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Common Fixed Point Theorems for Generalized $G-\eta-\chi-$ Contractive Type Mappings With Applications

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Abstract: Samet et. al. (Nonlinear Anal. 75, 2012, 2154-2165) introduced the concept of $\alpha-\psi-$ contractive type mappings in metric spaces. In 2013, Alghamdi et. al.[2] introduced the concept of $G-\beta-\psi-$contractive type mappings in G-metric spaces. Our aim is to introduce new concept of generalized $G-\eta-\chi-$contractive pair of mappings. Further, we study some fixed point theorems for such mappings in complete G-metric spaces. As an application, we further establish common fixed point theorems for G-metric spaces for cyclic contractive mappings.

Key Words: Common fixed point, Complete G-metric space, Contractive type mapping, Cyclic mappings.

Contents

1. Introduction

In the last few decades, fixed point theory has been one of the most interesting research fields in nonlinear functional analysis. It has wide applications in many disciplines like economics, sports, medical sciences etc. In 1922, Banach [5] gives a contraction principle, in which he proved that each contraction in a complete metric space has a unique fixed point. Later in 1968, Kannan [10] studied a new type of contractive mapping. Since then, many authors have directed their attention to this field and have generalized the Banach fixed point theorem in various ways (see, e.g., [1-32]). On the other hand, in 2006, Mustafa and Sims [15] introduced the notion of G-metric space and characterized the Banach fixed point theorem in the context of G-metric space.

In 2012, Samet et al. [27] introduced the concept of $\alpha-\psi-$contractive type mappings in metric spaces, which extends and generalizes the existing fixed point results in the literature, in particular the Banach contraction principle. Recently, Alghamdi et al. [2] introduced a new concept of $G-\beta-\psi-$contractive type mappings in G-metric spaces.

In this paper, some coincidence and common fixed point theorems are obtained for the generalized $G-\eta-\chi-$ contractive pair of mappings. Our results unify

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and generalize the results derived by Alghamdi et al. [2], Mustafa [14] and various other related results in the literature. Moreover, from our main results, we will derive results for cyclic contractive mappings.

Now, we introduce some notations and definitions that will be used subsequently.

Definition 1.1. (See [27]). Let Ψ denote the family of all functions $\chi : [0,\infty) \to [0,\infty)$ which satisfy the following:

(i) $\sum_{n=1}^{\infty} \chi^n(t) < \infty$ for each $t > 0$, where χ^n is the nth iterate of χ;

(ii)χ is non-decreasing.

Clearly $\chi(t) < t$ for any $t > 0$.

Recently, Samet et al. [27] introduced the following notions: Definition 1.2. Let (X, d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an $\alpha - \psi$–contractive mapping if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 1.3. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. T is said to be α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Let f, g be two self maps on a non-empty set X. We denote by $C(g, f)$ the set of coincidence points of g and f, that is,

$$C(g, f) = \{t \in X : gt = ft\}.$$

2. Main Results

We start the main results by introducing the new concepts of η-admissible w.r.t. g mapping and generalized $G - \eta - \chi$–contractive pair of mappings.

Definition 2.1. Let $f, g : X \to X$ and $\eta : X \times X \times X \to [0,\infty)$. We say that f is η-admissible w.r.t. g if for all x, y, z in X, we have

$$\eta(gx, gy, gz) \geq 1 \Rightarrow \eta(fx, fy, fz) \geq 1.$$

Remark 2.2. Clearly, every η-admissible mappings is η-admissible w.r.t. g when $g = I$.

The following example shows that a mapping which is η-admissible w.r.t. g may not be η-admissible.

Example 2.3. Let $X = [1,\infty)$. Define the mapping $\eta : X \times X \times X \to [0,\infty)$ by

$$\eta(x, y, z) = \begin{cases} e, & if\ x \geq y \geq z \\ \frac{1}{5}, & otherwise \end{cases}$$

Also, define the mappings $f, g : X \to X$ by $f(x) = e^x$ and $g(x) = \frac{1}{x+1}$ for all x in X.

Suppose that $\eta(x, y, z) \geq 1$. This implies from the definition of η that $x > y > z$ which further implies that $-x < -y < -z$, that is, $e^x < e^y < e^z$. Thus, $\eta(fx, fy, fz) \ngeq 1$, that is, f is not η-admissible.

