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Weighted Steklov Problem Under Nonresonance Conditions

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abstract: We deal with the existence of weak solutions of the nonlinear problem $-\Delta_p u + V|u|^{p-2}u = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ which is subject to the boundary condition $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f(x,u)$. Here $V \in L^\infty(\Omega)$ possibly exhibit both signs which leads to an extension of particular cases in literature and f is a Carathéodory function that satisfies some additional conditions. Finally we prove, under and between nonresonance conditions, existence results for the problem.

Key Words: Nonresonnance, p-Laplacian operator, Sobolev trace embedding, Steklov problem, First nonprincipal eigenvalue.

Contents

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with outward unit normal ν on the boundary $\partial\Omega$. For a given number $p > 1$, a bounded function V in Ω and a certain Carathéodory function f, we consider the following nonlinear problem with Steklov boundary condition

$$(P_f) : \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u &= 0 \qquad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(x,u) \qquad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$. This work is mainly motivated by the study of asymmetric elliptic problems with sign-changing weights carried out in [10]. The problem was actually considered recently in [7] for the p-Laplacian operator (in the case $V \equiv 0$), where the existence of the p-harmonic solutions was proved. Also in [5], the case $V \equiv 1$ was treated under and between the first two eigenvalues. In the present paper, we shall adapt and extend the approach in [7] in order to

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derive our main results for a homogeneous perturbation of $-\Delta_p$, the p-Laplacian operator, which is a prototype of quasilinear differential operator.

When $V \equiv 0$ it is known that (1.1) admits solutions, see e.g. [7], [3] and its references for the case $p = 2$. On the other hand, it can also be proved the existence of solutions for (1.1) when one introduces a positive potential V. Allowing V to change sign makes the problem more interesting and challenging as to ensure nontrivial solutions for (P_f), a lot of facts has to be put in consideration: the regularity of the domain and the lack of coercitivity of the functional energy.

The paper is organized as follows. In section two, we give a review of a certain tools and established results that help in our concern. We thereby state properties for the first nonprincipal eigenvalue for an asymmetric Steklov problem with respect to its weights. In the third section, we solve under nonresonance conditions, namely, conditions that involve not only a kind of nonresonance between the first two eigenvalues but also the ones under the first eigenvalue.

2. Relevant background

2.1. The functional framework

Let $\Omega \subset \mathbb{R}^N$ be an open set. The p-Laplace operator ($p > 1$) is the partial differential operator which to every function $u : \Omega \longrightarrow \mathbb{R}$ assigns the function

$$\Delta_p u(x) := div(|\nabla u(x)|^{p-2}\nabla u(x)), \quad x \in \Omega. \tag{2.1}$$

We simply write Δ instead of Δ_2 and call the 2-Laplace operator simply Laplace operator. Throughout this paper, Ω will be a bounded smooth domain of class $C^{2,\alpha}$ where $0 < \alpha < 1$ with outward unit normal ν on the boundary $\partial\Omega$. For a given $p > 1$,

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) | \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1, \ldots, N \right\} \tag{2.2}$$

denotes the usual Sobolev space equipped with the norm

$$||u|| = \left(||u||^p_{L^p(\Omega)} + \sum_{i=1}^{N} ||\frac{\partial u}{\partial x_i}||^p_{L^p(\Omega)} \right)^{1/p}.$$

It is well known that $(W^{1,p}(\Omega), ||\cdot||)$ is a Banach space that is separable and reflexive (see H. Brezis [6]). The value of any $u \in W^{1,p}(\Omega)$ on $\partial\Omega$ is to be understood in the sense of the trace i.e. there is a unique linear and continuous operator $\gamma : W^{1,p}(\Omega) \longrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ such that γ is surjective and for $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, we have $\gamma u = u_{|\partial\Omega}$. For each $u \in W^{1,p}(\Omega)$, one has

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right), \quad |\nabla u| = \left(\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

2.2. Detour on an asymmetric Steklov prob. with sign-changing weights

Here we give general results for an asymmetric Steklov problem with sign-changing weights of the form

$$(P_{V,m,n}) : \begin{cases} \Delta_p u & = & V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} & = & \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of class $C^{2,\alpha}$ where $0 < \alpha < 1$ with outward unit normal ν on the boundary $\partial\Omega$.

$\lambda \in \mathbb{R}$ is regarded as an eigenvalue. We assume that $m, n \in C^\alpha(\partial\Omega)$ for some $0 < \alpha < 1$. Finally, V is a given function in $L^\infty(\Omega)$ which may change sign and $u = u^+ - u^-$ where $u^\pm := \max\{\pm u, 0\}$. To solve (2.4), the authors in [10] have considered the C^1 functionals on $W^{1,p}(\Omega)$

$$E_V(u) := \int_\Omega (|\nabla u|^p + V|u|^p)dx \text{ and } B_{m,n}(u) := \int_{\partial\Omega} [m(u^+)^p + n(u^-)^p]d\sigma \tag{2.5}$$

and introduced the real parameters

$$\lambda_1^D(V) := \inf\left\{E_V(u);\ u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^p dx = 1\right\} > 0, \tag{2.6}$$

$$\beta(V,m) := \inf\left\{E_V(u);\quad ||u||^p_{L^p(\partial\Omega)} = 1 \quad \text{and} \quad B_{m,m}(u) = 0\right\} \tag{2.7}$$

$$\beta(V,n) := \inf\left\{E_V(u);\quad ||u||^p_{L^p(\partial\Omega)} = 1 \quad \text{and} \quad B_{n,n}(u) = 0\right\} \tag{2.8}$$

to bypass arisen coerciveness difficulties of the energy functional due to the sign-changing possibility of the potential V. In brief,

$$\lambda_{\pm 1}(V,m) := \pm \inf_{B_{m,m}(u)=\pm 1} E_V(u)$$

are the principal eigenvalues of $(P_{V,m,m})$ if and only if $\beta(V,m) \geq 0$. Furthermore, if $\beta(V,m) < 0$ then $\lambda_{\pm 1}(V,m) = -\infty$ (see [14]). It can be therefore seen that problem $(P_{V,m,n})$ has a nontrivial and one-signed solutions under suitable assumptions (see details in [10]) if and only if $\lambda = \lambda_1(V,m)$ or $\lambda = \lambda_1(V,n)$. Let φ_m and $-\varphi_n$ be the corresponding one-signed eigenfunctions associated respectively to $\lambda_1(V,m)$ and $\lambda_1(V,n)$.

