G has a minimal normal subgroup $N \leq H$ and N is an elementary abelian p-group, where p is the largest prime in $\pi(H)$.

By the hypothesis of the theorem, every maximal subgroup of any noncyclic Sylow subgroup of H either has the semi cover-avoiding property or is E-supplemented in G. Consequently, by Lemmas 2.1 and 2.2 every one also either has the semi cover-avoiding property or is E-supplemented in H. Applying Corollary 3.4, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and P a Sylow p-subgroup of H. Then P is normal in H. Obviously, P is normal in G. Therefore, G has a minimal normal subgroup $N \leq H$ and N is an elementary abelian p-group.

(2) $G/N \in \mathfrak{F}$ and $N = P$ is the Sylow p-subgroup of H.

First, we want to prove that G/N satisfies the hypothesis of the theorem. In fact, $(G/N)/(H/N) \cong G/H \in \mathfrak{F}$. Let P_1/N be a maximal subgroup of the Sylow p-subgroup P/N of H/N. Then P_1 is a maximal subgroup of the Sylow p-subgroup P of H. If P/N is noncyclic, then P is also noncyclic. By the hypothesis of the theorem, P_1 either has the semi cover-avoiding property or is E-supplemented in G. By Lemmas 2.1 and 2.2, P_1/N either has the semi cover-avoiding property or is E-supplemented in G/N. Let M_1/N be a maximal subgroup of the noncyclic Sylow q-subgroup QN/N of H/N, where $q \neq p$ and Q is a noncyclic Sylow q-subgroup of H. It is clear that $M_1 = Q_1N$, where Q_1 is a maximal subgroup of Q. By the hypothesis of the theorem, Q_1 either has the semi cover-avoiding property or is E-supplemented in G. Hence M_1/N either has the semi cover-avoiding property or is E-supplemented in G/N by Lemmas 2.1 and 2.2. We now have proved that G/N satisfies the hypothesis of the theorem. By the minimal choice of G, we have $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. By Lemma 2.7, it follows that $P = F(P) = N$.

(3) $|N| > p$.

This follows from Lemma 2.8.

(4) The final contradiction.

Let M be a maximal subgroup of N. By the hypothesis, M either has the semi cover-avoiding property or is E-supplemented in G.

Case I: M is E-supplemented in G. Then there is a subgroup T of G such that $G = MT$ and $M \cap T \leq M_{eG}$. Thus $G = NT$ and $N = N \cap MT = M(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is minimal normal in G, $N \cap T = N$. It follows that $T = G$ and so $M = M_{eG}$. In view of Lemma 2.11, M is s-quasinormal in G. By Lemma 2.10, $O^p(G) \leq N_G(M)$. Thus

$M \trianglelefteq G_pO^p(G) = G$. It follows that $M = 1$ and so $|N| = p$, which contradicts step (3).

Case II: M has the semi cover-avoiding property in G. Then there exists a chief series of G

$$1 = G_0 < G_1 < \cdots G_{n-1} < G_n = G$$

such that M covers or avoids every factor G_j/G_{j-1}. Since N is minimal normal in G, there exists j such that $G_j \cap N = N$ and $G_{j-1} \cap N = 1$. If M covers G_j/G_{j-1}, then $MG_j = MG_{j-1}$ and so $MG_j \cap N = MG_{j-1} \cap N$. Hence $M(G_j \cap N) = M(G_{j-1} \cap N)$, i.e., $MN = M$, a contradiction. If M avoids G_j/G_{j-1}, then $M \cap G_j = M \cap G_{j-1}$ and so $M \cap G_j \cap N = M \cap G_{j-1} \cap N$, i.e., $M = 1$. It follows $|N| = p$, a contradiction. □

Corollary 3.19 ([13, Theorem 3.6])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal Hall subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of H has the semi cover-avoiding property in G, then $G \in \mathfrak{F}$.*