Now, we prove that f is η-admissiblew.r.t. g. Let us suppose that $\eta(gx,gy,gz) \geq 1$. So, $\eta(gx,gy,gz) \geq 1 \Rightarrow gx > gy > gz \Rightarrow \frac{1}{x+1} > \frac{1}{y+1} > \frac{1}{z+1}$,that is, $z > y > x$, or, $-z < -y < -x$, which implies that $e^{-z} < e^{-y} < e^{-x}$, that is, $f(z) < f(y) < f(x)$, which implies that, $\eta(fx,fy,fz) \geq 1$.

Therefore, f is η-admissible w.r.t. g. In what follows, we present an example of η-admissible w.r.t. g mappings.

Example 2.4. Let $X = [1,\infty)$. Define the mapping $\eta : X \times X \times X \to [0,\infty)$ by

$$\eta(x,y,z) = \begin{cases} 3, & if\ x \geq y \geq z \\ 0, & otherwise. \end{cases}$$

Also, define the mapping $f,g : X \to X$ by $f(x) = Inx$ and $g(x) = x^3$ for all x in X. Thus, the mapping f is η-admissible w.r.t. g.

Next, we present the new notion of generalized $G-\eta-\chi-$contractive pair of mappings as follows:

Definition 2.5. Let (X,G) be a G-metric space and $f,g : X \to X$ be given mappings. We say that the pair (f,g) is generalized $G-\eta-\chi-$contractive pair of mappings if there exists two functions $\eta : X \times X \times X \to [0,\infty)$ and $\chi \in \psi$ such that for all x,y,z in X, we have

$$\eta(gx,gy,gz)G(fx,fy,fz) \leq \chi(M(x,y,z)),\ forall\ x,y,z \in X, \tag{2.1}$$

$$\begin{aligned} M(x,y,z) = max\{&G(gx,gy,gz), G(gx,fx,fx), \frac{1}{2}G(gx,fy,fy), \frac{1}{2}G(gx,fz,fz), \\ &G(gy,fy,fy), G(gy,fx,fx), G(gy,fz,fz), G(gz,fz,fz), \\ &G(gz,fx,fx), G(gz,fy,fy)\}. \end{aligned}$$

Our first result is the following coincidence point theorem.

Theorem 2.6. Let (X,G) be a G-metric space and $f,g : X \to X$ be such that $fX \subseteq gX$. Assume that the pair (f,g) is generalized $G-\eta-\chi-$contractive pair of mappings and the following conditions hold:

(i) f is η-admissible w.r.t. g.

(ii) there exists $x_0 \in X$ such that $\eta(gx_0,fx_0,fx_0) \geq 1$.

(iii) if $\{gx_n\}$ is a sequence in X such that $\eta(gx_n,gx_{n+1},gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in gX$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\eta(gx_{n(k)},gz,gz) \geq 1$ for all k.

Also suppose that gX is closed. Then f and g have a coincidence point.

Proof. In view of condition (ii), let $x_0 \in X$be such that $\eta(gx_0,fx_0,fx_0) \geq 1$. Since $fX \subseteq gX$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$ and $x_2 \in X$ where $fx_1 = gx_2$. By induction, we can define a sequence $\{y_n\}$ in X as follows:

$$y_n = fx_n = gx_{n+1},\ n = 0,1,2,... \tag{2.2}$$

Since f is η-admissible w.r.t. g, we have

$$\eta(gx_0,fx_0,fx_0) = \eta(gx_0,gx_1,gx_1) \geq 1 \Rightarrow \eta(fx_0,fx_1,fx_1) = \eta(gx_1,gx_2,gx_2) \geq 1$$