Remark 2.1. *Since the boundary weights lie in $C^\alpha(\partial\Omega)$, every solution of (2.4) belongs to $C^{1,\alpha}(\bar{\Omega})$, for $0 < \alpha < 1$ (see [12,14]). We note that if an eigenfunction u is positive in Ω, it is shown that u remains positive on $\partial\Omega$ (see the first part of the proof of Theorem 3.1 in [14]). Furthermore, one can state using Proposition 5.8 in [15] that if u changes sign in Ω then it is also a sign-changing function on $\partial\Omega$.*

Theorem 2.1. *[10] Assume $\lambda_1^D(V) > 0$, $\beta(V,m) > 0$ and $\beta(V,n) > 0$ and let*

$$\Gamma := \{\gamma \in C([0,1], \mathcal{M}_{m,n}) : \gamma(0) = \varphi_m \quad \text{and} \quad \gamma(1) = -\varphi_n\} \neq \emptyset$$

where

$$\mathcal{M}_{m,n} := \{u \in W^{1,p}(\Omega) : B_{m,n}(u) = 1\}. \tag{2.9}$$

Then

$$c(m,n,V) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \tilde{E_V}(u) \tag{2.10}$$

is a nonprincipal eigenvalue for $(P_{V,m,n})$ which satisfies

$$c(m,n,V) > \max\{\lambda_1(V,m), \lambda_1(V,n)\}.$$

Moreover $c(m,n,V)$ is the first nonprincipal eigenvalue of $(P_{V,m,n})$ in sense that there is no other eigenvalue of $(P_{V,m,n})$ between $\max\{\lambda_1(V,m), \lambda_1(V,n)\}$ and $c(m,n,V)$.

As some facts break down when (at least) one of the values $\beta(V,m)$ and $\beta(V,n)$ vanishes, the authors in [10] have proved that in this case, there is still a hope of obtaining existence solutions for $(P_{V,m,n})$. Indeed,

Theorem 2.2. *[10] Assume $\lambda_1^D(V) > 0$, $\beta(V,m) = 0$ or $\beta(V,n) = 0$.*

1. *There exist $u_1 \geq 0$ and $u_2 \leq 0$ in $\mathcal{M}_{m,n}$ such that*

$$E_V(u_1) < c(m,n,V) \quad \text{and} \quad E_V(u_2) < c(m,n,V).$$

2. *Define*

$$\overline{\Gamma}_0 := \{\gamma \in C\,([0,1], \mathcal{M}_{m,n}) : \gamma(0) = u_1 \quad \text{and} \quad \gamma(1) = u_2\} \neq \emptyset.$$

Then

$$\bar{c}(m,n,V) := \inf_{\gamma\in\bar{\Gamma}_0} \max_{t\in[0,1]} E_V(\gamma(t)) \tag{2.11}$$

is a nonprincipal eigenvalue for $(P_{V,m,n})$. Moreover

$$\bar{c}(m,n,V) = c(m,n,V).$$

Continuity and monotonicity properties concerning $c(m,n,V)$ with respect to its first two arguments (boundary weights) are given in [10].

Proposition 2.1. *Let $m,n \in L^q(\partial\Omega)$ with $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and $q \geq 1$ if $p \geq N$, and $V \in L^\infty(\Omega)$. Assume that $\lambda_1^D(V) > 0, \beta(V,m) \geq 0$ and $\beta(V,n) \geq 0$ is verified with $m_k \rightharpoonup m$, $n_k \rightharpoonup n$ and $V_k \rightharpoonup V$ as $k \to \infty$. If $\beta(V,m) = 0$ (resp. $\beta(V,n) = 0$), we then assume in addition that $\beta(V_k, m_k) \geq 0$ for all $k \in \mathbb{N}$ and $m^- \not\equiv 0$ (resp. $\beta(V_k, n_k) \geq 0$ for all $k \in \mathbb{N}$ and $n^- \not\equiv 0$). Hence, the following relations hold:*

1. *$\lambda_1(V_k, m_k) \longrightarrow \lambda_1(V, m)$ as $k \to \infty$ (resp. $\lambda_1(V_k, n_k) \longrightarrow \lambda_1(V, n)$ as $k \to \infty$).*

2. *$c(m_k, n_k, V_k) \longrightarrow c(m, n, V)$ as $k \to \infty$.*

Proposition 2.2. *1. If $m \leq \hat{m}$ and $n \leq \hat{n}$ then $c(m, n, V) \geq c(\hat{m}, \hat{n}, V)$.*

2. If $m \leq \hat{m}$, $n \leq \hat{n}$ in $\partial\Omega$ and

$$\int_{\partial\Omega} (\hat{m} - m)(u^+)^p d\sigma + \int_{\partial\Omega} (\hat{n} - n)(u^-)^p d\sigma > 0 \tag{2.12}$$

for at least one eigenfunction u associated to $c(m, n, V)$, then $c(m, n, V) > c(\hat{m}, \hat{n}, V)$.

We are guided to consider some basic results on the Nemytskii operator. Simple proofs of these facts can be found in (for instance) Kavian [11] or de Figueiredo [8].

2.3. On the Nemytskii operator

Let Ω be as in the beginning of Section 2. and $g : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function, i.e.:

- for each $s \in \mathbb{R}$, the function $x \longmapsto g(x, s)$ is Lebesgue measurable in $\partial\Omega$;
- for a.e. $x \in \partial\Omega$, the function $s \longmapsto g(x, s)$ is continuous in $\mathbb{R}$.

In the case of a Carathéodory function, the assertion $x \in \partial\Omega$ is to be understood in the sense a.e. $x \in \partial\Omega$. Let $\mathcal{M}$ be the set of all measurable function $u : \partial\Omega \longrightarrow \mathbb{R}$.

Proposition 2.3. *If $g : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory, then, for $u \in \mathcal{M}$, the function $N_g u : \partial\Omega \longrightarrow \mathbb{R}$ defined by*

$$(N_g u)(x) = g(x, u(x)) \quad \text{for } x \in \partial\Omega \tag{2.13}$$

is measurable in $\partial\Omega$.

In the light of this proposition, a Carathéodory function $g : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defines an operator $N_g : \mathcal{M} \longrightarrow \mathcal{M}$, which is called Nemytskii. The result below states sufficient conditions when a Nemytskii operator maps an L^{q_1} space into another L^{q_2}.

Proposition 2.4. *Assume $g : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory and the following growth condition is satisfied:*

$$|g(x, s)| \leq C|s|^r + b(x) \quad \text{for } x \in \partial\Omega,\ s \in \mathbb{R}, \tag{2.14}$$

where $C \geq 0$ is constant, $r > 0$ and $b \in L^{p_1}(\partial\Omega)$ with $1 \leq p_1 < \infty$.

Then $N_g(L^{p_1 r}(\partial\Omega)) \subset L^{p_1}(\partial\Omega)$. In addition, N_g is continuous from $L^{p_1 r}(\partial\Omega)$ into $L^{p_1}(\partial\Omega)$ and maps bounded sets into bounded sets.

We now give some important results concerning the Nemytskii operator that will be used later.