Corollary 3.20 ([21, Theorem 3.3])**.** *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-normal in G, then G is supersolvable.*

Corollary 3.21 ([16, Theorem 4.2])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 3.22 ([23, Theorem 4.1])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of H is c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 3.23 ([32, Theorem 3.3])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal subgroup H of a group G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of H is S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Corollary 3.24 ([14, Theorem 3.3])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. A group $G \in \mathfrak{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of H is either s-quasinormally embedded or c-normal in G.*

Theorem 3.25. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathfrak{F}$. If every maximal subgroup of each non-cyclic Sylow subgroup of $F(N)$ either has the semi cover-avoiding property or is E-supplemented in G, then $G \in \mathfrak{F}$.*

Proof: Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) $\Phi(G) \cap N = 1$.

Assume that $\Phi(G) \cap N \neq 1$. Then there exists a prime p dividing the order of $\Phi(G) \cap N$. Let P_0 be the Sylow p-subgroup of $\Phi(G) \cap N$. Then $P_o \trianglelefteq G$. Since $(G/P_0)/(N/P_0) \cong G/N$, it follows that $(G/P_0)/(N/P_0) \in \mathfrak{F}$. By [1, p.270 Satz 3.5], $F(N/P_0) = F(N)/P_0$. Let P_1/P_0 be a maximal subgroup of the Sylow p-subgroup P/P_0 of $F(N)/P_0$. Then P_1 is a maximal subgroup of the Sylow p-subgroup P of $F(N)$. If P/P_0 is non-cyclic, then P is non-cyclic. By the hypothesis, P_1 either has the semi cover-avoiding property or is E-supplemented in G. Hence P_1/P_0 either has the semi cover-avoiding property or is E-supplemented in G/P_0 by Lemmas 2.1 and 2.2. Set Q_*/P_0 be a maximal subgroup of the non-cyclic Sylow q-subgroup of $F(N)/P_0$, where $p \neq q$. It is clear that $Q_* = Q_1^* P_0$, where Q_1^* is a maximal subgroup of the non-cyclic Sylow q-subgroup of $F(N)$. Then Q_1^* either has the semi cover-avoiding property or is E-supplemented in G. Hence $Q_1^* P_0/P_o$ either has the semi cover-avoiding property or is E-supplemented in G/P_o by Lemmas 2.1 and 2.2. Now we have proved that G/P_0 satisfies the hypotheses of the theorem. Therefore $G/P_0 \in \mathfrak{F}$ by minimal choice of G. Since $P_o \leq \Phi(G)$ and $\mathfrak{F}$ is a saturated formation, we have that $G \in \mathfrak{F}$, a contradiction.

(2) $F(N) = L_1 \times L_2 \times \cdots \times L_n$, where every L_i is a minimal normal subgroup of G with prime order.

If $N = 1$, nothing need to be proved. So assume $N \neq 1$. Then $F(N) \neq 1$ by the solvability of N. By Lemma 2.7, $F(N)$ is the direct product of some minimal normal subgroups of G. Let P be the Sylow p-subgroup of $F(N)$. We can denote $P = R_1 \times R_2 \times \cdots \times R_m$, where every R_i is a minimal normal subgroup of G. We will show that $|R_i| = p$ $(i = 1, 2, \cdots, m)$. If not, then there exists an index i such that $|R_i| > p$. Without loss of generality, suppose that $i = 1$. Since $R_1 \nleq \Phi(G)$, there exist a maximal subgroup M of G such that $G = R_1 M$ and $R_1 \cap M = 1$. Then $G_p = R_1 M_p$. Pick a maximal subgroup G_p^* of G_p containing M_p. Then $|R_1 : G_p^* \cap R_1| = |R_1 G_p^*/G_p^*| = |G_p : G_p^*| = p$. Hence $R_1^* = G_p^* \cap R_1$ is a maximal subgroup of R_1. This implies that $P^* = R_1^* R_2 \cdots R_m$ is a maximal subgroup of P. Obviously, P is not cyclic. By the hypothesis, P^* either has the semi cover-avoiding property or is E-supplemented in G. Let $K = R_2 \times \cdots \times R_m$.