Using mathematical induction, we get $\eta(gx_n, gx_{n+1}, gx_{n+1}) \geq 1$, which implies that

$$\eta(y_{n-1}, y_n, y_n) \geq 1, \ n = 0, 1, 2, ... \tag{2.3}$$

If $y_n = y_{n+1}$ for some n, then by (2), we have

$$fx_{n+1} = gx_{n+1}, \ n = 0, 1, 2, ...$$

implies that, f and g have a coincidence point at $x = x_{n+1}$, and so we have finished the proof. So, suppose that, $y_n \neq y_{n+1}$ for any n. From (1), we have

$$\begin{aligned} G(y_n, y_{n+1}) &= G(fx_n, fx_{n+1}, fx_{n+1}) \\ &\leq \eta(gx_n, gx_{n+1}, gx_{n+1})G(fx_n, fx_{n+1}, fx_{n+1}) \\ &= \eta(y_{n-1}, y_n, y_n)G(y_n, y_{n+1}, y_{n+1}) \\ &\leq \chi(M(x_n, x_{n+1}, x_{n+1})), \end{aligned} \tag{2.4}$$

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= max\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, fx_n, fx_n), \\ &\quad \frac{1}{2}G(gx_n, fx_{n+1}, fx_{n+1}), G(gx_{n+1}, fx_n, fx_n), \\ &\quad G(gx_{n+1}, fx_{n+1}, fx_{n+1})\} \\ &= max\{G(y_{n-1}, y_n, y_n), \frac{1}{2}G(y_{n-1}, y_{n+1}, y_{n+1}), \\ &\quad G(y_n, y_n, y_n), G(y_n, y_{n+1}, y_{n+1})\}. \end{aligned} \tag{2.5}$$

We will have different cases: Case (1) If $M(x_n, x_{n+1}, x_{n+1}) = G(y_n, y_{n+1}, y_{n+1})$, then, from (4), we get

$$G(y_n, y_{n+1}, y_{n+1}) \leq \chi(G(y_n, y_{n+1}, y_{n+1})) < G(y_n, y_{n+1}, y_{n+1}),$$

since χ is monotonically decreasing, a contradiction.

Case (2) If $M(x_n, x_{n+1}, x_{n+1}) = \frac{1}{2}G(y_{n-1}, y_{n+1}, y_{n+1})$, then in this case, we have

$$max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1})\} < \frac{1}{2}G(y_{n-1}, y_{n+1}, y_{n+1}),$$

which implies that

$$G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}) < G(y_{n-1}, y_{n+1}, y_{n+1}). \tag{2.6}$$

But, from the property of G-metric, we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}). \tag{2.7}$$

Thus, from (6) and (7), we see that case (2) is impossible. Then, we must have the case

$$M(x_n, x_{n+1}, x_{n+1}) = G(y_{n-1}, y_n, y_n). \tag{2.8}$$

Thus, for all $n \geq 1$ and from (4), we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq \chi(G(y_{n-1}, y_n, y_n)).$$

Continuing this process inductively, we obtain

$$G(y_n, y_{n+1}, y_{n+1}) \leq \chi^n(G(y_0, y_1, y_1)), forall\ n \geq 1. \tag{2.9}$$

From (9), for all $k \geq 1$, we have

$$\begin{aligned} G(y_n, y_{n+k}, y_{n+k}) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad +... + G(y_{n+k-1}, y_{n+k}, y_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \chi^p(G(y_0, y_1, y_1)) \\ &\leq \sum_{p=n}^{\infty} \chi^p(G(y_0, y_1, y_1)). \end{aligned} \tag{2.10}$$

Letting $p \to \infty$ in (10), we obtain that $\{y_n\}$ is a Cauchy sequence in (X, G). Since by (2), we have $\{fx_n\} \subseteq \{gx_{n-1}\} \subseteq gX$ and gX is closed, there exists z in X such that

$$lim_{n\to\infty} gx_n = gZ. \tag{2.11}$$

Now, we show that z is a coincidence point of f and g. On the contrary, assume that $G(gz, fz, fz) > 0$. Since by condition (iii) and (11), we have $\eta(gx_{n(k)}, gz, gz) > 0$ for all k. From (1), we have

$$\begin{aligned} G(gz, fz, fz) &\leq G(gz, fx_{n(k)}, fx_{n(k)}) + G(fx_{n(k)}, fz, fz) \\ &\leq G(gz, fx_{n(k)}, fx_{n(k)}) + \eta(gx_{n(k)}, gz, gz)G(fx_{n(k)}, fz, fz) \\ &\leq G(gz, fx_{n(k)}, fx_{n(k)}) + \chi(M(x_{n(k)}, z, z)). \end{aligned}$$