Proposition 2.5. *Suppose $g : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is Carathéodory and it satisfies the growth condition:*

$$|g(x,s)| \leq C|s|^{p-1} + b(x) \quad \text{for } x \in \partial\Omega,\ s \in \mathbb{R}, \tag{2.15}$$

where $C > 0$ is constant, $p > 1$, $b \in L^{p'}(\partial\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $G : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$G(x,s) = \int_0^s g(x,t)dt. \tag{2.16}$$

Then we have:

1. *the function G is Carathéodory and there exist $C_1 \geq 0$ constant and $d \in L^1(\partial\Omega)$ such that*
$$|G(x,s)| \leq C_1|s|^p + d(x) \quad \text{for } x \in \partial\Omega,\ s \in \mathbb{R}; \tag{2.17}$$
2. *the functional $\Psi : L^p(\partial\Omega) \longrightarrow \mathbb{R}$ defined by*
$$\Psi(u) := \int_{\partial\Omega} N_G u = \int_{\partial\Omega} G(x,u)$$
is continuously Fréchet differentiable and $\Psi'(u) = N_G u$ for all $u \in L^p(\partial\Omega)$.

3. Assumptions and nonresonance results

The present article deals explicitly with a very known type of problem

$$(P_f) : \begin{cases} \Delta_p u &= V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(x,u) & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of class $C^{2,\alpha}$ where $0 < \alpha < 1$ with outward unit normal ν on the boundary $\partial\Omega$.

The functions $V(x)$ and $f(x,s)$ satisfy the following conditions:

$$(H_V) : V(x) \in L^\infty(\Omega) \text{ possibly indefinite,}$$

$$(H_C) : f : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is a Carathéodory function lying in } C^r$$

for some $0 < r < 1$.

We define the following functions and make the assumption that they have nontrivial positive parts:

$$(H_f) \quad k_\pm(x) := \liminf_{s \to \pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s \to \pm\infty} \frac{f(x,s)}{|s|^{p-2}s} := K_\pm(x)$$

$$(H_F) \quad l_\pm(x) := \liminf_{s \to \pm\infty} \frac{pF(x,s)}{|s|^p} \leq \limsup_{s \to \pm\infty} \frac{pF(x,s)}{|s|^p} := L_\pm(x)$$

with $F(x,s) = \int_0^s f(x,t)dt$. We also assume that

$$(H_S) \quad k_\pm, K_\pm, l_\pm, \text{ and } L_\pm \text{ are in } C^r(\partial\Omega)$$

for some $0 < r < 1$ and note that the aforementioned limits are held uniformly with respect to $x \in \partial\Omega$ that is for every $\varepsilon > 0$, there exist $a_\varepsilon \in L^{p'}(\partial\Omega)$, and $b_\varepsilon \in L^1(\partial\Omega)$ such that for a.e. $x \in \partial\Omega$ and $\forall s \in \mathbb{R}$,

$$(H_1) \quad -a_\varepsilon + \Big(k_+(x) + K_-(x) - \varepsilon\Big)|s|^{p-2}s \leq f(x,s) \leq \Big(k_-(x) + K_+(x) + \varepsilon\Big)|s|^{p-2}s + a_\varepsilon$$

$$(H_2) \quad -b_\varepsilon + \Big(l_+(x) + l_-(x) - \varepsilon\Big)\frac{|s|^p}{p} \leq F(x,s) \leq \Big(L_+(x) + L_-(x) + \varepsilon\Big)\frac{|s|^p}{p} + b_\varepsilon.$$

According to (H_S) and (H_1), we conclude that there exist $a_1 > 0$ and $b_1 \in L^{p'}(\partial\Omega)$ such that

$$(H_3) \quad |f(x,s)| \leq a_1|s|^{p-1} + b_1(x)$$

for a.e. $x \in \partial\Omega$ and all $s \in \mathbb{R}$.

In addition, we require the above functions to satisfy

$$(H_4) \quad \begin{cases} 0 < \lambda_1(V,k_\pm) \leq 1 \quad \text{and} \quad c(K_+,K_-,V) \geq 1 \\ 0 < \lambda_1(V,l_\pm) < 1 \quad \text{and} \quad c(L_+,L_-,V) > 1 \\ \lambda_1^D(V) > 0, \quad \beta(V,k_\pm) > 0 \quad \text{and} \quad \beta(V,l_\pm) > 0 \end{cases} \tag{3.2}$$

where $\lambda_1^D(V)$, $\beta(V,\cdot)$, $\lambda_1(V,\cdot)$ and $c(\cdot,\cdot,V)$ are related to the asymmetric Steklov problem (2.4).

Remark 3.1. *One easily checks from* (H_f) *and* (H_F) *that*

$$k_\pm(x) \leq l_\pm(x) \leq L_\pm(x) \leq K_\pm(x) \quad \text{a.e. on } \partial\Omega. \tag{3.3}$$

We state our first result concerning the strict monotonicity of $\lambda_1(V,\cdot)$ as a principal positive eigenvalue of $(P_{V,m,m})$.

Proposition 3.1. *If* $m_1(x) \lneqq m_2(x)$ *on* $\partial\Omega$ *(where* $\lneqq$ *means that one has a large inequality a.e in* $\partial\Omega$ *and a strict inequality in a subset with a positive measure) then* $\lambda_1(V,m_1) > \lambda_1(V,m_2)$.

Proof: Let $m_1(x)$ and $m_2(x)$ be two weight functions satisfying $m_1(x) \lneqq m_2(x)$ for a.e. x in $\partial\Omega$ and φ_{m_1} be an eigenfunction associated to $\lambda_1(V,m_1)$. We know that φ_{m_1} is positive and $\varphi_{m_1} > 0$ on $\partial\Omega$. One has

$$\frac{1}{\lambda_1(V,m_1)} = \frac{1}{E_V(\varphi_{m_1})}\int_{\partial\Omega} m_1(x)\varphi_{m_1}^p d\sigma \tag{3.4}$$

and then

$$\frac{1}{E_V(\varphi_{m_1})}\int_{\partial\Omega} m_1(x)\varphi_{m_1}^p d\sigma < \frac{1}{E_V(\varphi_{m_1})}\int_{\partial\Omega} m_2(x)\varphi_{m_1}^p d\sigma. \tag{3.5}$$

On the other hand,

$$\begin{aligned}\frac{1}{E_V(\varphi_{m_1})}\int_{\partial\Omega} m_2(x)\varphi_{m_1}^p d\sigma &\leq \sup\left\{\frac{1}{E_V(u)}\int_{\partial\Omega} m_2(x)|u|^p d\sigma:\ u\in W^{1,p}(\Omega)\right\} = \\ &= \frac{1}{\lambda_1(V,m_2)} \end{aligned}\tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we get $\frac{1}{\lambda_1(V,m_1)} < \frac{1}{\lambda_1(V,m_2)}$
i.e.