Case I: P^* has the semi cover-avoiding property in G. By Lemma 2.1, P^*/K has the semi cover-avoiding property in G/K. Suppose that P^*/K cover-avoids a chief series $1 = \overline{K} \lhd G_1/K = \overline{G_1} \lhd \cdots \lhd G/K = \overline{G_n}$ of G/K. Let i be the smallest number in $\{1, 2, \cdots, n-1\}$ such that $\overline{G_{i+1}}/\overline{G_i}$ was covered by P^*/K in above chief series. Then we have $G_i \cap P^* = K$ and $G_{i+1} \leq G_i P^* = G_i R_1^*$. Hence $G_{i+1} = G_i(R_1^* \cap G_{i+1})$ and $R_1^* \cap G_{i+1} > 1$. Since R_1 is a minimal normal subgroup of G, we have $R_1 \leq G_{i+1}$ and $R_1 \cap G_i = 1$. Hence $|R_1| = |G_{i+1}/G_i| = |R_1^* \cap G_{i+1}| < |R_1|$, a contraction. Therefore, P^*/K does not cover any chief factor in above chief series. It follows that $P^*/K = 1$ and $|R_1| = p$, a contraction.

Case II: P^* is E-supplemented in G. Then there exists a subgroup K such that $G = P^* T$ and $P^* \cap T \leq (P^*)_{eG}$. Obviously, $(P^*) \trianglelefteq G_p$. By Lemma 2.11, $(P^*)_{eG} = (P^*)_{sG}$. In view of [12, Lemma 2.10], $(P^*)_{sG} = (P^*)_G$. Denote $T_1 = KT$. Then $G = R_1^* T_1$ and $R_1^* \cap T_1 = R_1^* \cap T_1 \cap P^* = R_1 \cap K(P^* \cap T) \leq R_1 \cap K(P^*)_G = R_1 \cap K = 1$. Since $R_1 \cap T_1$ is normal in G, we have $R_1 \cap T_1 = 1$ or R_1 by the minimality of R_1. If the former holds, then $R_1 = R_1 \cap R_1^* T_1 = R_1^*(R_1 \cap T_1) = R_1^*$, a contraction. Hence $R_1 \cap T_1 = R_1$, i.e., $R_1 \leq T_1$. It follows that $R_1^* = 1$ and so $|R_1| = p$, a contraction.

(3) The final contradiction.

It is easy to see that $G/C_G(L_i)$ is abelian by step (2). Since $C_G(F(N)) = \bigcap_{i=1}^{n} C_G(L_i)$, we have that $G/C_G(F(N))$ is abelian. Hence $G/C_G(F(N)) \in \mathfrak{U} \subseteq \mathfrak{F}$. By the assumption, $G/N \in \mathfrak{F}$, which implies that $G/N \cap C_G(F(N)) = G/C_N(F(N)) \in \mathfrak{F}$ by the properties of formations. From the solvability of N, $C_N(F(N)) \leq F(N)$. In view of step (2), $F(N)$ is abelian. Then $F(N) \leq C_N(F(N))$. Thus $F(N) = C_N(F(N))$ and $G/F(N) \in \mathfrak{F}$. By Theorem 3.18, $G \in \mathfrak{F}$, a contradiction. □

Corollary 3.26 ([24, Theorem 1])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(N)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 3.27 ([18, Theorem 4.5])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(N)$ are c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 3.28 ([25, Theorem 1.6])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(N)$ are complemented in G, then $G \in \mathfrak{F}$.*

Corollary 3.29 ([32, Corollary 3.4])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(N)$ are complemented in G, then $G \in \mathfrak{F}$.*

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