Letting $k \to \infty$, we get

$$G(gz, fz, fz) \leq G(gz, gz, gz) + \chi(lim_{k\to\infty} M(x_{n(k)}, z, z)), \tag{2.12}$$

where

$$\begin{aligned} M(x_{n(k)}, z, z) = max\{&G(gx_{n(k)}, fx_{n(k)}, fx_{n(k)}), \frac{1}{2}G(gx_{n(k)}, fz, fz), \\ &G(gz, fz, fz), G(gz, fx_{n(k)}, fx_{n(k)})\}, \end{aligned}$$

that is,

$$\begin{aligned} \lim_{k\to\infty} M(x_{n(k)}, z, z) &= max\{G(gz, gz, gz), G(gz, gz, gz), \\ &\quad \frac{1}{2}G(gz, fz, fz), G(gz, fz, fz), G(gz, gz, gz)\} \\ &= G(gz, fz, fz). \end{aligned} \tag{2.13}$$

From (12) and (13), we have $G(gz, fz, fz) \leq 0+\chi(G(gz, fz, fz)) < G(gz, fz, fz)$, a contradiction. Hence, our supposition is wrong and $G(gz, fz, fz) = 0$, that is, $fz = gz$. This shows that f and g have a coincidence point. The next theorem shows that under additional hypothesis, we can deduce the existence and uniqueness of a common fixed point.

Theorem 2.7. In addition to the hypothesis of Theorem 2.6, suppose for all $u, v \in C(g, f)$, there exists $w \in X$ such that $\eta(gu, gw, gw) \geq 1$ $\eta(gv, gw, gw) \geq 1$ and f, g commute at their coincidence points. Then f and g have a unique common fixed point.

Proof. We need to consider following three steps:

Step 1. We claim that if $u, v \in C(g, f)$, then $gu = gv$. By hypothesis, there exists $w \in X$ such that

$$\eta(gu, gw, gw) \geq 1 \text{ and } \eta(gv, gw, gw) \geq 1. \tag{2.14}$$

Since $fX \subseteq gX$, we define the sequence $\{w_n\}$ in X by $gw_{n+1} = fw_n$ for all $n \geq 0$ and $w_0 = w$. Since f is η-admissible w.r.t. g, we have

$$\eta(gu, gw_n, gw_n) \geq 1 \text{ and } \eta(gv, gw_n, gw_n) \geq 1 \; forall \; n \geq 0. \tag{2.15}$$

From (1) and (15), we have

$$\begin{aligned} G(gu, gw_{n+1}, gw_{n+1}) &= G(fu, fw_n, fw_n) \\ &\leq \eta(gu, gw_n, gw_n)G(fu, fw_n, fw_n) \\ &\leq \chi(M(u, w_n, w_n)), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} M(u, w_n, w_n) &= max\{G(gu, gw_n, gw_n), G(gu, fu, fu), \frac{1}{2}G(gu, fw_n, fw_n), \\ &\quad G(gw_n, fw_n, fw_n), G(gw_n, fu, fu)\} \\ &= max\{G(gu, gw_n, gw_n), \frac{1}{2}G(gu, fw_n, fw_n), \\ &\quad G(gw_n, fw_n, fw_n), G(gw_n, gu, gu)\} \end{aligned}$$

We will have different cases: Case (i) If

$$\begin{aligned} M(u, w_n, w_n) &= \frac{1}{2}G(gu, fw_n, fw_n) \\ &= G(gu, gw_{n+1}, gw_{n+1}). \end{aligned}$$

Thus, from (16), we get

$$\begin{aligned} G(gu, gw_{n+1}, gw_{n+1}) &\leq \chi(\frac{1}{2}G(gu, gw_{n+1}, gw_{n+1})) \\ &< \frac{1}{2}G(gu, gw_{n+1}, gw_{n+1}), \end{aligned}$$

a contradiction.