$$\lambda_1(V,m_1) > \lambda_1(V,m_2).$$

□

3.1. Nonresonance between the first two eigenvalues

We study a nonresonance problem relating to Steklov boundary conditions and in addition we deal with some indefinite weight as a new feature. To fix one's ideas, problem (P_f) can be found in [4,7] where particular cases of weight were considered. Throughout this subsection, we work on gathering needed properties to apply a version of the classical "Mountain Pass Theorem" for a C^1 functional restricted to a C^1 manifold (see [1,8]). Our purpose is of course to obtain existence results for (P_f) and by doing so, extend some of the known results in [4,5,7]. In order to have things well defined in the context of variational approach, we consider for $u\in W^{1,p}(\Omega)$,

$$\Phi(u) := \frac{1}{p}E_V(u) - \int_{\partial\Omega} F(x,u)d\sigma \tag{3.7}$$

as the C^1 functional which allows to get the weak formulation of (3.1) as follows

$$\langle\Phi'(u),v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v + \int_\Omega V(x)|u|^{p-2}uv - \int_{\partial\Omega} f(x,u)v d\sigma = 0. \tag{3.8}$$

It follows readily that the critical points of Φ are precisely the weak solutions of (P_f). So the search for solutions of (3.1) is transformed in the investigation of critical points of Φ relying on standard arguments. For convenience, we recall a version of the well-known "Mountain Pass Theorem" in a useful and popular form (see [1]).

Proposition 3.2. *[1] Let E be a real Banach space and let $M := \{u\in E;\ g(u) = 1\}$, where $g\in C^1(E,\mathbb{R})$ and 1 is a regular value of g. Let $f\in C^1(E,\mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M. Let $u,v\in M$ with $u\neq v$ and suppose that*

$$H := \{h\in C([0,1],M) : h(0)=u \quad \text{and} \quad h(1)=v\} \tag{3.9}$$

is nonempty. Assume also that

$$c := \inf_{h\in H} \max_{w\in h([0,1])} f(w) > \max\{f(u), f(v)\}$$

and that $\tilde{f}$ *satisfies* (PS) *condition on* M*. Then* c *is a critical value of* $\tilde{f}$*.*

We state our first main result as follows:

Theorem 3.1. *Assume that* (H_V), (H_C), (H_f), (H_F), (H_S) *and* (H_4) *are satisfied. Then the problem* (P_f) *admits a solution in* $W^{1,p}(\Omega)$.

We will use Proposition 3.2 for the proof of Theorem 3.1 and we start with the following that proves the required Palais-Smale condition.

Proposition 3.3. Φ *satisfies the* (PS) *condition on* $W^{1,p}(\Omega)$ *that is for any sequence* (u_n) *such that*

$$\begin{cases} |\Phi(u_n)| \le c \\ |\langle \Phi'(u_n), \varphi\rangle| \le \varepsilon_n ||\varphi|| \quad \forall\, \varphi \in W^{1,p}(\Omega) \end{cases} \tag{3.10}$$

with c real constant and $\varepsilon_n \to 0$*, one has that* (u_n) *admits a convergent subsequence.*

Proof: The proof adopts the scheme in [7]. Let (u_n) be a Palais-Smale sequence, i.e. (3.10) is satisfied. Since $W^{1,p}(\Omega)$ is a Banach space that is reflexive, to prove that (u_n) has a convergent subsequence, it suffixes to prove its boundedness. To this end, let assume by contradiction that (u_n) is unbounded i.e. $||u_n|| \to \infty$ and set $v_n = \dfrac{u_n}{||u_n||}$. We now show that this is not the case, so arriving to contradiction. As (v_n) is bounded in the same space $W^{1,p}(\Omega)$, one can find some v_0 in $W^{1,p}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ and then in $L^p(\partial\Omega)$. Using (H_3) with $s = u_n(x)$ and divide it by $||u_n||^{p-1}$, one deduces that $f(x,u_n)/||u_n||^{p-1}$ is bounded in $L^{p'}(\partial\Omega)$ and then converges weakly to some f_0. Rewriting the second inequality of (3.10) by setting $\varphi = (v_n - v_0)$, we reach

$$\begin{aligned} \frac{\langle \Phi'(u_n), \varphi\rangle}{||u_n||^{p-1}} &= \frac{\langle \Phi'(u_n), (v_n - v_0)\rangle}{||u_n||^{p-1}} \\ &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla(v_n - v_0)dx + \int_\Omega V(x)|v_n|^{p-2}v_n(v_n - v_0)dx \\ &\quad - \int_{\partial\Omega} \frac{f(x,u_n)}{||u_n||^{p-1}}(v_n - v_0)d\sigma \longrightarrow 0. \end{aligned} \tag{3.11}$$

Applying Hölder inequality and taking into account the fact that $f(x,u_n)/||u_n||^{p-1}$ is bounded in $L^{p'}(\partial\Omega)$, one easily checks that

$$\left| \int_{\partial\Omega} \frac{f(x,u_n)}{||u_n||^{p-1}}(v_n - v_0)d\sigma \right| \longrightarrow 0 \tag{3.12}$$

and

$$\left|\int_\Omega V(x)|v_n|^{p-2}v_n(v_n - v_0)dx\right| \longrightarrow 0. \tag{3.13}$$

Thus

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla(v_n - v_0)dx \longrightarrow 0. \tag{3.14}$$

Moreover,

$$\int_\Omega \left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0\right) \cdot (\nabla v_n - \nabla v_0)\, dx \longrightarrow 0.$$

Applying the (S^+) property stated in Lemma 3.1 below and Hölder inequality, one easily derives that (v_n) converges strongly to v_0 in $W^{1,p}(\Omega)$ with $||v_0|| = 1$. From (3.11), we can write

$$\int_\Omega |\nabla v_0|^{p-2}\nabla v_0 \cdot \nabla \varphi dx + \int_\Omega V|v_0|^{p-2}v_0\varphi dx = \int_{\partial\Omega} f_0\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.15}$$

Based on (H_f) (see [9]), there exist α_1 and α_2 in $L^q(\partial\Omega)$ such that

$$f_0(x) = \alpha_1(x)(v_0^+)^{p-1} - \alpha_2(x)(v_0^-)^{p-1} \tag{3.16}$$

and almost for every $x \in \partial\Omega$,

$$\begin{cases} k_+(x) \leq \alpha_1(x) \leq K_+(x) \\ k_-(x) \leq \alpha_2(x) \leq K_-(x). \end{cases} \tag{3.17}$$

In view of (3.16) and since the values of $\alpha_1(x)$ (resp. $\alpha_2(x)$) on $\{x \in \partial\Omega : v_0(x) \leq 0\}$ (resp. on $\{x \in \partial\Omega : v_0(x) \geq 0\}$) are irrelevant, we follow [7] by assuming that

$$\begin{cases} \alpha_1(x) = L_+(x) \quad \text{on} \quad \{x \in \partial\Omega : v_0(x) \leq 0\} \\ \\ \alpha_2(x) = L_-(x) \quad \text{on} \quad \{x \in \partial\Omega : v_0(x) \geq 0\}. \end{cases} \tag{3.18}$$

Relying on Remark 2.1, we will distinguish the two cases where $v_0 \geq 0$ a.e. on $\partial\Omega$ or v_0 changes sign on $\partial\Omega$ and prove that $v_0 \geq 0$ almost everywhere on $\partial\Omega$ or v_0 changes sign on $\partial\Omega$, both lead to a contradiction and thereby get expected conclusion.

1. Suppose first that $v_0 \geq 0$ almost everywhere on $\partial\Omega$ and consider (3.15). One shows that $v_0 > 0$ on $\partial\Omega$. Indeed, assume that $v_0 = 0$ on $\partial\Omega$. Then $v_0 \in W_0^{1,p}(\Omega)$ and (3.15) becomes

$$\int_\Omega |\nabla v_0|^{p-2}\nabla v_0 \cdot \nabla\varphi dx + \int_\Omega V|v_0|^{p-2}v_0\varphi dx = 0, \quad \forall \varphi \in W^{1,p}(\Omega)$$

and for $\varphi = v_0 \in W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$, we have $E_V(v_0) = 0$. Moreover $\frac{v_0}{||v_0||^p_{L^p(\Omega)}}$ is admissible for $\lambda_1^D(V)$ and consequently,

$$\lambda_1^D(V) \leq E_V\left(\frac{v_0}{||v_0||^p_{L^p(\Omega)}}\right) = 0$$

which contradicts the assumption $\lambda_1^D(V) > 0$. Thus $v_0 > 0$ on $\partial\Omega$ and $f(x) = \alpha_1(x)(v_0^+)^{p-1} \neq 0$ since otherwise, we get into the previous case. We deduce that $B_{\alpha_1,\alpha_2}(v_0) \neq 0$ and then $\lambda_1(V, \alpha_1) \geq 1$ and combining monotonicity of $\lambda_1(V, \cdot)$, (3.2) and (3.17), one obtains $\lambda_1(V, k_+) = 1$ and then $\alpha_1 = k_+$ almost everywhere on $\partial\Omega$ by strict monotonicity. We have from the first condition in (3.10)

$$\frac{\Phi(u_n)}{||u_n||^p} \longrightarrow 0 \tag{3.19}$$

that is

$$E_V(v_0) = \lim_{n\to\infty} \int_{\partial\Omega} \frac{pF(x, u_n(x))}{||u_n||^p} d\sigma. \tag{3.20}$$

Using (H_2) and passing to the limit, we have

$$\int_{\partial\Omega} l_+ v_0^p d\sigma \leq \lim_{n\to\infty} \int_{\partial\Omega} \frac{pF(x, u_n(x))}{||u_n||^p} d\sigma = E_V(v_0) = \int_{\partial\Omega} \alpha_1 v_0^p d\sigma. \tag{3.21}$$

From Remark 3.1, one write $\alpha_1 = k_+ \leq l_+$ almost everywhere on $\partial\Omega$ and then $\alpha_1 = l_+$ since $v_0 > 0$. This implies $\lambda_1(V, l_+) = 1$ which contradicts the strict inequality in (3.2).

2. Suppose now that v_0 changes sign on $\partial\Omega$ and still consider (3.15). Then v_0 verifies (3.15) which means that v_0 is a solution of the following Steklov problem

$$\begin{cases} \Delta_p u &= V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \alpha_1(u^+)^{p-1} - \alpha_2(u^-)^{p-1} & \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Let us show that $B_{\alpha_1,\alpha_2}(v_0) \neq 0$. Assume by contradiction that

$$B_{\alpha_1,\alpha_2}(v_0) = \int_{\partial\Omega} \alpha_1(x)(v_0^+)^p d\sigma + \int_{\partial\Omega} \alpha_2(x)(v_0^-)^p d\sigma = 0.$$

Repeating similar arguments from the proof of [10, Proposition 3.10], we reach a contradiction and one can infer $c(\alpha_1, \alpha_2, V) \leq 1$. Moreover, monotonicity of $c(\cdot, \cdot, V)$ together with (3.17) and (3.2) lead to

$$c(\alpha_1, \alpha_2, V) = c(K_+, K_-, V) = 1.$$

Adapt ideas from the previous case, we have

$$\begin{aligned} \int_{\partial\Omega} (\alpha_1(v_0^+)^p + \alpha_2(v_0^-)^p) d\sigma &= E_V(v_0) = \lim_{n\to\infty} \int_{\partial\Omega} \frac{pF(x, u_n(x))}{||u_n||^p} d\sigma \\ &\leq \int_{\partial\Omega} (L_+(v_0^+)^p + L_-(v_0^-)^p) d\sigma \\ &\leq \int_{\partial\Omega} (K_+(v_0^+)^p + K_-(v_0^-)^p) d\sigma. \end{aligned} \tag{3.23}$$

Let assume by contradiction that

$$\int_{\partial\Omega}(\alpha_1(v_0^+)^p + \alpha_2(v_0^-)^p)d\sigma < \int_{\partial\Omega}(K_+(v_0^+)^p + K_-(v_0^-)^p)d\sigma.$$

Therefore

$$\int_{\partial\Omega}\left((K_+ - \alpha_1)(v_0^+)^p + (K_- - \alpha_2)(v_0^-)^p\right)d\sigma > 0$$

which leads to $c(\alpha_1, \alpha_2, V) > c(K_+, K_-, V)$ by the strict monotonicity of $c(\cdot, \cdot, V)$. We then reach a contradiction since we have established that $c(\alpha_1, \alpha_2, V) = c(K_+, K-, V)$. Finally, (3.23) reads as

$$\begin{aligned}\int_{\partial\Omega}(\alpha_1(v_0^+)^p + \alpha_2(v_0^-)^p)d\sigma &= E_V(v_0) = \lim_{n\to\infty}\int_{\partial\Omega}\frac{pF(x, u_n(x))}{||u_n||^p}d\sigma \\ &= \int_{\partial\Omega}(L_+(v_0^+)^p + L_-(v_0^-)^p)d\sigma \qquad (3.24)\\ &= \int_{\partial\Omega}(K_+(v_0^+)^p + K_-(v_0^-)^p)d\sigma\end{aligned}$$

and then from Remark 3.1 and (3.17),

$$\begin{cases} L_+(x) = K_+(x) & \text{on} \quad \{x \in \partial\Omega : v_0(x) > 0\} \\ L_-(x) = K_-(x) & \text{on} \quad \{x \in \partial\Omega : v_0(x) < 0\}\end{cases} \qquad (3.25)$$

and

$$\begin{cases} \alpha_1(x) = K_+(x) & \text{on} \quad \{x \in \partial\Omega : v_0(x) > 0\} \\ \alpha_2(x) = K_-(x) & \text{on} \quad \{x \in \partial\Omega : v_0(x) < 0\}\end{cases} \qquad (3.26)$$

hold. Considering (3.18), it follows that

$$c(\alpha_1, \alpha_2, V) = c(L_+, L_-, V) = 1$$

which contradicts the strict inequality in (3.2) and put an end to the proof.