Case (ii) If

$$M(u, w_n, w_n) = G(gw_n, fw_n, fw_n).$$

From (16), we have

$$\begin{aligned} G(gu, gw_{n+1}, gw_{n+1}) &\leq \chi(G(gw_n, fw_n, fw_n)) \\ &= \chi(G(gw_n, gw_{n+1}, gw_{n+1})) \\ &< G(gw_n, gw_{n+1}, gw_{n+1}). \end{aligned} \tag{2.17}$$

Using (11), we have

$$lim_{n\to\infty} gw_n = gz. \tag{2.18}$$

Letting $n \to \infty$ in (17), we get

$$lim_{n\to\infty} G(gu, gw_{n+1}, gw_{n+1}) = 0. \tag{2.19}$$

Case (iii) $M(u, w_n, w_n) = G(gw_n, gu, gu)$. From (16), we get

$$\begin{aligned} G(gu, gw_{n+1}, gw_{n+1}) &\leq \chi(G(gw_n, gu, gu)) \\ &< G(gw_n, gu, gu) \end{aligned}$$

Making $n \to \infty$ and using (18), we get $G(gu, gz, gz) < G(gz, gu, gu)$, a contradiction. Case (iv) If $M(u, w_n, w_n) = G(gu, gw_n, gw_n)$. From (16), we have $G(gu, gw_{n+1}, gw_{n+1}) \leq \chi(G(gu, gw_n, gw_n))$, for all n. This implies that

$$G(gu, gw_{n+1}, gw_{n+1}) \leq \chi^n(G(gu, gw0, gw0)), \ forall \ n \geq 1. \tag{2.20}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} G(gu, gw_{n+1}, gw_{n+1}) = 0. \tag{2.21}$$

Similarly, we can prove that

$$lim_{n\to\infty} G(gv, gw_{n+1}, gw_{n+1}) = 0. \tag{2.22}$$

From (19), (21) and (22), we get $gu = gv$.

Step 2. Existence of a common fixed point:

Let $u \in C(g, f)$, that is, $gu = fu$. Owing to the commuatativity of f and g at their coincidence points, we get

$$g^2u = gfu = fgu. \tag{2.23}$$

Let $gu = t$, then from (23), we have $gt = ft$. Thus, t is a coincidence point of f and g. Now, from step 1, we have $gu = gt = t = ft$. Then, t is a common fixed point of f and g.

Step 3. Uniqueness:

Suppose that t_1 is another common fixed point of f and g. Then, $t_1 \in C(g, f)$. By step 1, we have $t_1 = gt_1 = gt = t$. This completes the proof.

Example 2.8. Let $X = [0, \infty)$ equipped with the metric $G(x, y, z) = max\{|x-y|, |y-z|, |z-x|\}$ for all x, y, z in X. Suppose that $x \geq y \geq z$, then $G(x, y, z) = |z - x|$.

Define the mapping $f, g : X \to X$ by

$$fx = \begin{cases} 2x - \frac{5}{2}, \ if \ x > 1 \\ \frac{x}{10}, \ if \ 0 \leq x \leq 1 \end{cases}$$

and $gx = \frac{x}{3}$, for all $x \in X$. Now, we define the mapping $\eta : X \times X \times X \to [0, \infty)$ by

$$\eta(x, y, z) = \begin{cases} 1, \ if x, y, z \in [0, 1] \\ 0, \ otherwise \end{cases} .$$

Clearly, the pair (f, g) is a generalized $G - \eta - \chi-$contractive pair of mappings with $\chi(t) = \frac{t}{3}$, for all $t \geq 0$. In fact, for all x, y, z in X, we have

$$\begin{aligned} \alpha(gx, gy, gz)G(fx, fy, fz) &= |\frac{z}{10} - \frac{x}{10}| \\ &\leq \frac{1}{3}|\frac{z}{3} - \frac{x}{3}| \\ &= \frac{1}{3}G(gx, gy, gz) \quad \leq M(x, y, z) = \chi(M(x, y, z)). \end{aligned}$$