□

Lemma 3.1. *[13][(S^+) property] For all $x, y \in \mathbb{R}^N$, we have*

$$|x-y|^p \le c\left[\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y)\right]^{s/2}\left[|x|^p + |y|^p\right]^{1-s/2}$$

with $c = c(p)$, $s = p$ if $1 < p < 2$ and $s = 2$ if $p \ge 2$.

We now turn to the study of the geometry of Φ and first look for directions along which Φ goes to $-\infty$.

Lemma 3.2. *If δ_+ (resp. δ_-) is a positive eigenfunction associated to $\lambda_1(V, l_+)$ (resp. $\lambda_1(V, l_-)$), then $\lim_{r\to\infty} \Phi(r\delta_+) = -\infty$ (resp. $\lim_{r\to\infty} \Phi(-r\delta_-) = -\infty$).*

Proof: Let us show that $\Phi(r\delta_+) \to -\infty$ (similar argument for $\Phi(-r\delta_-) \to -\infty$). As in [1] and [7], we recall for simplicity, the meaning of (H_F) being holding uniformly with respect to $x \in \partial\Omega$. That is for every $\varepsilon > 0$, there exists $b_\varepsilon \in L^1(\partial\Omega)$ such that

$$\begin{aligned} &\frac{1}{p}l_+(x)|s^+|^p + \frac{1}{p}l_-(x)|s^-|^p - \frac{\varepsilon}{p}|s|^p - b_\varepsilon \\ \leq\ & F(x,s) \leq \frac{1}{p}L_+(x)|s^+|^p + \frac{1}{p}L_-(x)|s^-|^p + \frac{\varepsilon}{p}|s|^p + b_\varepsilon \end{aligned} \tag{3.27}$$

for almost every $x \in \partial\Omega$ and all $s \in \mathbb{R}$. Let take $r > 0$ and get back to Φ to write

$$\begin{aligned} \Phi(r\delta_+) &= \frac{r^p}{p}E_V(\delta_+) - \int_{\partial\Omega} F(x, r\delta_+) \\ &\leq \frac{r^p}{p}E_V(\delta_+) - \frac{r^p}{p}\int_{\partial\Omega}\left(l_+\delta_+^p - \varepsilon\delta_+^p\right)d\sigma + \int_{\partial\Omega} b_\varepsilon d\sigma \\ &\leq \frac{r^p}{p}E_V(\delta_+) - \frac{r^p}{p}.\frac{E_V(\delta_+)}{\lambda_1(V,l_+)} + \frac{\varepsilon r^p}{p}\int_{\partial\Omega}\delta_+^p d\sigma + \int_{\partial\Omega} b_\varepsilon d\sigma \\ &\leq \frac{r^p}{p}\left(1 - \frac{1}{\lambda_1(V,l_+)}\right)E_V(\delta_+) + \frac{\varepsilon r^p}{p}\int_{\partial\Omega}\delta_+^p d\sigma + \int_{\partial\Omega} b_\varepsilon d\sigma \\ &\leq \frac{r^p}{p}\left(1 - \frac{1}{\lambda_1(V,l_+)} + \varepsilon\frac{\int_{\partial\Omega}\delta_+^p d\sigma}{E_V(\delta_+)}\right)E_V(\delta_+) + \int_{\partial\Omega} b_\varepsilon d\sigma. \end{aligned}$$

As $\lambda_1(V, l_+) < 1$, we just have to choose ε less than $\frac{(1-\lambda_1(V,l_+))E_V(\delta_+)}{\lambda_1(V,l_+)\int_{\partial\Omega}\delta_+^p d\sigma}$ to get $\Phi(r\delta_+) \to -\infty$ when $r \to +\infty$. □

Proposition 3.4. *There exists $r_0 > 0$ such that for all $r \geq r_0$ and for all $\gamma \in \Gamma_r$ with*

$$\Gamma_r := \{\gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = r\delta_+ \quad \textit{and} \quad \gamma(1) = -r\delta_-\},$$

one obtains

$$\max_{w\in\gamma([0,1])} \Phi(w) > max\{\Phi(r\delta_+), \Phi(-r\delta_-)\}.$$

Once Proposition 3.4 is proved, we can pick $r \geq r_0$ and apply Proposition 3.2 by setting $H = \Gamma_r$ and $f \equiv \Phi$ in Proposition 3.2 to conclude on the solvability of (P_f).

Proof: [Proof of Proposition 3.4.] Keeping b_ε as in (3.27), it follows that there is a possibility to pick $r_0 > 0$ by Lemma 3.2 and get for all $r > r_0$,

$$-\int_{\partial\Omega} b_\varepsilon d\sigma > max\{\Phi(r\delta_+), \Phi(-r\delta_-)\}. \tag{3.28}$$

Thus let $r > r_0$ and take $\gamma \in \Gamma_r$. We now face the two cases that arise here that is either $B_{L_+,L_-}(\gamma(t)) > 0$ for all $t \in [0,1]$ or there exists $t_0 \in [0,1]$ such that $B_{L_+,L_-}(\gamma(t_0)) \leq 0$.

1. In the first case, $B_{L_+,L_-}(\gamma(t)) > 0$ for all $t \in [0,1]$ and we set

$$\tilde{\gamma}(t) := \frac{\gamma(t)}{\left(B_{L_+,L_-}(\gamma(t))\right)^{1/p}}$$

which is a path in $\mathcal{M}_{L_+,L_-}$ (which is defined in (2.10)) that verifies

$$\max_{w \in \tilde{\gamma}([0,1])} E_V(w) \geq c(L_+, L_-, V). \tag{3.29}$$

On the other hand

$$\begin{aligned}
\Phi(w) &\geq \frac{1}{p}E_V(w) - \frac{1}{p}\int_{\partial\Omega}\left(L_+(x)(w^+)^p + L_-(x)(w^-)^p\right)d\sigma \\
&- \frac{\varepsilon}{p}\int_{\partial\Omega}|w|^p d\sigma - \int_{\partial\Omega} a_\varepsilon d\sigma \\
&\geq \frac{1}{p}E_V(w) - \frac{1}{p}B_{L_+,L_-}(w) - \varepsilon\int_{\partial\Omega}|w|^p d\sigma - \int_{\partial\Omega} b_\varepsilon d\sigma \quad (3.30)
\end{aligned}$$

and from (3.29),

$$\begin{aligned}
\max_{w\in\tilde{\gamma}([0,1])}\frac{1}{B_{L_+,L_-}(w)} \quad \times \quad &\Big[p\Phi(w) + pB_{L_+,L_-}(w) + \varepsilon\int_{\partial\Omega}|w|^p d\sigma \\
&+p\int_{\partial\Omega} b_\varepsilon d\sigma\Big] \\
&\geq c(L_+, L_-, V)
\end{aligned}$$

which means that one can find some w_0 in $\gamma([0,1])$ such that

$$\Phi(w_0)+\frac{1}{p}B_{L_+,L_-}(w_0)+\frac{\varepsilon}{p}\int_{\partial\Omega}|w_0|^p d\sigma+\int_{\partial\Omega} b_\varepsilon d\sigma \geq c(L_+, L_-, V)B_{L_+,L_-}(w_0)$$

that is

$$\Phi(w_0) \geq \left(c(L_+, L_-, V) - \frac{1}{p}\right)B_{L_+,L_-}(w_0) - \varepsilon\int_{\partial\Omega}|w_0|^p d\sigma - \int_{\partial\Omega} b_\varepsilon d\sigma,$$

for all $\varepsilon > 0$. Thus one can choose ε small enough, to get

$$\left(c(L_+, L_-, V) - \frac{1}{p}\right)B_{L_+,L_-}(w_0) - \varepsilon\int_{\partial\Omega}|w_0|^p d\sigma > 0$$

and then

$$\Phi(w_0) > -\int_{\partial\Omega} b_\varepsilon d\sigma.$$

Consequently,

$$\Phi(w_0) > -\int_{\partial\Omega} b_\varepsilon d\sigma > max\{\Phi(r\delta_+), \Phi(-r\delta_-)\} \tag{3.31}$$

and it reads $\max_{w\in\gamma([0,1])} \Phi(w) > max\{\Phi(r\delta_+), \Phi(-r\delta_-)\}$.

2. In the second case, there exists $t_0 \in [0,1]$ such that $B_{L_+,L_-}(\gamma(t_0)) \le 0$. From (3.27), we get

$$\begin{aligned} &\max_{w\in\gamma([0,1])} \Phi(w) \ge \Phi(\gamma(t_0)) \\ \ge\ & \frac{1}{p}E_V(\gamma(t_0)) - \frac{1}{p}B_{L_+,L_-}(\gamma(t_0)) - \frac{\varepsilon}{p}\int_{\partial\Omega} |\gamma(t_0)|^p d\sigma \\ & -\int_{\partial\Omega} b_\varepsilon d\sigma. \end{aligned} \tag{3.32}$$

- If $B_{L_+,L_-}(\gamma(t_0)) \le 0$ and $\int_{\partial\Omega} |\gamma(t_0)|^p d\sigma = 0$ then $\gamma(t_0) = 0$ almost everywhere on $\partial\Omega$ and $\gamma(t_0) \in W_0^{1,p}(\Omega)$. Assuming that $\gamma(t_0) = 0$ in Ω, the relation (3.32) reads

$$\max_{w\in\gamma([0,1])} \Phi(w) \ge -\int_{\partial\Omega} b_\varepsilon d\sigma \quad \text{(which is our expected result).}$$

Now suppose rather that $\gamma(t_0) \neq 0$ in Ω. Hence we can normalize the path $\gamma(t_0)$ and get

$$\zeta_0 := \frac{\gamma(t_0)}{||\gamma(t_0)||_{L^p(\Omega)}^{1/p}}$$

as an admissible function for the definition of $\lambda_1^D(V)$ and write

$$0 < \lambda_1^D(V) \le E_V(\zeta_0) = \frac{E_V(\gamma(t_0))}{||\gamma(t_0)||_{L^p(\Omega)}}.$$

This leads to $E_V(\gamma(t_0)) > 0$ and we can conclude that

$$\frac{1}{p}E_V(\gamma(t_0)) - \frac{1}{p}B_{L_+,L_-}(\gamma(t_0)) > 0.$$

As a consequence,

$$\max_{w\in\gamma([0,1])} \Phi(w) \ge -\int_{\partial\Omega} b_\varepsilon d\sigma$$

and the result follows.

- If $B_{L_+,L_-}(\gamma(t_0)) \leq 0$ and $\int_{\partial\Omega} |\gamma(t_0)|^p d\sigma > 0$ then one can define

$$\tilde{\gamma} := \frac{\gamma(t_0)}{||\gamma(t_0)||_{L^p(\partial\Omega)}}$$

and easily check that $B_{1,1}(\tilde{\gamma}) = 1$ refering to the problem tackled in [14] with constant weight 1 on the boundary. The function $\tilde{\gamma}$ is therefore admissible for $\lambda_1(V, 1)$ and plugging a right $\varepsilon > 0$ in (3.32), we obtain again

$$\max_{w\in\gamma([0,1])} \Phi(w) \geq -\int_{\partial\Omega} b_\varepsilon d\sigma$$

and consequently

$$\max_{w\in\gamma([0,1])} \Phi(w) > max\{\Phi(r\delta_+), \Phi(-r\delta_-)\}.$$

This achieves the proof of Proposition 3.4 and also of Theorem 3.1. □

3.2. Nonresonance under the First Eigenvalue

We are mainly interested in this subsection in the situation where the condition of nonresonance lies below the first eigenvalue. Precisely, for a Carathéodory function $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (H_C) and define $F(x, s) = \int_0^s f(x, t)dt$, we assume

$$g(x) := \limsup_{|s|\to+\infty} \frac{pF(x, s)}{|s|^p} \tag{3.33}$$

to have nontrivial positive parts, lying in $C^r(\partial\Omega)$ and verifies

$$\lambda_1(V, g) > 1. \tag{3.34}$$

Remark 3.2. *The conditions* $g(x) := \limsup_{|s|\to+\infty} \frac{pF(x, s)}{|s|^p}$ *imply that for all* $\varepsilon > 0$, $\exists\, d_\varepsilon \in L^{p'}(\partial\Omega)$ *such that*

$$F(x, s) \leq (g(x) + \varepsilon) \frac{|s|^p}{p} + d_\varepsilon(x) \tag{3.35}$$

for a.e. x *in* $\partial\Omega$ *and* $\forall s \in \mathbb{R}$.

Theorem 3.2. *If* $\lambda_1^D(V) > 0$, $\beta(V, g) > 0$, (H_C), (H_S) *and* (3.34) *are satisfied then the problem* (P_f) *has (at least) one solution in* $\mathcal{M}_{g,g}$.