Moreover, there exists $x_0 \in X$ such that $\eta(gx_0, fx_0, fx_0) \geq 1$. Infact, for $x_0 = 1$, we have $\eta(\frac{1}{3}, \frac{1}{10}, \frac{1}{10}) = 1$. Now, it remains to show that f is η-admissible w.r.t. g. In so doing, let $x, y, z \in X$ such that $\eta(gx, gy, gz) \geq 1$. This implies that $gx, gy, gz \in [0, 1]$ and by the definition of g, we have $x, y, z \in [0, 1]$. Therefore, by the definition of f and η, we have $fx = \frac{x}{10} \in [0, 1], fy = \frac{y}{10} \in [0, 1], fz = \frac{z}{10} \in [0, 1]$ and $\eta(fx, fy, fz) = 1$. Thus, f is η-admissible w.r.t. g . Clearly, $fX \subseteq gX$ and gX is closed. Finally, let $\{x_n\}$ be a sequence in X such that$\eta(gx_n, gx_{n+1}, gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in gX$ as $n \to \infty$. Since $\eta(gx_n, gx_{n+1}, gx_{n+1}) \geq 1$ for all n, by the definition of η, we have $gx_n \in [0, 1]$ for all n and $z \in [0, 1]$. Then, $\eta(gx_n, gz, gz) \geq 1$. Now, all the hypothesis of Theorem 2.6 are satisfied. Consequently, f and g have a coincidence point. Here, 0 is the coincidence point of f and g. Also, clearly all the hypothesis of Theorem 2.7 are satisfied. In this example 0 is the unique common fixed point of f and g.

3. Consequences

In this section, we will show that many existing results in the literature can be easily obtained from our Theorem 2.7.

3.1. Standard fixed point theorems: By taking $\eta(x, y, z) = 1$ for all x, y, z in X in Theorem 2.7, we obtain immediately the following fixed point theorems.

Corollory 3.1. Let (X, G) be a complete G-metric space and $f, g : X \to X$

be such that $fX \subseteq gX$. Suppose that there exists a function $\chi \in \psi$ such that $G(fx, fy, fz) \leq \chi(M(x, y, z))$, for all x, y, z in X. Also, suppose gX is closed. Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

By taking $g = I$ in corollory 3.1, we obtain immediately the following fixed point results.

Corollory 3.2.(see Mustafa[14]): Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all x, y, z in X.

$$G(Tx, Ty, Tz) \leq kG(x, y, z). \tag{3.1}$$

where $0 \leq k \leq 1$. Then, T has a unique fixed point.

Corollory 3.3.(see Mustafa[14]): Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all x, y in X:

$$G(Tx, Ty, Tz) \leq kG(x, y, y), \tag{3.2}$$

where $0 \leq k < 1$. Then, T has a unique fixed point.

Remark 3.4. The condition 24 implies condition 25. The converse is true only if $k \in [0, \frac{1}{2})$. For details see [14].

3.2. Cyclic contraction:

In 2003, Kirk et al.[13] generalizes the Banach contraction principle by introducing cyclic representations and cyclic contractions. A mapping $T : A_1 \cup A_2 \to A_1 \cup A2$ is called cyclic if $T(A_1) \subseteq A_2$ and $T(A_1) \subseteq A_2$, where A_1 and A_2 are non-empty subsets of a metric space (X, d). In the same way, we can introduce for G-metric space (X, G). Moreover, T is called cyclic contraction if there exists $k \in (0, 1)$ such that $G(Tx, Ty, Ty) \leq kG(x, y, y)$ for all $x \in A_1$, $y \in A_2$.

Now, we prove our results for cyclic contractive mappings in a G-metric space.

Theorem 3.5. Let (X, G) be a complete G-metric space, A and B are two non-empty closed subsets of X, and $f, g : Y \to Y$ be two mappings, where $Y = A \cup B$. Suppose that the following conditions hold:

(i) $g(A)$ and $g(B)$ are closed;

(ii)$f(A) \subseteq g(B)$ and $f(B) \subseteq g(A)$;

(iii)g is one-to-one;

(iv)there exists a function $\chi \in \Psi$ such that $G(fx, fy, fy) \leq \chi(M(x, y, y))$, for all $x \in A$, $y \in B$.

Then, f and g have a coincidence point $z \in A \cap B$. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point that belongs to $A \cap B$.

Proof. Due to the fact that g is one-to-one, condition (iv) is equivalent to $G(fx, fy, fy) \leq \chi(M(x, y, y))$, for all $gx \in gA$, $gy \in gB$. Notice that (Y, G) is a complete G-metric space because A, B are closed subsets of a complete G-metric space (X, G). Define the mapping $\eta : Y \times Y \times Y \to [0, \infty)$ by

$$\eta(x, y, y) = \begin{cases} 1, & if\ (x, y) \in (gA \times gB) \cup (gB \times gA) \\ 0, & otherwise \end{cases}$$

Due to the definition of η and condition (iv), we can write

$$\alpha(gx, gy, gy)G(fx, fy, fy) \leq \chi(M(x, y, y)),$$

for all $gx \in gA$, $gy \in gB$. Thus, the pair (f, g) is generalized $G-\eta-\chi-$contractive pair of mappings. By using condition (ii), we can show that $fY \subseteq gY$. Moreover, gY is closed. Next, we proceed to show that f is η-admissible w.r.t. g. Let $(gx, gy) \in Y \times Y$ such that $\eta(gx, gy, gy) \geq 1$, that is,

$$(gx, gy) \in (gA \times gB) \cup (gB \times gA).$$

Since g is one-to-one, this implies that

$$(x, y) \in (A \times B) \cup (B \times A).$$

So, from condition (ii), we have $(fx, fy) \in (gB \times gA) \cup (gA \times gB)$, that is, $\eta(gx, gy, gy) \geq 1$. This implies that f is η-admissible w.r.t. g. Now, let $\{xn\}$ be a sequence in X such that $\eta(gx_n, gx_{n+1}, gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in gX$ as $n \to \infty$. From the definition of η, we have

$$(gx_n, gx_n+1) \in (gA \times gB) \cup (gB \times gA).$$

Since $(gA \times gB) \cup (gB \times gA)$ is a closed set, we get that $(gz, gz) \in (gA \times gB) \cup (gB \times gA)$, which implies that, $gz \in gA \cap gB$. Therefore, we get that $\eta(gx_n, gz, gz) \geq 1$ for all n. Now, let $a \in A$. We need to show that $\eta(ga, fa, fa) \geq 1$. Indeed, from condition (ii), we have $fa \in gB$. Since, $ga \in gA$, we get $(ga, fa) \in gA \times gB$, which implies that $\eta(ga, fa, fa) \geq 1$.

Now, all the hypotheses of Theorem 2.6 are satisfied. Hence, we deduce that f and g have a coincidence point $z \in A \cap B$, that is, $fz = gz$. If $z \in A$, from (ii), $fz \in gB$. On the other hand, $fz = gz \in gA$. Then, we have $gz \in gA \cap gB$, which implies from the one-one property of g that $z \in A \cap B$. Similarly, if $z \in B$, we obtain that $z \in A \cap B$.

Notice that if x is a coincidence point of f and g, then $x \in A \cap B$. Finally, let $x, y \in C(g, f)$, that is, $x, y \in A \cap B$, $gx = fx$ and $gy = fy$. Now, from our observation, we have $w = x \in A \cap B$, which implies that, $gw \in g(A \cap B) = gA \cap gB$ due to the fact that g is one-to-one. Then, we get that $\eta(gx, gw, gw) \geq 1$ and $\eta(gy, gw, gw) \geq 1$. Then our claim holds.

Now, all the hypotheses of Theorem 2.7 are satisfied. So, we deduce that $z \in A \cap B$ is the unique common fixed point of f and g. The following result is the immediate consequence of Theorem 3.4.

Corollary 3.5. Let (X, G) be a complete G-metric space, A and B are two non-empty closed subsets of X and $f, g : Y \to Y$ be two mappings, where $Y = A \cup B$. Suppose that the following conditions hold:

(i)gA and gB are closed,

(ii)$fA \subseteq gB$ and $fB \subseteq gA$,

(iii)g is one-to-one,

(iv)there exists a function $\chi \in \psi$ such that $G(fx, fy, fy) \leq \chi(G(gx, gy, gy))$, for all $x \in A$, $y \in B$.

Then, f and g have a coincidence point $z \in A \cap B$. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point that belongs to $A \cap B$.

Remark 3.6. Letting $g = IX$ in corollary 3.5, we obtain Theorem 3.5 in [2].

Conflict of Interests

The authors declares that there is no conflict of interests regarding the publication of this paper.

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