Proof: Let us show first that the energy functional Φ is coercive. Indeed, assume by contradiction that there exists a sequence (u_n) in $\mathcal{M}_{g,g}$ such that

$$||u_n||_{W^{1,p}} := \int_\Omega |\nabla u_n|^p dx + \int_{\partial\Omega} |u_n|^p d\sigma \to \infty \tag{3.36}$$

(where the norm $||\cdot||_{W^{1,p}}$ is a equivalent to the usual norm on $W^{1,p}(\Omega)$) and

$$K \geq \frac{1}{p}E_V(u_n) - \int_{\partial\Omega} F(x,u_n)d\sigma = \Phi(u_n) \tag{3.37}$$

for some constant K. One shows by contradiction that $t_n := \left(\int_\Omega |u_n|^p dx\right)^{1/p} \to \infty$. Indeed, assume by contradiction that (t_n) is bounded. We first deduce that $\int_{\partial\Omega} |u_n|^p d\sigma$ is bounded and in addition we write from (3.7),

$$\int_\Omega |\nabla u_n|^p dx = p\Phi(u_n) - \int_\Omega V(x)|u_n|^p dx + p\int_{\partial\Omega} F(x,u_n)d\sigma \tag{3.38}$$

and using (3.35), it reads

$$\int_\Omega |\nabla u_n|^p dx \leq p\Phi(u_n) - \int_\Omega V|u_n|^p dx + \int_{\partial\Omega}(g+\varepsilon)|u_n|^p d\sigma + p\int_{\partial\Omega} d_\varepsilon d\sigma. \tag{3.39}$$

Recall that

$$\int_\Omega V(x)|u_n|^p dx \leq ||V||_{L^\infty(\Omega)} t_n^p$$

and

$$\int_{\partial\Omega} F(x,u_n)d\sigma \leq \int_{\partial\Omega}(g(x)+\varepsilon)\frac{|u_n|^p}{p}d\sigma + \int_{\partial\Omega} d_\varepsilon(x)d\sigma$$

which make $\int_\Omega V(x)|u_n|^p dx$ and $\int_{\partial\Omega} F(x,u_n)d\sigma$ bounded and therefore $\int_\Omega |\nabla u_n|^p dx$ also is bounded. Thus this contradicts (3.36) and we conclude that

$$t_n = \left(\int_\Omega |u_n|^p dx\right)^{1/p} \longrightarrow \infty.$$

Define $v_n = \dfrac{u_n}{t_n}$ and note that $||v_n||_{L^p(\Omega)} = 1$. Dividing (3.39) by t_n^p, we can easily see that $\int_\Omega |\nabla v_n|^p dx$ becomes bounded and then (v_n) is a bounded sequence in $W^{1,p}(\Omega)$ and by standard arguments, one derives that (v_n) converges weakly to some v in $W^{1,p}(\Omega)$ and $v_n \to v$ in $L^p(\Omega) \cap L^p(\partial\Omega)$.

Using (3.35), we have

$$\Phi(u_n) \geq \frac{1}{p}E_V(u_n) - \int_{\partial\Omega}(g(x)+\varepsilon)\frac{|u_n|^p}{p}d\sigma - \int_{\partial\Omega} d_\varepsilon(x)d\sigma. \tag{3.40}$$

Choosing $\varepsilon > 0$ such that $\lambda_1(V, g+\varepsilon) > 1$, it comes

$$K \geq \frac{1}{p}E_V(u_n)\left(1 - \frac{1}{\lambda_1(V,g+\varepsilon)}\right) - \int_{\partial\Omega} d_\varepsilon(x)d\sigma. \tag{3.41}$$

Dividing (3.41) by t_n^p and passing to the limit, one writes $\liminf_{n\to\infty} E_V(v_n) \le 0$. Moreover, we face two cases regarding $\int_{\partial\Omega} |v|^p d\sigma$ that are $\int_{\partial\Omega} |v|^p d\sigma = 0$ and $\int_{\partial\Omega} |v|^p d\sigma > 0$.

- In the case $\int_{\partial\Omega} |v|^p d\sigma = 0$, one has $v \equiv 0$ a.e. in $\partial\Omega$ and as $||v||_{L^p(\Omega)} = 1$, we get $v \in W_0^{1,p}(\Omega)$ and v is therefore admissible for $\lambda_1^D(V)$. This leads to

$$\lambda_1^D(V) \le E_V(v) \le \liminf_{n\to\infty} E_V(v_n) \le 0 \tag{3.42}$$

  which contradicts the assumption $\lambda_1^D(V) > 0$.

- Taking the case $\int_{\partial\Omega} |v|^p d\sigma > 0$, we have $\int_{\partial\Omega} |v_n|^p d\sigma > 0$ and then $\int_{\partial\Omega} |u_n|^p d\sigma > 0$. Let us set $s_n := \left(\int_{\partial\Omega} |u_n|^p d\sigma\right)^{1/p}$ and show that $s_n \to \infty$. By contradiction, assume that the sequence (s_n) is bounded. Then one shows (using (3.39)) that it is so for $\int_\Omega |\nabla u_n|^p dx$ but this, once again, contradicts (3.36) and the result follows. Now we define $w_n := \dfrac{u_n}{s_n}$ and get $||w_n||_{L^p(\partial\Omega)} = 1$. We show by contradiction that $\dfrac{\int_\Omega |u_n|^p dx}{s_n^p}$ is bounded and as

$$\left|\int_{\partial\Omega} g(x)|w_n|^p d\sigma\right| \le \frac{1}{s_n^p} \longrightarrow 0 \quad \text{when } n \to \infty, \tag{3.43}$$

  we conclude by dividing (3.39) by s_n^p that $\int_\Omega |\nabla w_n|^p dx$ is bounded and as a consequence (w_n) is a bounded sequence in $W^{1,p}(\Omega)$ and there exists $w \in W^{1,p}(\Omega)$ such that $w_n \rightharpoonup w$. By standard argument, one reaches $w_n \to w$ in $L^p(\Omega) \cap L^p(\partial\Omega)$ and write $||w||_{L^p(\partial\Omega)} = 1$ and $B_{g,g}(w) = 0$ which mean that w can be seen as an admissible function in the definition of $\beta(V, g)$. Furthermore, dividing (3.41) by s_n^p and passing to the limit, one gets $\liminf_{n\to\infty} E_V(w_n) \le 0$ and consequently

$$\beta(V, g) \le E_V(w) \le \liminf_{n\to\infty} E_V(w_n) \le 0. \tag{3.44}$$

  This leads to a contradiction with the assumption $\beta(V, g) > 0$ and we get expected result that is Φ is coercive on $\mathcal{M}_{g,g}$. Since Φ is sequentially weakly lower semicontinuous, Φ attains a minimum value and this ends the proof.

□

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